Restricted Partition Pairs

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The generating function for sets of pairs of partitions that have an ordering relation between parts of the two pairs is obtained by an inclusion-exclusion argument. Some of the sets of pairs can, by virtue of the extra ordering relation, be interpreted as single partitions. When this is the case an identity between the two generating functions is established. A number of identities including Euler’s theorem and the Rogers–Ramanujan identities are obtained by these means in a doubly bounded form. Each doubly bounded identity combines two known but previously unrelated identities into a single one. The generating functions can be summed to produce another of the same type but with different restrictions. By using this summation an infinite number of identities between a single and a multiple summation can be established. © 1993 Academic Press, Inc.

INTRODUCTION

The polynomials that occur in Schur’s proof of the Rogers–Ramanujan identities [9] have been shown by Andrews [3] to be generating functions for sets of partitions in which the successive ranks are limited to 0 or 1. Andrews and Bressoud [2, 6], using a sieving technique, subsequently showed that the polynomials arising from a more general restriction on the successive ranks also yield partition identities. These generating functions have been further generalized [4] to partitions that have restrictions on their off-diagonal hook differences.

This paper establishes a similar generating function for the pairs of partitions formed by removing the Durfee square from partitions having restrictions along their off-diagonals.

Each partition of the pair is limited by both number of parts and largest part. When the generating function can also be obtained by an alternative argument the two generating functions become the two sides of an identity. A number of identities including Euler’s theorem and the Rogers–Ramanujan identities are obtained in a doubly bounded form. Each doubly bounded identity combines two well known but previously unrelated...
identities into a single one. A summation formula similar in nature to the Bailey transform [5] for these generating functions is established, and by virtue of the dual nature of the restrictions, in which the roles of the hook differences and diagonals become interchanged, we are led to a "Bailey tree" of generating functions by analogy with the Bailey lattice [1, 7].

**Notation.** The Gaussian polynomial will be written

$$
\left[ \begin{array}{c} N + M \\ N \end{array} \right] = \frac{(q)_{N+M}}{(q)_N (q)_M},
$$

where

$$(q)_i = (1 - q)(1 - q^2)(1 - q^3) \cdots (1 - q^i).$$

**Definition 1.** A partition with no more than $N$ parts whose largest part is not greater than $M$ will be called a partition in $(N, M)$, or an $(N, M)$-partition. Let $p(N, M, a, b, \alpha, \beta, n)$ be the number of pairs of partitions with total sum $n$, namely $p_N \geq p_{N-1} \geq \cdots \geq p_1$ and $q_N \geq q_{N-1} \geq \cdots \geq q_1$, both in $(N, M)$ and possibly containing parts of magnitude zero, which for $\alpha, \beta, a, b \geq 0$ have the restrictions

\begin{align*}
p_i - q_i + 1 - a &\geq 1 - \alpha \\
q_i - p_i + 1 - b &\geq 1 - \beta.
\end{align*}

In order to define comparisons outside the range, both partitions will be extended with parts $M$ at the high end and zeros at the low end. Let

$$P(N, M, a, b, \alpha, \beta) = \sum_{n \geq 0} p(N, M, a, b, \alpha, \beta) q^n.$$

**Theorem 1.** If

$$g(x, y) = \left[ \begin{array}{c} N + x + M - y \\ N + x \end{array} \right] \left[ \begin{array}{c} N - x + M + y \\ N - x \end{array} \right]$$

then

$$P(N, M, a, b, \alpha, \beta)$$

$$= \sum_{\mu = -\infty}^{\infty} q^{(a + \beta)(a + b)\mu^2 + (a\beta - ab)\mu} g((a + b)\mu, (\alpha + \beta)\mu)$$

$$- \sum_{\mu = -\infty}^{\infty} q^{(a + \beta)(a + b)\mu^2 + (2a\alpha + a\beta + ab)\mu + a\alpha} g((a + b)\mu + a, (\alpha + \beta)\mu + \alpha).$$

**Proof of Theorem 1.** The generating function for these restricted partition pairs is obtained by subtracting the generating function for all pairs
that do not conform to the restrictions from that for all partition pairs. Any pair of partitions \( p_s \geq p_{s-1} \geq \cdots \geq p_1 \) and \( q_t \geq q_{t-1} \geq \cdots \geq q_1 \) in which \( t \geq s \) can be classified by the pattern of its "faults." A fault is either an \( a \)-fault at position \( p_i \) in which \( p_i - q_{i+1} - a < 1 - \alpha \) or a \( b \)-fault at position \( q_i \) in which \( q_i - p_{i+1} + b < 1 - \beta \).

Suppose the two partitions are merged into a sequence \( q_1, q_{t-1}, \ldots, p_t, q_s, \ldots, p_2, q_2, p_1, q_1 \) and the first fault from the left is found. If the first fault from the left is an \( a \)-fault at \( p_i \) then find the first \( b \)-fault starting at, and to the right of \( q_{i+2} - a \), say at \( q_j \), then find the first \( a \)-fault starting at \( q_{j+2} - b \), etc., and similarly when the first fault reading from right to left is a \( b \)-fault. The result will be a sequence of alternating \( a \)-faults and \( b \)-faults which will be called the \( a-b \)-sequence of the partition pair. If there are no faults then the partition pair belongs to the set that is to be enumerated. This set is made up of all partition pairs less those that have at least one fault.

Suppose that \( A_1 \) is the set of pairs that have at least one \( a \)-fault, and that \( A_2 \) is the set whose \( a-b \)-sequences contain \( ab \). Suppose in general that the \( a-b \)-sequences of members of an \( A_{2i} \) set contain a sequence \( abab \cdots ab \) of length \( 2i \), and the \( a-b \)-sequences of members of an \( A_{2i-1} \) set contain \( abab \cdots a \) of length \( 2i-1 \). Also let \( B_{2i} \) be the set of partition pairs whose \( a-b \)-sequences contain \( baba \cdots ba \) of length \( 2i \), and \( B_{2i-1} \) be the set whose \( a-b \)-sequences include \( baba \cdots b \) of length \( 2i-1 \).

The \( a-b \)-sequence of the set \( A_{2i-1} - A_{2i} \) is either \( abab \cdots ba \) of length \( 2i-1 \) or \( baba \cdots ba \) of length \( 2i \) and the \( a-b \)-sequence of the set \( B_{2i-1} - B_{2i} \) is either \( baba \cdots ab \) of length \( 2i-1 \) or \( abab \cdots ab \) of length \( 2i \).

The set \( A_{2i-1} - A_{2i} + B_{2i-1} - B_{2i} \) therefore contains all those partitions whose \( a-b \)-sequences have length either \( 2i-1 \) or \( 2i \).

The sum \( \sum_{i=1}^{\infty} A_{2i-1} - A_{2i} + B_{2i-1} - B_{2i} \) therefore contains all those partitions that have a fault. If \( A_0 \) is the set of all partition pairs then the set of partitions with the restrictions

\[
\begin{align*}
p_i - q_{i+1} - a &\geq 1 - \alpha \\
q_i - p_{i+1} + b &\geq 1 - \beta
\end{align*}
\]

is

\[
A_0 - \sum_{i=1}^{\infty} A_{2i-1} - A_{2i} + B_{2i-1} - B_{2i}.
\]

The generating functions for the sets \( A_i \) and \( B_i \) are established by using a 1-1 correspondence between \( A_i \) and \( B_{i-1} \) and one between \( B_i \) and \( A_{i-1} \). Consider an \( A_i \) partition pair

\[
\begin{align*}
p_s &\geq p_{s-1} \geq \cdots \geq p_1 \\
q_t &\geq q_{t-1} \geq \cdots \geq q_1
\end{align*}
\]
in which \( t \geq s \) and let \( j \) be the position of its first \( a \)-fault so that \( p_j - q_{j+1-a} < 1 - \alpha \).

Now rearrange the pair to

\[
\begin{align*}
q_i - \alpha & \geq q_{i-1} - \alpha \geq \cdots \geq q_{j+1-a} - \alpha \geq p_j \geq \cdots \geq p_1 \\
p_s + \alpha & \geq p_{s-1} + \alpha \geq \cdots \geq p_{j+1} + \alpha \geq q_{j-a} \geq \cdots \geq q_1.
\end{align*}
\]

The result is a partition pair because \( q_{j+1-a} - \alpha \geq p_j \), and \( p_{j+1-a} \geq q_j \). Either \( p_{j+1} \) is not an \( a \)-fault and so \( p_{j+1} + \alpha \geq 1 + q_{j+2-a} \geq 1 + q_{j-a} \) or \( j = s \) and \( p_{s+1} \) does not exist. The \( a \)-fault at \( p_j \) and all faults to the right continue to exist as before because \( p_{j+1} \geq p_j \) and so \( p_j - (p_{j+1} + \alpha) < 1 - \alpha \) unless \( j = s \) and the first \( a \)-fault has been removed. In this case the result is a \( B_{i-1} \) partition-pair. To reverse the correspondence take any \( B_{i-1} \) partition pair, find the position of the first \( b \)-fault in the last \( B_{i-1} \) sequence, and then look for the transformed \( a \)-fault in the segment before this position. In other words, in a partition, find the \( j \) such that \( q_{j-a} - p_{j+1} < 1 + \alpha \) and for \( k > j \), \( q_{k-a} - p_{k+1} \geq 1 + \alpha \). Now transform

\[
\begin{align*}
p_s & \geq p_{s-1} \geq \cdots \geq p_1 \\
q_i & \geq q_{i-1} \geq \cdots \geq q_1
\end{align*}
\]

to

\[
\begin{align*}
q_i - \alpha & \geq \cdots q_{j+1-a} - \alpha \geq p_j \geq p_{j-1} \geq \cdots \geq p_1 \\
p_s + \alpha & \geq \cdots \geq p_{j+1} + \alpha \geq q_{j-a} \geq \cdots \geq q_1.
\end{align*}
\]

If no transformed fault is found then a new \( a \)-fault is introduced by changing

\[
\begin{align*}
p_s & \geq p_{s-1} \geq \cdots \geq p_1 \\
q_i & \geq q_{i-1} \geq \cdots \geq q_1
\end{align*}
\]

to

\[
\begin{align*}
p_{t+a} & \geq p_{t+a-1} \geq \cdots \geq p_1 \\
p_s + \alpha & \geq p_{s-1} + \alpha \geq p_{t+a+1} + \alpha \geq q_i \geq \cdots \geq q_1
\end{align*}
\]

When an \( A_i \) partition pair is transformed to a \( B_{i-1} \) pair the number of parts of the top partition is changed from \( s \) to \( t+a \) and the size of the bottom partition is changed from \( t \) to \( s-a \). If \( u \) and \( v \) are the largest parts of the top and bottom partitions they are changed to \( v - \alpha \) and \( u + \alpha \), respectively. Also the number \( \alpha(t - j + a) - \alpha(s - j) = \alpha(a + t - s) \) has been subtracted.
Let $A_i(s, u, t, v, a, b, \alpha, \beta) = \sum_{j=0}^{\infty} a_j q^j$ be the generating function for $A_i$ pairs in $(s, u)$ and $(t, v)$, and $B_i(s, u, t, v, a, b, \alpha, \beta) = \sum_{j=0}^{\infty} b_j q^j$ be the generating function for $B_i$ pairs in $(s, u)$ and $(t, v)$. Then

$$A_i(s, u, t, v, a, b, \alpha, \beta) = q^{a(s - t + a)} B_{i-1}(s + a, u - \alpha, t - a, v + \alpha, a, b, \alpha, \beta)$$

and similarly

$$B_i(s, u, t, v, a, b, \alpha, \beta) = q^{b(s - t + b)} A_{i-1}(s + b, u - \beta, t - b, v + \beta, a, b, \alpha, \beta).$$

The generating function for pairs without any restrictions is

$$A_0(s, u, t, v, a, b, \alpha, \beta) = B_0(s, u, t, v, a, b, \alpha, \beta) = [s + u \left\lfloor \frac{t + v}{t} \right\rfloor].$$

The $A_i$ and $B_i$ generating functions may now be deduced step by step as follows:

$$A_1(N, M, N, M, a, b, \alpha, \beta) = q^{a\alpha} B_0(N + a, M - \alpha, N - a, M + \alpha, a, b, \alpha, \beta) = q^{a\alpha} g(a, \alpha);$$

$$B_2((N, M, N, M, a, b, \alpha, \beta) = q^{b\beta} A_1(N + b, M - \beta, N - b, M + \beta, a, b, \alpha, \beta) = q^{(N + b - (N - b) + b)\beta} q^{a\alpha} g(a + b, \alpha + \beta).$$

In general

$$A_{2\mu}(N, M, N, M, a, b, \alpha, \beta) = B_{-2\mu}(N, M, N, M, b, a, \beta, \alpha) = q^{(a + \beta)(a + b)\mu^2 + (a\beta - ab\mu)} g((a + b)\mu, (\alpha + \beta)\mu)$$

and

$$A_{2\mu + 1}(N, M, N, M, a, b, \alpha, \beta) = B_{-2\mu - 1}(N, M, N, M, b, a, \beta, \alpha) = q^{(a + \beta)(a + b)\mu^2 + (2a\alpha + a\beta + ab)\mu + a a g((a + b)\mu + a, (\alpha + \beta)\mu + \alpha).$$

The generating function for partition pairs, both in $(N, M)$ and subject to the restrictions

$$p_i - q_i + a - 1 \leq a - 1$$

$$q_i - p_i + b - 1 \leq b - 1,$$

is

$$P(N, M, a, b, \alpha, \beta) = A_0 - \sum_{i=1}^{\infty} A_{2i-1} - A_{2i} + B_{2i-1} - B_{2i}. $$
which may be rearranged to a bilateral series to provide Theorem 1:

\[ P(N, M, a, b, \alpha, \beta) = \sum_{\mu = -\infty}^{\infty} q^{(\alpha + \beta)(a + b)\mu^2 + (ab - \alpha b)\mu} \left( (a + b)\mu, (\alpha + \beta)\mu \right) - \sum_{\mu = -\infty}^{\infty} q^{(\alpha + \beta)(a + b)\mu^2 + (2\alpha a + \alpha b + \alpha b)\mu + \alpha a} \left( (a + b)\mu + a, (\alpha + \beta)\mu + \alpha \right). \]

At this point we note from the symmetry of the generating function that not only is

\[ P(N, M, a, b, \alpha, \beta) = P(N, M, b, a, \beta, \alpha) \]

as might have been expected by interchanging the two partitions, but also

\[ P(N, M, a, b, \alpha, \beta) = P(M, N, \beta, \alpha, a, b). \]

As a consequence we have the following theorem.

**Theorem 2.** The number of partition pairs of \( n \), both in \((N, M)\), with the restrictions

\[ p_i - q_{i+1} - a \geq 1 - a \]
\[ q_i - p_{i+1} - \beta \geq 1 - b \]

is equal to the number of partition pairs of \( n \), both in \((M, N)\), with the restrictions

\[ p_i - q_{i+1} - a \geq 1 - \alpha \]
\[ q_i - p_{i+1} - b \geq 1 - \beta. \]

This can be proved directly because if a pair \((p, q)\) satisfies the first restriction then the pair of conjugate partitions satisfies the second. Each polynomial \( P(N, M, a, b, \alpha, \beta) \) is reciprocal because the coefficients of \( q^i \) and \( q^{2NM-i} \) are equal. For if

\[ p_i - q_{i+1} - a \geq 1 - \alpha \]

then when all parts are subtracted from \( M \) and reversed

\[ (M - q_{i+1} - a) - (M - p_i) \geq 1 - \alpha. \]
PARTITION PAIRS THAT ARE ALSO PARTITIONS

Some of the partition pairs can be interpreted as single partitions and $P(N, M, a, b, \alpha, \beta)$ can be established using an alternative argument implying identities between the two generating functions.

**Euler's Theorem.** In the case $P(N, M, 1, 2, 1, 1)$ the partition pair can be rearranged to form a single partition in $(2N, M)$ because $p_N \geq q_N \geq p_{N-1} \geq q_{N-1} \cdots$. This implies the identity

$$
\left[ \begin{array}{c} 2N + M \\ 2N \end{array} \right] = P(N, M, 1, 2, 1, 1)
$$

$$
= \sum_{\mu = -\infty}^{\infty} q^{6\mu^2 - \mu} \left[ \begin{array}{c} N + M + \mu \\ N + 3\mu \end{array} \right] \left[ \begin{array}{c} N + M - \mu \\ N - 3\mu \end{array} \right] - \sum_{\mu = -\infty}^{\infty} q^{6\mu^2 + 5\mu + 1} \left[ \begin{array}{c} N + M + \mu \\ N + 3\mu + 1 \end{array} \right] \left[ \begin{array}{c} N + M - \mu \\ N - 3\mu - 1 \end{array} \right].
$$

This is a doubly bounded version of Euler's theorem which becomes two known identities when either $N$ or $M$ tend to infinity. When $N$ tends to infinity we have

$$
\frac{1}{(q)_M} = \sum_{\mu = -\infty}^{\infty} \frac{q^{6\mu^2 - \mu}}{(q)_{M + 2\mu}(q)_{M - 2\mu}} - \sum_{\mu = -\infty}^{\infty} \frac{q^{6\mu^2 + 5\mu + 1}}{(q)_{M + 2\mu + 1}(q)_{M - 2\mu - 1}}.
$$

When $M$ tends to infinity we have

$$
\frac{1}{(q)_{2N}} = \sum_{\mu = -\infty}^{\infty} \frac{q^{6\mu^2 - \mu}}{(q)_{N + 3\mu}(q)_{N - 3\mu}} - \sum_{\mu = -\infty}^{\infty} \frac{q^{6\mu^2 + 5\mu + 1}}{(q)_{N + 3\mu + 1}(q)_{N - 3\mu - 1}}.
$$

When both $N$ and $M$ tend to infinity it becomes Euler's theorem,

$$
\prod_{n=1}^{\infty} (1 - q^n) = \sum_{i = -\infty}^{\infty} (-1)^i q^{i(i-1)/2}.
$$

The following two identities are closely related:

$$
P(N, M, 1, 2, 0, 2) = q^N \left[ \begin{array}{c} 2N + M - 1 \\ 2N \end{array} \right]
$$

$$
P(N, M, 0, 3, 1, 1) = q^M \left[ \begin{array}{c} 2N + M - 2 \\ 2N - 2 \end{array} \right].
$$
RESTRICTED PARTITION PAIRS

The restrictions in the case \( a = b = \alpha = \beta = 1 \) imply that \( p_i = q_i \) and the pairs in this case may be mapped into a partition into even parts in \((N, 2M)\). The corresponding identity is

\[
\begin{bmatrix} N + M \\ N \end{bmatrix}_{q^2} = P(N, M, 1, 1, 1, 1) = \sum_{\mu = -\infty}^{\infty} (-1)^\mu q^{\mu^2} \begin{bmatrix} N + M \\ N + \mu \end{bmatrix} \begin{bmatrix} N + M \\ N - \mu \end{bmatrix}.
\]

The identity corresponding to partitions into odd numbers may be obtained by adding \( N \) ones to such a partition:

\[
q^N \begin{bmatrix} N + M - 1 \\ N \end{bmatrix}_{q^2} = P(N, M, 1, 1, 0, 2)
\]

\[
= \sum_{\mu = -\infty}^{\infty} q^{4\mu^2 + 2\mu} \begin{bmatrix} N + M \\ N + 2\mu \\ N - 2\mu \end{bmatrix} - \sum_{\mu = -\infty}^{\infty} q^{4\mu^2 + 2\mu} \begin{bmatrix} N + M + 1 \\ N + 2\mu + 1 \\ N - 2\mu - 1 \end{bmatrix}.
\]

The generating function \( P(N, M, 1, 2, 0, 1) \) can also be easily obtained by an alternative argument. In this case the parts may be merged into a sequence in which \( p_i > q_i \) and \( q_i > p_{i-1} \) and \( p_N > q_N \geq p_{N-1} > q_{N-1} \cdots \). It is possible to remove the sequence \( 1 + 3 + 5 + \cdots + 2N-1 \) from such a partition, leaving a partition in \((2N, M-N)\) and resulting in the identity

\[
q^N \begin{bmatrix} N + M - 1 \\ 2N \end{bmatrix}_{q^2} = P(N, M, 1, 2, 0, 1)
\]

\[
= \sum_{\mu = -\infty}^{\infty} q^{3\mu^2 + \mu} \begin{bmatrix} N + M + 2\mu \\ N + 3\mu \\ N - 3\mu \end{bmatrix} - \sum_{\mu = -\infty}^{\infty} q^{3\mu^2 + \mu} \begin{bmatrix} N + M + 2\mu + 1 \\ N + 3\mu + 1 \\ N - 3\mu - 1 \end{bmatrix}.
\]

Again as \( N \) or \( M \) tend to infinity these become known identities; see, for example, Slater [10].

SUMMATION FORMULAE

The generating function \( P(N, M, a + \alpha, b + \beta, \alpha, \beta) \) can be expressed in terms of \( P(N, M, \alpha, \beta, a, b) \) as follows:

\[
P(N, M, a + \alpha, b + \beta, \alpha, \beta) = P(M, N, \alpha, \beta, a + \alpha, b + \beta)
\]

\[
= \sum_{i=0}^{M} q^i \begin{bmatrix} 2N + M - i \\ 2N \end{bmatrix} P(i, N - i, \alpha, \beta, a, b).
\]
This summation was first derived by a combinatorial argument, then proved for all integers $a$, $b$, $\alpha$, $\beta$ by George Andrews using the $q$-Pfaff–Saalschütz formula. Given a generating function for pairs of type $(\alpha, \beta, a, b)$ we can produce two generating functions, one of type $(a + \alpha, b + \beta, a, b)$ and the other of type $(a + \alpha, b + \beta, a, b)$. The unit for partition pairs is $P(N, M, 1, 1, 0, 1) = \delta_{N0}$ and from it one can generate an infinite binary tree of generating functions and an infinite number of identities between single summations of the form $P(N, M, a, b, \alpha, \beta)$ and multiple summations. The first few single summations are

\[
P(N, M, 2, 3, 1, 1) = \sum_{i=0}^{M} q^{i^2} \left[ \begin{array}{c} 2N + M - i \\ 2N \end{array} \right] \frac{p(i, N - i, 1, 1, 0, 1)}{q^{N+1}}
\]

\[
P(N, M, 1, 4, 1, 1) = \sum_{i=0}^{M} q^{i^2} \left[ \begin{array}{c} 2N + M - i \\ 2N \end{array} \right] \frac{p(i, N - i, 1, 1, 0, 3)}{q^{N+1}}
\]

\[
P(N, M, 2, 3, 1, 2) = \sum_{i=0}^{M} q^{i^2} \left[ \begin{array}{c} 2N + M - i \\ 2N \end{array} \right] \frac{p(i, N - i, 1, 2, 1, 1)}{q^{N+1}}
\]

\[
P(N, M, 2, 2, 1, 1) = \sum_{i=0}^{M} q^{i^2} \left[ \begin{array}{c} 2N + M - i \\ 2N \end{array} \right] \frac{p(i, N - i, 1, 1, 1, 1)}{q^{N+1}}
\]

\[
P(N, M, 1, 3, 0, 2) = \sum_{i=0}^{M} q^{i^2} \left[ \begin{array}{c} 2N + M - i \\ 2N \end{array} \right] \frac{p(i, N - i, 0, 2, 1, 1)}{q^{N+1}}
\]

\[
P(N, M, 1, 3, 1, 1) = \sum_{i=0}^{M} q^{i^2} \left[ \begin{array}{c} 2N + M - i \\ 2N \end{array} \right] \frac{p(i, N - i, 1, 1, 0, 2)}{q^{N+1}}
\]
\[ P(N, M, 1, 3, 0, 1) = \sum_{i=0}^{M} q^{2i} \left[ \frac{2N + M - i}{2N} \right] P(N, M, 0, 1, 1, 2) \]
\[ = \sum_{i=0}^{M} q^{2i + (N - i)^2} \left[ \frac{2N + M - i}{2N} \right] \int_{2N - 2i}^{N} \]

\[ P(N, M, 1, 3, 1, 2) = \sum_{i=0}^{M} q^{2i} \left[ \frac{2N + M - i}{2N} \right] P(N, M, 1, 2, 0, 1) \]
\[ = \sum_{i=0}^{M} q^{2i} \left[ \frac{2N + M - i}{2N} \right] \int_{2N - 2i}^{N}. \]

The coefficient of \( q^n \) in \( p(N, M, 2, 3, 1, 1) \) is the number of sequences \( s_i \) of length no greater than \( 2N \) whose sum is \( n \) that are both 2-ordered \( (s_i \geq s_{i+2}) \) and 3-ordered \( (s_i \geq s_{i+3}) \). Alternatively, by taking the conjugates of the two partitions it also enumerates the pairs in which \(-1 < p_i - q_i \leq 2: \)

\[ \sum_{i=0}^{M} q^{2i} \left[ \frac{2N + M - i}{2N} \right] \int_{2N - 2i}^{N} \]

\[ = \sum_{\mu = -\infty}^{\infty} q^{10\mu^2 - \mu} \left[ \frac{N + M + 3\mu}{N + 5\mu} \right] \int_{N - 5\mu}^{N} \]
\[ - \sum_{\mu = -\infty}^{\infty} q^{10\mu^2 + 9\mu + 2} \left[ \frac{N + M + 3\mu + 1}{N + 5\mu + 2} \right] \int_{N - 5\mu - 2}^{N} \]

This is again a doubly bounded identity which tends to two known but previously unrelated identities. When \( N \) tends to infinity it becomes essentially Roger's proof [8] of the Rogers–Ramanujan identities:

\[ \sum_{i=0}^{M} \frac{q^{i^2}}{(q)_{M-i}} (q)_{i} = \sum_{\mu = -\infty}^{\infty} q^{10\mu^2 - \mu} \left[ \frac{N + M + 3\mu}{N + 5\mu} \right] \int_{N - 5\mu}^{N} \]
\[ - \sum_{\mu = -\infty}^{\infty} q^{10\mu^2 + 9\mu + 2} \left[ \frac{N + M + 3\mu + 1}{N + 5\mu + 2} \right] \int_{N - 5\mu - 2}^{N} \]

When \( M \) tends to infinity it becomes Schur's proof of the Rogers–Ramanujan identities:

\[ \sum_{i=0}^{\infty} q^{i^2} \left[ \frac{2N - i}{i} \right] = \sum_{\mu = -\infty}^{\infty} q^{10\mu^2 - \mu} \left[ \frac{2N}{N + 5\mu} \right] \int_{N + 5\mu}^{N} \]
\[ - \sum_{\mu = -\infty}^{\infty} q^{10\mu^2 + 9\mu + 2} \left[ \frac{2N}{N + 5\mu + 2} \right] \int_{N + 5\mu + 2}^{N} \]

When both \( M \) and \( N \) tend to infinity, and after an application of Jacobi's triple product identity, it becomes the first Rogers–Ramanujan identity:

\[ 1 + \sum_{n=1}^{\infty} q^{n^2} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})}. \]
The doubly bounded form of the second Rogers–Ramanujan identity is given by the expression for \( P(N, M, 1, 4, 1, 1) \) above.

In general when both \( N \) and \( M \) tend to infinity by using Jacobi’s triple product theorem,

\[
(q)_{\infty} (q)_{\infty} P(\infty, \infty, a, b, \alpha, \beta, n) = \prod_{n=0}^{\infty} (1 - q^{(2n + 2)(a + b)(\alpha + \beta)})
\times (1 - q^{(2n + 1)(a + b)(\alpha + \beta) + a(\alpha + \beta) - a(a + b)})
\times (1 - q^{(2n + 1)(a + b)(\alpha + \beta) + a(a + b) - a(\alpha + \beta)})
- q^{ax} \prod_{n=0}^{\infty} (1 - q^{(2n + 2)(a + b)(\alpha + \beta)})
\times (1 - q^{(2n + 1)(a + b)(\alpha + \beta) + a(\alpha + \beta) + a(a + b)})
\times (1 - q^{(2n + 1)(a + b)(\alpha + \beta) - a(\alpha + \beta) - a(a + b)}).
\]

In certain circumstances the difference of the two theta series can be expressed as a single product, see [4]. In the limit the identities are between generating functions for partition pairs and those for two partitions, one restricted and the other not.

It is possible to generalize the generating function \( P \) by removing the constraint that the two partitions of the pair be limited to the same bounds, \( N \) and \( M \). Let \( R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta) \) be the generating functions for partition pairs having the same restrictions \( (a, b, \alpha, \beta) \) as before but with partition \( q \) in \( (N_1, M_1) \) and partition \( p \) in \( (N_2, M_2) \). By arguments similar to those above it can be shown that

\[
g(x, y) = \begin{bmatrix} N_1 + x + M_1 - y \\ N_1 + x \\ N_2 - x + M_2 + y \\ N_2 - x \end{bmatrix}
R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)
=
\sum_{\mu = -\infty}^{\infty} q^{(x + \beta)(a + b)\mu^2 + (a\beta - ab + (N_1 - N_2)(\alpha + \beta))\mu} g((a + b)\mu, (\alpha + \beta)\mu)
- \sum_{\mu = -\infty}^{\infty} q^{(x + \beta)(a + b)\mu^2 + (2a\alpha + a\beta + ab)\mu + (N_1 - N_2)((\alpha + \beta))\mu + a} \times g((a + b)\mu, (\alpha + \beta)\mu + \alpha).
\]

The generating function \( P \) above may be defined in terms of \( R \) as follows:

\[
P(N, M, a, b, \alpha, \beta) = R(N, M, N, M, a, b, \alpha, \beta).
\]
By using $R(N + 1, M, N, M, a, b, \alpha, \beta)$ the companion odd cases of the identities above may be produced. For example the companion to the doubly bounded Euler's theorem is

$$R(N + 1, M, N, M, 1, 2, 1, 1)$$

$$= \left[ \begin{array}{c}
2N + 1 + M \\
2N + 1
\end{array} \right]$$

$$= \sum_{\mu = -\infty}^{\infty} q^{6\mu^2 + 2\mu} \left[ \begin{array}{c}
N + M + \mu + 1 \\
N + 3\mu + 1
\end{array} \right] \left[ \begin{array}{c}
N + M - \mu \\
N - 3\mu
\end{array} \right]$$

$$- \sum_{\mu = -\infty}^{\infty} q^{6\mu^2 + 7\mu + 2} \left[ \begin{array}{c}
N + M + \mu + 1 \\
N + 3\mu + 2
\end{array} \right] \left[ \begin{array}{c}
N + M - \mu \\
N - 3\mu - 1
\end{array} \right].$$

In this case the summation depends on the following lemma, applied term by term to the series.

**Lemma 1.**

$$q^{i^2 + (a-b)i} \left[ \begin{array}{c}
m_1 + n - r + b \\
n + r + a
\end{array} \right] \left[ \begin{array}{c}
m_2 + n + r + a \\
n - r + b
\end{array} \right]$$

$$= \sum_{i=0}^{n} q^{i^2 + (a+b)i} \left[ \begin{array}{c}
m_1 + m_2 + n - i \\
i + r + a
\end{array} \right] \left[ \begin{array}{c}
m_1 \\
i + r + b
\end{array} \right].$$

When this is applied term by term to the series for $R$ we obtain the more general summation formula

$$R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)$$

$$= \sum_{i=0}^{M_2} q^{i^2 + (M_1 - M_2)i} \left[ \begin{array}{c}
N_1 + N_2 + M_2 - i \\
M_2 - i
\end{array} \right]$$

$$\times R(i + M_1 - M_2, N_1 - i, i, N_2 - i + M_1 - M_2, a, b, \alpha, b - \beta),$$

a formula that is true for all integer $(a, b, \alpha, \beta)$. The summation can be interpreted in two ways: (1) as a recurrence between partition pairs, or (2) as a recurrence between single partitions having restrictions on their hook differences as in [4]. The second interpretation arises from the addition of a Durfee rectangle with sides $i$ and $i + M_1 - M_2$ to a partition pair in $(N_1 - i, i + M_1 - M_2)$, $(N_2 + M_2 - M_1 - i, i)$ to create a partition in $(N_1, N_2)$, which is then applied to a partition in $(N_1 + N_2, M_2 - i)$ to create a pair in $(N_1, M_1)$, $(N_2, M_2)$, to which may be attached a Durfee rectangle of sides $N_1$, $N_2$ or $M_1$, $M_2$. In this case the recurrence is between a partition in $(N_1, N_2)$ of type $(\alpha, \beta, a-\alpha, b-\beta)$ and one in $(N_1 + M_1, N_2 + M_2)$ of type $(a, b, \alpha, \beta)$.
There are certain classes of \((a, b, \alpha, \beta)\) that are of interest. When \(a = b = 1\) the relations become
\[
1 - \beta \geq p_i - q_i \geq 1 - \alpha
\]
which correspond to partitions having restrictions on hook differences along the main diagonal.

When \(a = \alpha\) and \(b = \beta\) the generating functions take a particularly simple form and the pairs and the pairs of their conjugates belong to the same class.

When \(b\) and \(\beta\) are larger than the dimensions of the partitions the Gaussian polynomials containing them vanish and we are left with pairs of partitions with a single restriction, namely
\[
p_i - q_{i+1-a} \geq 1 - \alpha,
\]
whose generating function degenerates to
\[
R(N_1, M_1, N_2, M_2, a, \infty, b, \infty)
=
\begin{bmatrix}
N_1 + M_1 & N_2 + M_2 \\
N_1 & N_2
\end{bmatrix} - q^{\alpha a}
\begin{bmatrix}
N_1 + M_1 + a - \alpha & N_2 + M_2 - a + \alpha \\
N_1 + a & N_2 - a
\end{bmatrix}
\]

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**REFERENCES**