Abstract

We first give an intrinsic characterization of the $\lambda$-rings which are representation rings of compact connected Lie groups. Then we show that the representation ring of a compact connected Lie group $G$, with its $\lambda$-structure, determines $G$ up to a direct factor which is a product of groups $\text{Sp}(l)$ or $\text{SO}(2l+1)$. This result is used to show that a compact connected Lie group is determined by its classifying space; different (and independent) proofs of this result have been given by Zabrodsky-Harper, Notbohm and Møller. Our technique also yields a new proof of a result of Jackowski, McClure and Oliver on self-homotopy equivalences of classifying spaces. © 1997 Elsevier Science B.V.


1. Introduction

Fix a compact connected Lie group $G$ and let $R(G)$ denote the free abelian group generated by the isomorphism classes of complex irreducible representations of $G$. With the multiplication induced by tensor products of representations, $R(G)$ becomes a commutative ring with unit; it is called the (complex) representation ring of $G$. The correspondence $G \rightarrow R(G)$ is functorial, so that $R(G)$ is an invariant of the Lie group $G$. The importance of this invariant is best illustrated by the following result of Atiyah: if $BG$ denotes the classifying space of $G$, then the topological $K$-theory ring $K^0(BG)$ is isomorphic to a completion of the ring $R(G)$.

As well-known, it is much more rewarding to consider $R(G)$ with the $\lambda$-ring structure provided by the exterior powers of the representations. We refer to the next section for the definition and main properties of $\lambda$-rings. In our case, an important result of Weyl gives a complete description of the $\lambda$-structure $R(G)$:
Theorem (see Adams [1, Theorem 6.20]). Let $G$ be a compact connected Lie group, $T$ a maximal torus of $G$ and $W = N_G(T)/T$ the Weyl group with respect to $T$. The finite group $W$ acts (by $\lambda$-automorphisms) on $R(T)$ and the ring of invariants $R(T)^W$ is $\lambda$-isomorphic to $R(G)$.

Motivated by the search for homotopical characterizations of classifying spaces of Lie groups, we will devote the first part of this article to the following problem:

Find an intrinsic characterization of the $\lambda$-rings which are representation rings of compact connected Lie groups.

In other words, we are looking for algebraic ways of recognizing representation rings of compact connected Lie groups. The obvious observation is that Weyl's structure theorem furnishes many algebraic properties of the rings $R(G)$. Some of them have been selected to define the notion of a normal $\lambda$-ring. By adapting the method of [5, Section 1.6] to our context and using some Galois theory, we show that these normal $\lambda$-rings are exactly of the form $R(T^n)^W$, where $T^n$ is an $n$-dimensional torus and $W$ a finite group of $\lambda$-automorphisms of $R(T^n)$. This reduces our problem to the characterization of the normal $\lambda$-rings for which the finite group $W$ is a Weyl group. To deal with this, we consider the so-called $\gamma$-filtration on the normal $\lambda$-rings and observe that the $W$-action on $R(T^n)$ becomes "linear" on the associated graded rings. We are then in position of applying the well-known result of Chevalley, Sheppard and Todd on rings of invariants. Our characterization result is stated in the text as

**Theorem 3.1.** For a $\lambda$-ring $R$, the following statements are equivalent:

1. $R$ is $\lambda$-isomorphic to the representation ring of a compact connected Lie group;
2. $R$ is a normal $\lambda$-ring and the ring $\text{Gr}_\gamma(R) \otimes \mathbb{Q}$, where $\text{Gr}_\gamma(R)$ is the graded ring associated to the $\gamma$-filtration of $R$, is polynomial over $\mathbb{Q}$.

It has been known for a long time that two different compact Lie groups may have $\lambda$-isomorphic representation rings. The usual examples are the groups $SO(2l + 1)$ and $Sp(l)$, for $l \geq 1$ [10, Appendix 1]. In the second part of this paper, we show that these are essentially the only cases where the $\lambda$-ring $R(G)$ fails to characterize the group $G$. More precisely, we prove that the $\lambda$-ring $R(G)$ determines the group $G$ up to a direct factor which is isomorphic to a product of groups $SO(2l + 1)$ and $Sp(l)$ ($l \geq 1$). In particular, the Lie groups $\text{Spin}(2l+1)$ and $\text{PSp}(l)$ ($l \geq 3$) are completely classified by their representation rings; this shows that the Weyl group is not the sole responsible for the troubles...

As expected, the next question is about the extra piece of information needed for the complete reconstruction of the group $G$. Among the invariants which distinguish the groups $SO(2l + 1)$ from the $Sp(l)$'s, we first consider the real or symplectic representation groups. The latters are obtained in the same way as $R(G)$, but from the real or symplectic representations of the group $G$; they are, respectively, denoted by
RO(G) and RSp(G). We will identify (via the usual "complexification" maps) RO(G) and RSp(G) to additive subgroups of the complex representation ring \( R(G) \). Our main reconstruction theorem (see Theorem 4.3) says that a compact connected Lie group \( G \) is uniquely determined by the \( \lambda \)-ring \( R(G) \) and either the subgroup \( RO(G) \subset R(G) \) or \( RSp(G) \subset R(G) \). It may be worth remarking here that there is no analogue of Weyl’s structure Theorem for \( RO(-) \) or \( RSp(-) \); easy counterexamples are given by the groups \( SO(3) \) and \( Sp(1) \).

Another way to tell a group \( SO(2I + 1) \) apart from \( Sp(I) \) is to look at the normalizers of maximal tori. We exploit this observation to give a new proof of a celebrated result of Curtis, Wiederhold and Williams.

The notion of root diagram plays a central role in all our arguments; for this reason, the first part of Section 4.1 is devoted to some recollections about these objects. The most important result states that a compact connected Lie group is completely classified by its root diagram.

The last part of the paper deals with some rigidity properties of classifying spaces. As a first result, we show that a compact connected Lie group is uniquely determined by its classifying space. This provides a positive answer to a question of Mislin (see [19]). The present proof is (hopefully) simpler than the one in [18]. We also refer to [21, 17, 16] for independent (and different) proofs of the same result. Let us also mention that Notbohm [17] has extended the result to all the compact Lie groups. Our second application is another proof of the "disappointing" (sic) result of [15]: the classifying space of a compact connected Lie group has, up to homotopy, no exotic self-homotopy equivalence.

2. Normal \( \lambda \)-rings

The theory of \( \lambda \)-rings is first reviewed, mainly to fix the notations for the sequel. Then the normal \( \lambda \)-rings are introduced and studied. Our main result states that they are exactly the rings of invariants \( \mathcal{A}_n^W \), where \( \mathcal{A}_n \) is a Laurent polynomials ring \( \mathbb{Z}[x_1^\pm 1, \ldots, x_n^\pm 1] \) and \( W \) is a finite subgroup of \( GL_n(\mathbb{Z}) \).

2.1. Definition

Let \( R \) be a commutative ring with unit 1. A \( \lambda \)-structure on \( R \) is given by a family of maps \( \{ \lambda^n : R \to R \}_{n \geq 0} \) satisfying the following axioms:

(i) For all \( x \in R \), \( \lambda^0(x) = 1 \) and \( \lambda^1(x) = x \).
(ii) For all \( x, y \in R \) and for all \( n \geq 0 \), \( \lambda^n(x + y) = \sum_{k=0}^n \lambda^k(x) \cdot \lambda^{n-k}(y) \).
(iii) For all \( n \geq 2 \), \( \lambda^n(1) = 0 \).
(iv) For all \( x, y \in R \) and \( m, n \geq 0 \),

\[
\lambda^n(x \cdot y) = P_n(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^n(y)),
\]

\[
\lambda^m(\lambda^n(x)) = P_{n,m}(\lambda^1(x), \ldots, \lambda^{n+m}(x)),
\]
where $P_n$ and $P_{nm}$ are certain universal polynomials defined, for example, in [5, Section 1.1].

The maps $\lambda^n$ are called $\lambda$-operations and $R$ is called a $\lambda$-ring. For any $x \in R$, it is customary to write

$$\lambda_t(x) = \sum_{n \geq 0} \lambda^n(x)t^n \in R[[t]].$$

With this notation, axioms (ii) and (iii), respectively, become

$$\lambda_t(x + y) = \lambda_t(x) \cdot \lambda_t(y), \quad \forall x, y \in R \quad \text{and} \quad \lambda_t(1) = 1 + t.$$

**Definition 2.1.** (i) Let $n$ be a nonnegative integer. An element $x \in R$ is of $\lambda$-dimension $n$ if $\lambda_t(x)$ is a polynomial of degree $n$ in the variable $t$; we shall then write $\lambda$-dim$(x) = n$.

(ii) The $\lambda$-ring $R$ is of finite $\lambda$-dimension if every element of $R$ can be written as the difference of two elements of finite $\lambda$-dimension.

(iii) Let $R$ and $S$ be $\lambda$-rings. A $\lambda$-morphism $f : R \to S$ is a ring homomorphism which satisfies the additional property:

$$f(\lambda^n(x)) = \lambda^n(f(x)), \quad \forall x \in R \quad \forall n \geq 0.$$  

A $\lambda$-morphism which is also bijective is called a $\lambda$-isomorphism. The symbol $\text{Aut}_\lambda(R)$ will stand for the group of $\lambda$-isomorphisms $f : R \to R$.

The most important examples for us are (of course) the representation rings of compact Lie groups, where the $\lambda$-operations are induced by the exterior powers of representations; the verification of the above axioms are done in [5, p. 260]. For the $n$-dimensional torus $T^n = S^1 \times \cdots \times S^1$ ($n$ times), $R(T^n)$ is the Laurent polynomial ring $\mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$, where $\alpha_1^\pm : T^n \to S^1$, $(x_1, \ldots, x_n) \mapsto x_i^\pm$ for $i = 1, \ldots, n$. The $\lambda$-structure of $R(T^n)$ is completely determined by the formulas

$$\lambda_t(x_i^\pm) = 1 + x_i^\pm \cdot t, \quad i = 1, \ldots, n.$$  

It is easily shown that the monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $m_1, \ldots, m_n \in \mathbb{Z}$ are the only elements of $\lambda$-dimension $1$ in $R(T^n)$. It follows that the group $\text{Aut}_\lambda(R(T^n))$ is isomorphic to $\text{GL}_n(\mathbb{Z})$.

**Definition 2.2.** Let $R$ be a $\lambda$-ring. The Adams operations on $R$ are the maps $\Psi^k : R \to R$ ($k \geq 1$) defined by

$$\forall x \in R, \quad \sum_{k \geq 1} \Psi^k(x)t^k = -t \cdot \frac{d}{dt} [\log \lambda_{-t}(x)].$$
The interested reader may refer to [5, Section 1.5] for the proofs of the following important properties of the Adams operations:

1. For every \( k \), \( \Psi^k \) is a \( \lambda \)-morphism.
2. For every \( k, l \geq 1 \), \( \Psi^k \circ \Psi^l = \Psi^{k+l} \).
3. For any \( x \in R \) and any prime number \( p \), \( \Psi^p(x) - x^p \in pR \).

Before introducing the normal \( \lambda \)-rings, let us record some easy and useful consequences of the definitions above.

**Proposition 2.1.** Assume \( R \) is \( \lambda \)-ring with no torsion and set

\[
R(1) = \{ x \in R \text{ such that } \lambda \dim(x) = 1 \}.
\]

(i) For any \( x \in R \), \( x \) is in \( R(1) \) if and only if \( x \) is nonzero and \( \Psi^k(x) = x^k \) for all \( k \geq 1 \).

(ii) If \( x \in R \) is of \( \lambda \)-dimension \( n < \infty \), then \( \lambda^n(x) \) is in \( R(1) \).

(iii) \( R(1) \) is a multiplicative subset of \( R \), that is, a subset closed under multiplication and containing the identity.

**Definition 2.3.** We shall say that a \( \lambda \)-ring \( R \) is normal if the following conditions are satisfied:

1. \( R \) is a finitely generated ring.
2. \( R \) is an integrally closed domain.
3. \( R \) is of finite \( \lambda \)-dimension.
4. Every element of \( \lambda \)-dimension 1 is invertible in \( R \).

The rank of \( R \) is defined as the integer \( \dim(R) - 1 \), where \( \dim(R) \) denotes the Krull dimension of \( R \).

Of course Weyl’s structure theorem implies that the representation rings of compact connected Lie groups are normal \( \lambda \)-rings. If one wishes to avoid Weyl’s result, one quickly discovers that condition (2) is the most difficult to verify. Unfortunately this condition does not follow from the others. Indeed, the \( \lambda \)-subring \( \mathbb{Z}[x^2, x^3] \) of \( R(\text{SU}(2)) = \mathbb{Z}[x] \), where \( \lambda_t(x) = 1 + xt + t^2 \), is not integrally closed; but it satisfies all the other conditions of the definition. More general examples of normal \( \lambda \)-rings may be constructed as follows. Fix an integer \( n \geq 1 \) and a finite subgroup \( W \) of \( \text{GL}_n(\mathbb{Z}) \). Then \( W \) acts by \( \lambda \)-automorphisms on \( \mathcal{A}_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). By the same argument as before, the ring of invariants \( \mathcal{A}_n^W \) is a normal \( \lambda \)-ring. On the other hand and by the result of Atiyah, the \( K \)-theory ring \( K^0(BG) \) (where \( BG \) is the classifying space of a non-trivial compact Lie group \( G \)) does not satisfy the first condition.

**Remark.** In Commutative Algebra, a noetherian and integrally closed ring is usually called normal. This terminology is kept here to stress the importance of the second condition above.

The next result will be very important in Section 3; here, it will provide the classification of the normal \( \lambda \)-rings of rank 0.
Proposition 2.2. Let R be a normal λ-ring. There exists a unique λ-morphism ε: R → Z; it is determined by the following rule:

if x ∈ R is of λ-dimension k < ∞, then ε(x) = k.

The kernel of ε is the augmentation ideal of R; it will be denoted I(R).

Proof. Straightforward or adapt the proof of [2, Lemma 1.5]. □

Corollary 2.1. The ring of integers, with its canonical λ-structure, is the only normal λ-ring of rank 0.

Proof. Let R be a normal λ-ring of rank 0. Its augmentation ideal I(R) is prime, but cannot be maximal. As R is a domain of Krull dimension 1, we must have I(R) = 0; that is R = Z. □

2.2. The classification

Fix a normal λ-ring of rank n (n ≥ 1) and let x₁, x₂, ..., xₙ be a generating set for the ring R, such that

λ-dim(xᵢ) = nᵢ < ∞ for i = 1, 2, ..., r.

Let K be the field of fractions of R and L the splitting field (over K) of the polynomials

Qᵢ(ξ) = ξⁿᵢ - λ₁(ξ) · ξⁿᵢ₋₁ + ... + (-1)ⁿᵢ · λⁿᵢ(ξ) (i = 1, 2, ..., r).

Thus, L is a finite Galois extension of K; the Galois group of this extension will be denoted by W. Let A be the subring of L generated by R and the roots ξᵢ,j of the polynomials Qᵢ(ξ). It is clear that the ξᵢ,j's generate A as a ring and that the latter is an integral (ring) extension of R. Since the elements λⁿᵢ(xᵢ) are invertible in R (we are using here the fourth condition of Definition 2.3), we can also say that the inverses ξ⁻¹ᵢ,j lie in A. It follows that the multiplicative group Γ, generated by the ξᵢ,j's is contained in A. Let us observe that, by construction, the groups W, Γ and the ring A depend on the choice of the generating system x₁, x₂, ..., xₙ. But we will see in a moment that this dependence does not seriously matter.

Lemma 2.1. There is a unique λ-structure on A, extending the one on R and such that every xᵢ splits as a sum of elements of λ-dimension 1.

Proof. For any i ∈ {1, 2, ..., r}, we can write

λ₋ξ(xᵢ) = ξⁿᵢ · Qᵢ(1/ξ) = \prod_{j=1}^{nᵢ} (1 + ξᵢ,j · (-ξ)).

Since xᵢ = ξᵢ,1 + ... + ξᵢ,nᵢ and L[t] is factorial, this factorization forces the equalities

λᵢ(ξᵢ,j) = 1 + ξᵢ,j · t, \quad ∀j = 1, ..., nᵢ.
It is easy to see that these equalities define a \( \lambda \)-structure on \( A \) and that the required conditions are satisfied.

**Lemma 2.2.** The group \( \Gamma \) (generated by the \( \xi_{ij} \)'s) is free abelian of rank \( n \) and its group ring \( \mathbb{Z}[\Gamma] \) coincide with \( A \).

**Proof.** Note first that any element of \( \Gamma \) is of \( \lambda \)-dimension 1 for the \( \lambda \)-structure given by the preceding lemma. By construction, \( \Gamma \) is a finitely generated abelian group. Suppose that \( F \) is a finite subgroup of \( \Gamma \) and consider the element of \( A \) defined by \( z = \sum_{y \in F} y \). Note that \( z \) is not an element of \( \Gamma \), since the latter is not closed under the addition of \( A \). By construction \( z^2 = |F| \cdot z \) and \( \lambda_t(z) \) is a polynomial of degree \( |F| > 0 \); so we must have \( z = |F| \) and

\[
\lambda_t(z) = \prod_{y \in F} (1 + y \cdot t) = (1 + t)^{|F|}.
\]

It follows that \( F = \{1\} \) and we have shown that \( \Gamma \) is torsion-free.

Now let \( \gamma_1, \gamma_2, \ldots, \gamma_s \) be mutually distinct elements of \( \Gamma \) and suppose that there are \( a_1, a_2, \ldots, a_s \in \mathbb{Z} \) such that

\[
a_1 \cdot \gamma_1 + \cdots + a_s \cdot \gamma_s = 0.
\]

By successively applying the Adams operations \( \Psi^1, \Psi^2, \ldots, \Psi^s \) to this relation, we obtain the linear system

\[
\begin{align*}
a_1 \cdot \gamma_1 + a_2 \cdot \gamma_2 + \cdots + a_s \cdot \gamma_s &= 0, \\
a_1 \cdot \gamma_1^2 + a_2 \cdot \gamma_2^2 + \cdots + a_s \cdot \gamma_s^2 &= 0, \\
& \vdots \\
a_1 \cdot \gamma_1^s + a_2 \cdot \gamma_2^s + \cdots + a_s \cdot \gamma_s^s &= 0.
\end{align*}
\]

The matrix of the system is non-singular (Vandermonde), therefore \( a_1 = \cdots = a_s = 0 \) and we can deduce that \( A \) is indeed the group ring \( \mathbb{Z}[\Gamma] \). Since \( A \) is integral over \( R \), we have

\[
\text{rank}(\Gamma) = \text{dim}(A) - 1 = \text{dim}(R) - 1 = n.
\]

Consequently there exists a basis \( \{x_1, \ldots, x_n\} \subset \{\xi_{ij}\} \) of \( \Gamma \) such that \( A = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), the Laurent polynomial ring on the variables \( x_1, \ldots, x_n \).

**Lemma 2.3.** The Galois group \( W \) of the field extension \( L/K \) acts (by \( \lambda \)-automorphisms) on the \( \lambda \)-ring \( A \); the ring of invariants for this action is equal to \( R \). Moreover, if \( f : A \rightarrow A \) is a ring homomorphism such that \( f(x) = x \) for all \( x \in R \), then \( f \) is an element of the group \( W \).

**Proof.** By construction, \( W \) permutes the \( \xi_{ij} \)'s; this means that \( A \) is stable under the action of this finite group on the field \( L \). Let \( x \in A \) be such that \( w \cdot x = x \) for all \( w \in W \). Galois theory tells us that \( x \), viewed as an element of \( L \), should lie in the fraction field \( K \) of \( R \). Since \( x \) is integral over \( R \) and the later is integrally closed, \( x \)
has to be an element of $R$. This shows that $R = A^W$, the ring of invariants for the $W$ action on $A$. For the last statement, observe that the homomorphism $f$ is necessarily injective, thus can be extended to a field homomorphism $\tilde{f} : L \rightarrow L$ which fixes $K$. By Galois theory, $\tilde{f}$ is an element of $W = \text{Gal}(L/K)$; this imply that its restriction $f$ is in $W \subset \text{Aut}_L(A)$. □

For the proof of the classification theorem, we shall also need the next result:

**Lemma 2.4** (Wilkerson [20]). Let $R, A, B$ be integral domains. Assume that $R$ is a subring of $A$ and $A$ is integral over $R$. Let $\Phi : R \rightarrow B$ be a ring homomorphism satisfying the following conditions:

1. There exists a subset $\{y_i\}$ of $A$ such that, for each $y_i$, there is a monic polynomial $f_i(\xi) \in R[\xi]$ with $f_i(y_i) = 0$.

2. As an $R$-module, $A$ is generated by polynomials in the $y_i$'s.

3. For any $i$, the polynomial $\Phi \cdot f_i(\xi) \in B[\xi]$, obtained by applying $\Phi$ to the coefficients of $f_i(\xi)$, splits as a product of linear factors in $B[\xi]$.

Then there's a ring homomorphism $\tilde{\Phi} : A \rightarrow B$ satisfying $\tilde{\Phi}(x) = \Phi(x)$ for all $x \in R$.

**Proof.** See [20, p. 326]. Let us note that the lemma is well-known when $R, A, B$ are fields and $B$ algebraically closed. □

Everything is now in place for the main result of this section, namely,

**Theorem 2.2.** Let $R$ be a normal $\lambda$-ring of rank $n > 0$. There exists an injective $\lambda$-morphism

$$\varphi : R \rightarrow \mathcal{A}_n^{\mathbb{Z}}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]$$

and a finite subgroup $W_\varphi$ of $\text{Aut}_L(\mathcal{A}_n) = \text{GL}_n(\mathbb{Z})$ such that $\varphi(R) = \mathcal{A}_n^{\mathbb{Z}}$, the ring of invariants for the $W_\varphi$-action on $\mathcal{A}_n$. We shall say that $\varphi$ is a Galois $\lambda$-extension of $R$; it has the following properties:

1. If $f : \mathcal{A}_n \rightarrow \mathcal{A}_n$ is a ring homomorphism such that

$$f(\varphi(x)) = \varphi(x), \quad \forall x \in R,$$

then $f$ is in the group $W_\varphi \subset \text{GL}_n(\mathbb{Z})$.

2. For any positive integer $m$ and any $\lambda$-morphism $\theta : R \rightarrow \mathcal{A}_m$, there exists a $\lambda$-morphism $g : \mathcal{A}_n \rightarrow \mathcal{A}_m$ such that $g(\varphi(x)) = \theta(x), \forall x \in R$.

3. If $\varphi' : R \rightarrow \mathcal{A}'_n = \mathbb{Z}[\beta_1^{\pm 1}, \ldots, \beta_n^{\pm 1}]$ is another Galois $\lambda$-extension of $R$, then there is a $\lambda$-isomorphism $h : \mathcal{A}_n \rightarrow \mathcal{A}'_n$ such that

$$h(\varphi'(x)) = \varphi(x), \quad \forall x \in R.$$ 

This implies, when $\text{Aut}_L(\mathcal{A}_n)$ and $\text{Aut}_L(\mathcal{A}'_n)$ are identified to $\text{GL}_n(\mathbb{Z})$, that the Galois groups $W_\varphi$ and $W_{\varphi'}$ are conjugate in $\text{GL}_n(\mathbb{Z})$. 

Proof. The existence of $\varphi$ and property (1) have already been shown in Lemmas 2.1–2.3. The second property will follow from Wilkerson's lemma, by taking $A = \mathcal{A}_n$, $B = \mathcal{A}_m$, $\Phi = \theta$, $\{y_i\} = \{\xi_{ij}^{-1}\}$ with $f_{ij}(\xi) = \lambda_{-\xi}(x_i) \in R[\xi]$. Let us check that the last two conditions of this lemma are satisfied. Condition (2) is fine since

$$
\xi_{ij} = \lambda_1^1(x_i) - \lambda_2^2(x_i) \cdot \xi_{ij}^{-1} + \cdots + (-1)^{n_i-1} \cdot \lambda_{n_i}^n(x_i) \cdot \xi_{ij}^{-n_i+1}.
$$

For condition (3), we use the fact that $\theta$ is a $\lambda$-morphism to compute, for any $i, j$:

$$
\begin{align*}
\theta \ast f_{ij}(\xi) &= \sum_{k=0}^{n_i} \theta \left( \lambda^k(x_i) \right) \cdot (-\xi)^k \\
&= \sum_{k=0}^{n_i} \lambda^k(\theta(x_i)) \cdot (-\xi)^k \\
&= \lambda_{-\xi}(\theta(x_i)).
\end{align*}
$$

But for any $i$, $\theta(x_i)$ is of $\lambda$-dimension $n_i$ in $\mathcal{A}_m$ so that we can find $v_1, \ldots, v_{n_i}$ in $\mathcal{A}_m$ with each $v_j$ of $\lambda$-dimension 1 and $\theta(x_i) = v_1 + \cdots + v_{n_i}$. Consequently,

$$
\theta \ast f_{ij}(\xi) = \lambda_{-\xi}(\theta(x_i)) = \prod_{j=1}^{n_i} (1 - v_j \cdot \xi),
$$

which means that condition (3) is also satisfied.

Let $g : \mathcal{A}_n \rightarrow \mathcal{A}_m$ denote the ring homomorphism given by Wilkerson's lemma. For $i$ fixed and $\theta(x_i) = v_1 + \cdots + v_{n_i}$ as above, the image $g(\xi_{ij})$ is necessarily one of the $v_j$'s and we can deduce that $g$ is a $\lambda$-morphism.

Let $\varphi' : R \rightarrow \mathcal{A}_n' = \mathbb{Z}[[\beta_1^{\pm 1}, \ldots, \beta_n^{\pm 1}]]$ be another Galois $\lambda$-extension. By applying property (2) twice, we obtain $\lambda$-morphisms $F : \mathcal{A}_n \rightarrow \mathcal{A}_n'$ and $G : \mathcal{A}_n' \rightarrow \mathcal{A}_n$ such that $F \circ \varphi = \varphi'$ and $G \circ \varphi' = \varphi$. Thus, $(G \circ F) \circ \varphi = \varphi$ and we can use property (1) to conclude that $G \circ F \in W_{\varphi'}$; in particular, $G \circ F$ is invertible. Similarly $F \circ G$ is in the group $W_{\varphi'}$. So we can safely take $h = F$. 

Remark. The second and third points of the theorem are due to C. Wilkerson (see [20]). But the existence of $\varphi$ seems to be new...

3. An intrinsic characterization for $R(G)$

In Section 2, we gave necessary and sufficient conditions for a $\lambda$-ring to be a ring of invariants $\mathcal{A}_n^W$, where $\mathcal{A}_n = \mathbb{Z}[[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]]$ and $W$ is a finite subgroup of $\text{Aut}_1(\mathcal{A}_n)$. According to Weyl's structure theorem, we now need to look for supplementary conditions which force the group $W$ to be generated by reflections. And this will solve the first problem of the introduction, since a finite subgroup of $\text{GL}_n(\mathbb{Z})$ is a Weyl group.
if and only if it is generated by reflections [6, p. 180]. Note here that we are dealing with “exponential” actions, so that the well-known theorem of Chevalley, Sheppard and Todd cannot be directly applied. Fortunately, these actions become “linear” when we pass to the graded ring associated to the $\gamma$-filtration of our $\lambda$-rings.

Let $R$ be a normal $\lambda$-ring of rank $n > 0$. By Proposition 2.2, $R$ has a unique augmentation ideal $I$. Let us fix a Galois $\lambda$-extension

$$\varphi : R \hookrightarrow \mathcal{A}_n = \mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}],$$

with Galois group $W \subset \text{Aut}_A(\mathcal{A}_n)$.

Recall that the $\gamma$-operations on $R$ are maps $\{\gamma^k : R \rightarrow R\}$ $(k \geq 0)$ defined by the formulas:

$$\gamma_t(x) = \sum_{k \geq 0} \gamma^k(x) \cdot t^k = \lambda_t((1-t)x), \quad \forall x \in R.$$ 

It is not hard to check (or see [5, Section 1.4]) that these maps satisfy the following properties:

1. For all $x, y \in R$, $\gamma^0(x) = 1$, $\gamma^1(x) = x$ and $\gamma_t(x + y) = \gamma_t(x) \cdot \gamma_t(y)$.
2. For all $x \in R$ and all $k \geq 0$, $\gamma^k(x) = \lambda^k(x + k - 1)$.
3. If $x \in R$ is of $\lambda$-dimension 1, then

$$\gamma_t(x - 1) = 1 + (x - 1) \cdot t \quad \text{and} \quad \gamma_t(1 - x) = \sum_{j \geq 0} (1 - x)^j \cdot t^j.$$ 

The $\gamma$-filtration of $R$ is defined as the decreasing sequence

$$F^0 = R \supset F^1 \supset F^2 \supset \cdots,$$

where, for any $k \geq 1$, $F^k = F^k R$ is the $\mathbb{Z}$-submodule generated by the elements $\gamma^1(x_1) \cdot \gamma^2(x_2) \cdots \gamma^k(x_k)$, with $x_1, \ldots, x_k \in I$ and $\sum_s r_s \geq k$. It can be shown that the $F^k$'s are $\lambda$-ideals of $R$ [5, Proposition 4.1]; the graded ring associated to this filtration is

$$\text{Gr}^n_{\gamma}(R) = \bigoplus_{k \geq 0} \text{Gr}^k_{\gamma}(R) = \bigoplus_{k \geq 0} F^k / F^{k+1}.$$ 

Of course, the definition of $I$ implies that $\text{Gr}^0_{\gamma}(R) = \mathbb{Z}$. To handle the situation in degree 1, let us first recall that $R(1)$ denotes the (multiplicative) group of the elements of $\lambda$-dimension 1 in $R$.

**Proposition 3.1.** The map

$$R(1) \longrightarrow R, \quad x \mapsto x - 1$$

induces an isomorphism of $R(1)$ onto the additive group $\text{Gr}^1_{\gamma}(R) = F^1 R / F^2 R$.

**Proof.** It is a consequence of Theorem 1.7 [12, p. 73]. □
For $k > 1$, there is no such simple description of the group $Gr^k(R)$, except for the Laurent ring $R$. In that particular case, we have

**Proposition 3.2.** The $\gamma$-filtration of $R = \mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]$ coincides with its $I$-adic filtration, that is:

$$F^k R = (F^1 R)^k \quad \forall k > 1.$$

For $i = 1, \ldots, n$, let $t_i \in Gr^1_i(R)$ denote the image of $\alpha_i \in R(1)$ by the isomorphism of Proposition 3.1. Then the elements $t_1, \ldots, t_n$ are algebraically independent over $\mathbb{Z}$, so that the graded ring associated to the $\gamma$-filtration of $R$ is

$$Gr_\gamma(R) = \mathbb{Z}[t_1, \ldots, t_n].$$

**Proof.** We refer to [12, p. 49] for the first assertion. The rest of the proposition is a well-known fact of Commutative Algebra; it follows for example from Corollary 2.4 [12, p. 75]. □

Clearly, the action of the Galois group $W$ respect the $\gamma$-filtration of $R$. Moreover, Propositions 3.1 and 3.2 say that the induced $W$-action on $Gr_\gamma(R)$ is "linear"; in other words, $W \subset \text{GL}_n(\mathbb{Z})$ acts linearly on

$$Gr^1_\gamma(R) = \mathbb{Z} \cdot t_1 \oplus \cdots \oplus \mathbb{Z} \cdot t_n$$

and by graded ring automorphisms on the entire $Gr_\gamma(R)$. Since the $\lambda$-morphism $\phi$ also respects the $\gamma$-filtrations, it induces a graded ring homomorphism $Gr(\phi) : Gr_\gamma(R) \rightarrow Gr_\gamma(R)$, whose image is contained in the ring of invariants $(Gr_\gamma(R))^W$. In general, the map $Gr(\phi)$ is not injective; for instance, J.F. Adams [3, p. 3] has detected non-trivial 3-torsion in the graded ring $Gr_\gamma(R(F_4))$ corresponding to the representation ring of the exceptional Lie group $F_4$. And we have just seen that $Gr_\gamma(R)$ is always torsion-free. Nevertheless, we will see in a moment that $\text{Ker}(Gr(\phi))$ contains only torsion elements. To this end, it is more convenient to tensor everything in sight by the rationals; so we set

$$R Q := R \otimes \mathbb{Q}, \quad F^k Q := F^k \otimes \mathbb{Q} \quad \forall k \geq 1,$$

$$I Q := I \otimes \mathbb{Q} \quad \text{and} \quad J Q := F^1 R \otimes \mathbb{Q}.$$

We will also need the Chern character

$$ch : R \rightarrow \prod_{j \geq 0} \left( Gr^j_\gamma(R) \otimes \mathbb{Q} \right), \quad x \mapsto (ch^j(x))_{j \geq 0}.$$

This map is defined in [12, Sections 1.4 and 3.2]; here we just recall that it is a natural ring homomorphism. Its obvious extension to $R Q$ will be written $ch_Q$.

**Lemma 3.1.** Fix a positive integer $k$ and let $x$ be an element of $R Q$. Then $x$ lies in $F^k Q$ if and only if $ch^j_Q(x) = 0$ for any $j < k$. 

Proof. Since $F^k$ is a $\lambda$-ideal of $R$, the quotient $R/F^k$ inherits a canonical $\lambda$-structure. The $\gamma$-filtration on $R/F^k$ is given by $R/F^k \supset F^1/F^k \supset F^2/F^k \supset \cdots \supset F^k/F^k = 0$, hence

$$\text{Gr}_\gamma(R/F^k) = \bigoplus_{i=0}^{k-1} \text{Gr}_\gamma^i(R).$$

Moreover, the Chern character of $R/F^k$ is just the truncation at level $k$ of $\text{ch}$. By Theorem 3.5 [12, p. 611], this Chern character induces a ring isomorphism

$$R/F^k \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{i=0}^{k-1} (\text{Gr}_\gamma^i(R) \otimes \mathbb{Q}),$$

and we can now conclude. $\square$

Lemma 3.2. The Chern character $\text{ch}$ extends to a ring isomorphism

$$\hat{\text{ch}} : \hat{R} \xrightarrow{\cong} \prod_{j \geq 0} (\text{Gr}_\gamma^j(R) \otimes \mathbb{Q}),$$

with $\hat{R}$ denoting the completion of $R$ with respect to the filtration $\{F^k\}$.

Proof. By definition, this completion of $R$ is

$$\hat{R} = \varinjlim R/F^k \otimes \mathbb{Q}.$$

Using the naturality of the Chern character, we see that the announced isomorphism is obtained as the limit of the isomorphisms (*) above. $\square$

Lemma 3.3. Let $\hat{\varphi} : \hat{R} \rightarrow \hat{\mathcal{A}}$ be the completion (with respect to the rationalized $\gamma$-filtration) of the homomorphism $\varphi \otimes \mathbb{Q} : R \hookrightarrow \mathcal{A} \otimes \mathbb{Q}$. Then $\hat{\varphi}$ is injective and its image is the ring of invariants $(\hat{\mathcal{A}})$

Note that the completion $\hat{\mathcal{A}}$ is isomorphic to the power series ring $\mathbb{Q}[[x_1 - 1, \ldots, x_n - 1]]$.

Proof. We use $\varphi \otimes \mathbb{Q}$ to identify $R$ to the ring of invariants $(\mathcal{A} \otimes \mathbb{Q})^{\mathbb{W}}$ and we consider another filtration $E^0_\mathcal{Q} = R_\mathcal{Q} \supset E^1_\mathcal{Q} \supset E^2_\mathcal{Q} \supset \cdots$, defined by

$$E^k_\mathcal{Q} = (J_\mathcal{Q})^k \cap R_\mathcal{Q} \quad \text{for} \quad k = 0, 1, \ldots$$

We claim that there exists a positive integer $m$ such that

$$E^m_\mathcal{Q} \subset (J_\mathcal{Q})^k \subset F^k_\mathcal{Q} \subset E^k_\mathcal{Q}, \quad \forall k \geq 1.$$

The last two inclusions are obvious and the first one, together with the integer $m$, are provided by Lemma 9 of [11]. Strictly speaking, this lemma of [11] is stated for algebras over $\mathbb{C}$, but its proof only uses the fact that $\mathbb{C}$ is of characteristic zero.

Because of the inclusions (**), we can say that the completion of $R_\mathcal{Q}$ with respect to the $E$-filtration coincide with the $\gamma$-completion $\hat{R}_\mathcal{Q}$. Granted this fact, we can now appeal to [11, Lemma 10] for the conclusion. $\square$
Proposition 3.3. The graded ring homomorphism

\[ Gr(\varphi) : Gr_\gamma(R) \otimes \mathbb{Q} \rightarrow Gr_\gamma(\mathcal{A}_n) \otimes \mathbb{Q} \]

is injective; its image is equal to the ring of invariants \((Gr_\gamma(\mathcal{A}_n) \otimes \mathbb{Q})^W\). In other words we have

\[ Gr_\gamma(R) \otimes \mathbb{Q} \cong \mathbb{Q} [t_1, \ldots, t_n]^W. \]

Proof. By the very definition of the Chern character, there is a commutative diagram

Thanks to the preceding lemmas, the two top vertical maps are bijective and \(\hat{\phi}_Q\) is injective; thus we can conclude that \(Gr(\varphi) \otimes \mathbb{Q}\) is injective.

Assume now that \(\tilde{y} \in Gr^k_\gamma(\mathcal{A}_n)\) is invariant under the \(W\)-action. Let us choose \(y \in F^k \mathcal{A}_n\) such that \([y] = \tilde{y}\) in \(Gr^k_\gamma(\mathcal{A}_n)\). The hypothesis on \(\tilde{y}\) implies that, for each \(w \in W\), there exists \(y_w \in F^{k+1} \mathcal{A}_n\) such that \(w \cdot y = y + y_w\). Summing over \(W\), we get

\[ \sum_{w \in W} w \cdot y = |W| \cdot y + \sum_{w \in W} y_w. \]

Therefore, we have an element \(x \in R\) such that \(\varphi(x) \in F^k \mathcal{A}_n\) and \([\varphi(x)] = |W| \cdot \tilde{y}\) in \(Gr^k_\gamma(\mathcal{A}_n)\). In particular, \(ch^j_Q(\varphi(x)) = 0\) for any \(j < k\). The injectivity of \(Gr(\varphi) \otimes \mathbb{Q}\) and the commutativity of the diagram above imply that \(ch^j_Q(x) = 0\) for any \(j < k\). By Lemma 3.1 this last property says that \(x\) in an element of \(F^k_Q\); we then have

\[ |W| \cdot \tilde{y} = (Gr(\varphi) \otimes \mathbb{Q})([x]), \]

where \([x]\) denotes the class of \(x\) in \(Gr^k_\gamma(R) \otimes \mathbb{Q}\). And we have shown that the ring of invariants \((Gr_\gamma(\mathcal{A}_n) \otimes \mathbb{Q})^W\) is contained in the image of \(Gr(\varphi) \otimes \mathbb{Q}\). \(\square\)
At this point, we just put together the preceding proposition and the theorem of Chevalley, Sheppard and Todd (see e.g. [6, p. 180]) to complete the proof of

**Theorem 3.1.** For a $\lambda$-ring $R$, the following are equivalent:

1. $R$ is $\lambda$-isomorphic to the representation ring of a compact connected Lie group;
2. $R$ is a normal $\lambda$-ring and the graded ring $Gr_{\gamma}(R) \otimes \mathbb{Q}$ is polynomial over $\mathbb{Q}$.

4. A reconstruction theorem

In this section, we will examine how the $\lambda$-ring $R(G)$ fails to characterize the compact connected Lie group $G$. Surprisingly, the indeterminacy only involves the groups $\text{Sp}(l)$ and $\text{SO}(2l+1)$. This is used to show that the group $G$ can be reconstructed from anyone of the following data:

(i) the $\lambda$-ring $R(G)$ and its subgroup $RO(G)$;
(ii) the $\lambda$-ring $R(G)$ and its subgroup $RSp(G)$.

4.1. Root diagrams

All the materials in this subsection comes from the [7]. The interested reader may refer to it for the proofs. We will also assume some familiarity with the notion of *root system*, for which [6, Ch. VI] is a good reference.

**Definition 4.1** (cf. [7, Section 4.8]). A *root diagram* is a triple $D = (M, M_0; \Phi)$ with the following properties:

1. $M$ is a finitely generated free $\mathbb{Z}$-module and $M_0$ a submodule which is also a direct factor of $M$.
2. $\Phi$ is a finite subset of $M$ and $\Phi \cup M_0$ generates the $\mathbb{R}$-vector space $V = M \otimes \mathbb{R}$.
3. For any any $\alpha \in \Phi$ and any $k \in \mathbb{Z}$, $k\alpha \in \Phi$ if and only if $k = \pm 1$.
4. For any $\alpha \in \Phi$, there exists linear form $\alpha^\vee \in M^* = \text{Hom}(M, \mathbb{Z})$ such that:
   (i) $\alpha^\vee(M_0) = 0$ and $\alpha^\vee(\alpha) = 2$;
   (ii) the homomorphism
   $$s_\alpha : M \to M, \quad x \mapsto s_\alpha(x) = x - \alpha^\vee(x) \cdot \alpha,$$
   called *reflection of vector* $\alpha$, globally fixes the set $\Phi$.

Let $D = (M, M_0; \Phi)$ be a root diagram. It is not difficult to show that, for each $\alpha \in \Phi$, the element $\alpha^\vee$ as above is uniquely determined. Moreover, the $\mathbb{R}$-vector space $V = M \otimes \mathbb{R}$ decomposes as a direct sum

$$V = (M_0 \otimes \mathbb{R}) \oplus V(\Phi),$$

where $V(\Phi)$ is the $\mathbb{R}$-span of $\Phi$. The couple $(V(\Phi); \Phi)$ is a reduced root system whose Weyl group is isomorphic to the subgroup $W(D)$ of $\text{Aut}(M)$ generated by the
reflections $s_\alpha$ ($\alpha \in \Phi$); $W(D)$ is called the Weyl group of the root diagram $D$. Without loss of generality, we can assume that $V = M \otimes \mathbb{R}$ is equipped with a $W(D)$-invariant inner product $(\cdot, \cdot)$. Then, for each $\alpha \in \Phi$, the reflection $s_\alpha$ extends to $V$ where it is given by the formula

$$s_\alpha : x \mapsto x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \cdot \alpha \quad \forall x \in V.$$ 

Consequently, the fourth condition of the preceding definition translates into the more familiar conditions

(4.i)' For any $\alpha \in \Phi$ and any $x \in M_0$, we have $(x, \alpha) = 0$.
(4.ii)' For any $\alpha, \beta \in \Phi$, we have

$$S_\alpha(\Phi) = \Phi \quad \text{and} \quad \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}.$$ 

We now pause to recall how root diagrams arise in Lie theory. Let us fix a compact connected Lie group $G$ and a maximal torus $T$ of the latter. The set of (global) roots of $G$, with respect to $T$, will be denoted by $\Phi(G, T)$ and the symbol $D(G)$ will stand for the commutator subgroup of $G$. Let also $X(T)$ (respectively $X_0(T)$) be the group of characters of $T$ (respectively $T/(T \cap D(G))$). The triple

$$A(G, T) = (x(T), x_0(T); \Phi(G, T))$$ 

is a root diagram [7, p. 381]; its Weyl group is (as expected) naturally isomorphic to the quotient $W(G, T) = N(T)/T$, $N(T)$ being the normalizer of $T$ in $G$. It can be shown that $X_0(T)$ is equal to the group of elements of $X(T)$ which are invariant under the action of $W(G, T)$. If $G'$ is another compact connected Lie group and $T'$ a maximal torus of $G'$, then the root diagram of $G \times G'$ with respect $T \times T'$ is

$$A(G \times G', T \times T') = (X(T) \oplus X(T'), X_0(T) \oplus X_0(T'); \Phi(G, T) \sqcup \Phi(G', T')).$$ 

We return to the abstract theory by introducing the following:

**Definition 4.2.** Let $D_1 = (M_1, (M_1)_0, \Phi_1)$ and $D_2 = (M_2, (M_2)_0, \Phi_2)$ be root diagrams. A $\mathbb{Z}$-linear bijective map $f : M_1 \to M_2$ is called an isomorphism of $D_1$ onto $D_2$ if $f((M_1)_0) \subset (M_2)_0$ and $f(\Phi_1) = \Phi_2$.

**Lemma 4.1.** Assume that $f : M_1 \to M_2$ is an isomorphism from a root diagram $D_1 = (M_1, (M_1)_0, \Phi_1)$ onto a root diagram $D_2 = (M_2, (M_2)_0, \Phi_2)$. For any $\alpha \in \Phi_1$ we have

$$f \circ S_\alpha \circ f^{-1} = S_{f(\alpha)} \in W(D_2).$$

In particular, for all $\alpha, \beta \in \Phi_1$, $f(\alpha)^\vee [f(\beta)] = \alpha^\vee(\beta)$.

**Proof.** Let us fix $\alpha \in \Phi_1$ and denote the $\mathbb{R}$-linear extension of $s_ \alpha$ and $f$ by the same letters. Let also $P_\alpha$ be the fixed hyperplane of the reflection $s_\alpha$; by definition, $(M_2)_0$
is contained in the hyperplane \( f(P_\alpha) \). The \( \mathbb{R} \)-linear bijection \( f \circ s_\alpha \circ f^{-1} : V_2 \to V_2 \) (where \( V_2 = M_2 \otimes \mathbb{R} \)) satisfies the following properties:

(i) \( (f \circ s_\alpha \circ f^{-1})(v_2) = v_2 \), for all \( v_2 \in f(P_\alpha) \).

(ii) \( (f \circ s_\alpha \circ f^{-1})[f(\alpha)] = -f(\alpha) \).

(iii) \( (f \circ s_\alpha \circ f^{-1})(\Phi_2) = \Phi_2 \).

Properties (i) and (ii) say that \( f \circ s_\alpha \circ f^{-1} \) is a reflection of \( V_2 \), with fixed hyperplane containing \((M_2)_0\). By combining this with property (iii), we can apply Lemma 1 [7, p. 142] to conclude that \( f \circ s_\alpha \circ f^{-1} \) is equal to the reflection \( s_{f(\alpha)} \) of \( W(D_2) \).

As a consequence of this lemma, any isomorphism \( f : D_1 \to D_2 \) induces a group isomorphism

\[
f_\# : W(D_1) \to W(D_2), \quad w \mapsto f \circ w \circ f^{-1}.
\]

When \( D_1 = D_2 = D \) the isomorphisms of \( D \) onto itself form a group noted \( Aut(D) \); the preceding observation implies that the Weyl group \( W(D) \) is normal in \( Aut(D) \).

**Example.** Let \( G \) be a compact connected Lie group and \( T, T' \) maximal tori in \( G \). The conjugacy of \( T \) and \( T' \) (in \( G \)) implies that the root diagrams \( \Delta(G,T) \) and \( \Delta(G,T') \) are isomorphic. In other words there is, up to isomorphism, a unique root diagram \( \Delta(G) \) attached to \( G \).

We are now in position to state the central ingredient of all our future arguments:

**Theorem 4.1** (cf. [7, p. 40]). For any root diagram \( D \), there exist a compact connected Lie group \( G \) with \( \Delta(G) \) isomorphic to \( D \). Moreover, two compact connected Lie groups are isomorphic if and only if their root diagrams are isomorphic.

For later use, we quote the useful and well-known

**Proposition 4.1.** Let \( D = (M,M_0;\Phi) \) be a root diagram. Any reflection in the Weyl group \( W(D) \) is of the form \( s_\alpha \) for some \( \alpha \in \Phi \).

**Proof.** See Proposition 1.14 [13, p. 24]

**4.2. Reconstructing \( G \) from \( R(G) \)**

Let \( R \) be a \( \lambda \)-ring and assume that we have shown (using, for example, Theorem 3.1) that \( R \) is \( \lambda \)-isomorphic to the representation ring of an unknown compact connected Lie group \( G \). Our goal is to find all the informations on \( G \) which are encoded in the \( \lambda \)-ring \( R \).

By Theorem 2.2, we can construct a Galois \( \lambda \)-extension

\[
\varphi : R(G) \to R(T) = \mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]
\]
with Galois group $W \subset Aut_{\mathbb{Z}}(R(T)) = \text{GL}_n(\mathbb{Z})$, where $n = \text{dim}(R(G)) - 1 = \text{rank}(G)$ and $T = S^1 \times \cdots \times S^1$ ($n$ factors) is identified to a maximal torus in $G$. The same theorem also tells us that $W$ is conjugate in $\text{GL}_n(\mathbb{Z})$ to the Weyl group of $G$.

If $W = \{1\}$ then $R$ is equal to $R(T)$, hence all irreducible (complex) representations of $G$ are one dimensional. It follows that $G$ must be isomorphic to $T$. From now on we assume that $W$ is not trivial.

Let us set

$$X(T) := R(T)(1) = \{a_1^{m_1} \cdot a_2^{m_2} \cdots a_n^{m_n}; \ m_i \in \mathbb{Z}\}.$$ 

The multiplication and $W$-action on $R(T)$ induce a $W$-module structure on $X(T)$. Until further notice, $X(T)$ will be identified with the free abelian group $\mathbb{Z}^n$ (via the map $a_1^{m_1} \cdot a_2^{m_2} \cdots a_n^{m_n} \mapsto (m_1, \ldots, m_n)$); in particular, the group operation of $X(T)$ will be written additively. The $W$-module structure on $X(T)$ extends (by $\mathbb{R}$-linearity) to the $\mathbb{R}$-vector space $V := X(T) \otimes \mathbb{R}$. For simplicity, we will also choose a $W$-invariant inner product $(\cdot, \cdot)$ on $V$.

Since $W$ and the Weyl group $W(G)$ are conjugate in $\text{GL}_n(\mathbb{Z})$, the $W$-action on $X(T)$ is equivalent to the usual action of $W(G)$ on $X(T)$. By invoking Proposition 4.1, we can deduce that the reflections of $W \subset \text{GL}(V)$ are in one to one correspondence with the pairs of roots of the Lie group $G$. More precisely, for each reflection $s \subset W$, the intersection

$$\delta_s = \{x \in V; s(x) = -x\} \cap X(T)$$

is an infinite cyclic group and its two generators $\pm a_s$ are proportional to the two roots associated to the reflection $s$. Put in another way, $\pm a_s$ are the minimal vectors of $X(T)$ which are in the span of the roots of $G$ corresponding to $s$. Let us define

$$X_0(T) := X(T)^W = \{x \in X(T); \ w \cdot x = x \ \forall w \in W\},$$

$$\Phi_{\text{min}} = \{\pm a_s; \ s \ \text{reflection of } W\}.$$ 

**Claim 1.** The triple $D_{\text{min}} = (X(T), X_0(T); \Phi_{\text{min}})$ is a root diagram with Weyl group $W$.

**Proof.** The elementary divisors theorem, combined with the definition of $X_0(T)$, implies that $X_0(T)$ is a direct factor of $X(T)$. The second condition in Definition 4.1 is also satisfied, since the roots of $G$ are nonzero multiples of the $a_s$. The third condition is obviously satisfied. For the last one, it is more convenient to check the equivalent conditions $(4.ii)'$ and $(4.ii)'$. If $\alpha = \pm a_s \in \Phi_{\text{min}}$ and $x \in X(T)$ are fixed, then

$$s_s(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)} \cdot \alpha = s(x) \in X(T).$$

This relation shows that $2(\alpha, x)/(\alpha, \alpha)$ is an integer and $(\alpha, \alpha) = 0$ whenever $x$ lies in $X_0(T)$. Finally, if $\beta$ is another element of $\Phi_{\text{min}}$ and $\gamma = s_\beta$, then Lemma 4.1 says...
that the element \( s \circ s_\beta \circ s^{-1} \) of \( W \) coincide with the reflection \( s_\gamma \). It follows that 
\[ \gamma = s(\beta) = s_\gamma(\beta) \] 
lies in \( \Phi_{\text{min}} \) and we are done. \( \square \)

**Remark.** The idea of considering *minimal vectors* is of course part of the folklore. The proof given above is already present in [11] or [7].

In general, \( D_{\text{min}} \) is different from the diagram \( \Delta(G) \) that we are looking for. Nonetheless, we have shown that \( \Delta(G) \) is necessarily of the form \( (X(T), X_0(T); \Phi(G)) \), where the set of roots \( \Phi(G) \) is related to \( \Phi_{\text{min}} \) as follows:

For any reflection \( s \in W \), there is a nonzero integer \( n_s \) such that \( \theta_s = \pm n_s \cdot a_s \) is the pair of roots of \( \Phi(G) \) which corresponds to \( s \).

As \( \Delta(G) \) is a root diagram and the \( a_s \)'s are in \( X(T) \), the last condition in Definition 4.1 gives us:

\[ \frac{2(a_r, \theta_s)}{(\theta_s, \theta_s)} = \frac{1}{n_s} \cdot \frac{2(a_r, a_s)}{(a_s, a_s)} \in \mathbb{Z} \quad \text{for any two reflections } r, s \in W. \]

In order to fully exploit these integrality conditions, we consider the decomposition

\[ \Phi_{\text{min}} = \Phi_1 \coprod \cdots \coprod \Phi_k \]

of the root system \( (V(\Phi_{\text{min}}); \Phi_{\text{min}}) \) into its irreducible components. For \( j = 1, \ldots, k \), let us choose a basis \( \mathcal{B}_j \) for the irreducible root system \( (V(\Phi_j); \Phi_j) \). Fixing \( j \) and making \( a_r, a_s \) run over \( \mathcal{B}_j \) in the above integrality conditions, we see that, for each reflection \( s \), the integer \( n_s \) must divide all the coefficients in some column of the Cartan matrix for \( \Phi_j \). A quick glance through the list of irreducible Cartan matrices (for example in [6, Planches]) reveals that \( n_s \) is either \( \pm 1 \) or \( \pm 2 \). Moreover, the case \( n_s = \pm 2 \) is only possible when \( \Phi_j \) is of type \( B_l \) \( (l \geq 1) \) and \( A_1 = B_1 \) and \( \pm a_s \) are the short roots of the basis \( E_j \). In other words, all the \( \Phi_j \)'s of types different from \( B_l \) \( (l \geq 1) \) are already components of the root system \( (V(\Phi(G)); \Phi(G)) \). Thus, we only have to take a closer look at components of type \( B_l \) in \( \Phi_{\text{min}} \).

Let \( \Phi_j \) be such a component. If we multiply its short roots by 2, \( \Phi_j \) becomes an irreducible root system \( \tilde{\Phi}_j \) of type \( C_l \) \( (l \geq 1, \text{ with } C_1 = B_1 \text{ and } C_2 = B_2) \). We will then say that the component \( \Phi_j \) has been *stretched*.

**Claim 2.** Let us write \( \tilde{\Phi} \) for the disjoint union of \( \Phi_1, \ldots, \Phi_{j-1}, \tilde{\Phi}_j, \Phi_{j+1}, \ldots, \Phi_k \). The triple \( \tilde{\Delta} = (X(T), X_0(T); \tilde{\Phi}) \) is a root diagram if and only if the subgroup \( \langle \Phi_j \rangle \), generated by the roots of \( \Phi_j \) is a direct factor of \( X(T) \).

**Proof.** Let us write \( \langle \tilde{\Phi}_j \rangle \) for the subgroup (of \( X(T) \)) generated by the roots of \( \tilde{\Phi}_j \), and \( X_j \) for the image of \( X(T) \) by the orthogonal projection of \( V \) onto \( V(\Phi_j) \). We have \( \langle \tilde{\Phi}_j \rangle \subset (\Phi_j) \subset X_j \) and, by construction, \( \langle \tilde{\Phi}_j \rangle \) is of index 2 in \( (\Phi_j) \). If \( x \in X(T) \) and \( x_j \) is its orthogonal projection, then

\[ \frac{2(x_j, \alpha)}{(\alpha, \alpha)} = \frac{2(x, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \quad \text{for any } \alpha \in \tilde{\Phi}_j. \]
This means that $X_j$ is contained in the weight lattice of $\Phi_j$. But for any root system of type $C_l$, the subgroup generated by the roots is of index 2 in the weight lattice ([6, Planches II]). Consequently $X_j$ is equal to $\langle \Phi_j \rangle$ and we have shown the non trivial part of our claim. □

Let us rearrange (if necessary) the indices of the $\Phi_j$ in such a way that $\Phi_1, \ldots, \Phi_r$ ($0 \leq r \leq k$) are the components of type $B$ in $(V(\Phi_{\text{min}}); \Phi_{\text{min}})$ which satisfy the equivalent conditions of Claim 2. Clearly these components (if there is any!) are the only candidates for the stretching of the short roots described before. If we set

$$M_d = \langle \Phi_1 \rangle \oplus \cdots \oplus \langle \Phi_r \rangle,$$

then Claim 2 says that there is a unique submodule $M_d$ of $X(T)$ such that $D_{\text{min}}$ decomposes as

$$D_{\text{min}} = D_d \oplus D_u = \left( M_d \oplus M_u, \{0\} \oplus X_0(T); \Phi_d \coprod \Phi_u \right),$$

where $\Phi_d = \Phi_1 \coprod \cdots \coprod \Phi_r$ and $\Phi_u = \Phi_{\text{min}} \setminus \Phi_d$ (the complementary set).

The root diagram $D_u$ constructed above is uniquely determined by our initial datum, namely the $\lambda$-ring $R$; in other words all the indeterminacies are contained in the diagram $D_d$. Let us observe that the latter is a direct sum of root diagrams of some groups $SO(2l+1)$ and that the stretching of a component in $D_d$ transforms the corresponding factor into the root diagram of some $Sp(l)$. It suffices now to invoke Theorem 4.1, in order to complete the proof of

**Theorem 4.2.** With its $\lambda$-ring structure, the representation ring of a compact connected Lie group $G$ uniquely determines a compact connected Lie group $K$ such that

(i) $K$ has no direct factor isomorphic to some $SO(2l+1)$ or $Sp(l)$ with $l \geq 1$;

(ii) $G$ is isomorphic to a direct product $H_1 \times \cdots \times H_r \times K$ where each $H_j$ is equal either to $SO(2l_j+1)$ or $Sp(l_j)$ for some $l_j \geq 0$.

**Example.** For any integer $l \geq 3$, the group $\text{Spin}(2l+1)$ is completely determined by its representation ring (as a $\lambda$-ring). In particular, $R(\text{Spin}(2l+1))$ is not $\lambda$-isomorphic to $R(\text{Sp}(l))$. Of course we have $\text{Spin}(3) = \text{Sp}(1)$ and $\text{Spin}(5) = \text{Sp}(2)$. Similarly, $\text{PSp}(l)$ can be reconstructed from its representation ring.

We are now ready for the announced reconstruction result:

**Theorem 4.3**. A compact connected Lie group $G$ is uniquely determined by the $\lambda$-ring $R(G)$ and either the subgroup $RO(G) \subseteq R(G)$ or $RSp(G) \subseteq R(G)$.

**Proof.** Starting with the normal $\lambda$-ring $R$, we proceed as in the proof of Theorem 4.2 to construct root diagrams $D_{\text{min}} = (X(T), X_0(T); \Phi_{\text{min}}), D_d$ and $D_u$ with $D_{\text{min}} = D_d \oplus D_u$

$^1$ D. Notbohm has a different proof of this result (private communication).
and \( D_d \) the product of all the "stretchable" components in \( D_{\text{min}} \). In order to recover the root diagram of \( G \), hence the group \( G \) itself, we only need to know the components of \( D_d \) which have to be effectively stretched.

Let \( \Phi_j \) be a fixed irreducible component of \( D_d \) and \( B_j = \{a_1, \ldots, a_l\} \) a basis of \( \Phi_j \). We may assume that the matrix \([2(a_i, a_j)/(a_j, a_j)]\) is equal to the standard Cartan matrix for type \( B_l \). As we have seen, \( \Phi_j \) corresponds to a direct factor \( H_j \) of \( G \), isomorphic to either \( \text{SO}(2l + 1) \) or \( \text{Sp}(l) \).

For the rest of the proof, we have to switch back to the multiplicative notation for the group operation in \( X(T) \). In other words we are viewing again \( X(T) \) as the multiplicative group of the elements of \( \lambda \)-dimension 1 in \( R(T) \). Let us define \( \omega \in R(T) \) by

\[
\omega = \prod_{i=1}^l a_i.
\]

Thanks to Sections VI.5 and V.6 of [8], \( \omega \) is a (global) weight for the representation

\[
q : G \xrightarrow{\pi} H_j \xrightarrow{\tilde{\varrho}} \text{GL}(W),
\]

where \( \pi \) is the canonical projection and \( \tilde{\varrho} \) the standard (complex) representation of the group \( H_j \). The little miracle here is that \( \omega \) does not depend on whether \( H_j \) is \( \text{SO}(2l + 1) \) or \( \text{Sp}(l) \). The representation \( q \) is irreducible and its character is given by

\[
i^*(q) = \begin{cases} 
S(\omega) & \text{if } H_j = \text{Sp}(l) \\
S(\omega) + 1 & \text{if } H_j = \text{SO}(2l + 1)
\end{cases},
\]

where \( S(\omega) \in R(T)^W = R(G) \) denotes the sum of the elements in the orbit \( W \cdot \omega \subset X(T) \). From the representation theory of the \( \text{SO}(2l + 1) \)'s and \( \text{Sp}(l) \)'s, one can deduce that \( S(\omega) \) is in \( RO(G) \subset R(G) \) if and only if \( H_j = \text{SO}(2l + 1) \) (and this is equivalent to \( S(\omega) \notin R\text{Sp}(G) \subset R(G) \)). In other words, the component \( \Phi_j \) will be stretched only if \( S(\omega) \) does not lie in \( RO(G) \subset R(G) \) (or equivalently only if \( S(\omega) \in R\text{Sp}(G) \subset R(G) \)).

There is another way to tell a group \( \text{SO}(2l + 1) \) apart from \( \text{Sp}(l) \): their normalizer of maximal tori are very different (see [9, Ch. 11]). We will combine this observation with Theorem 4.2 to present a new proof of a reconstruction result of Curtis, Wiederhold and Williams. In [10] the result is only stated and proved in the semi-simple case. With Theorem 4.1 at hand, it is easy to see that their proof extends to all compact connected Lie groups. However, we believe that the proof presented here might shed some light on their remarkable result.

**Theorem 4.3** (Curtis [10]). A compact connected Lie group can be reconstructed from the normalizer of one of its maximal tori. Equivalently, two compact connected Lie groups are isomorphic if and only if the normalizers of their maximal tori are isomorphic.
Proof. Suppose that we are given the normalizer $N$ of a maximal torus $T$, in some compact connected Lie group $G$. As is well-known there is a short exact sequence

$$1 \rightarrow T \rightarrow N \overset{\pi}{\rightarrow} W \rightarrow 1,$$

where $T = N_0$ (the component of the identity) and $W = \pi_0(N)$ is the Weyl group with respect to $T$. This exact sequence also gives the action of $W$ on $T$; so we can proceed as in the proof of Theorem 4.2 to construct a compact connected Lie group $K$ such that

(i) $K$ is uniquely determined by the $W$-action on $T$;
(ii) $G = I_1 \times \cdots \times I_r \times K$ where each $I_j$ is either $SO(2l_j + 1)$ or $Sp(l_j)$ for some $l_j \geq 1$.

Property (ii) implies that the normalizer $N$ must decompose as

$$N = N(I_1) \times \cdots \times N(I_r) \times N(K),$$

where $N(X)$ denotes the normalizer of a maximal torus in the group $X$. At this stage neither this decomposition nor the groups $I_j$ are known; but the $W$-action on $T$ does give us the decomposition of Weyl groups

$$W = W(I_1) \times \cdots \times W(I_r) \times W(K).$$

Let us fix $j \in \{1, \ldots, r\}$ and set $N_j = \pi_j^{-1}(W(H_j))$; thus we have a short exact sequence

$$1 \rightarrow T \rightarrow N_j \overset{\pi_j}{\rightarrow} W(H_j) \rightarrow 1,$$

where $\pi_j$ is the restriction of $\pi$ to $N_j$. Clearly, the exact sequence (†) is determined by our initial datum. By the above decompositions, we must have $N_j = N(I_j) \times T_j$, where $T_j$ is a subtorus of $T$. Consequently, the sequence (†) splits if and only if the normalizer’s short exact sequence

$$1 \rightarrow T(H_j) \rightarrow N(H_j) \rightarrow W(H_j) \rightarrow 1$$

splits. In [9, Ch. 11], it is proved that the normalizer’s sequence splits for $SO(2l_j + 1)$ but does not for $Sp(l_j)$. Therefore, the sequence (†) splits if and only if $H_j = SO(2l_j + 1)$; this criterion shows that our datum is indeed sufficient for the reconstruction of the group $G$. 

5. Applications to classifying spaces

Our first application deals with a question of Mislin. He asked (in [19]) if a compact connected Lie group is determined by its classifying space. The answer is affirmative and several independent (and very different!) proofs are now available ([16–18], and also [21]). Moreover, it is shown in [17] that the connectedness hypothesis is superfluous. Here we would like to stay with the connected case and present (yet!) another proof of
Theorem 5.1. The classifying space $BG$ of a compact connected Lie group $G$ uniquely determines the isomorphism class of $G$. Equivalently, two compact connected Lie groups are isomorphic if and only if their classifying spaces are homotopy equivalent.

Proof. Suppose that the classifying space $BG$ of a compact connected Lie group $G$ is given. This datum determines the (complex) $K$-theory ring $K^0(BG)$ with its $\lambda$-structure and the real $K$-theory $KO^0(BG) \subset K^0(BG)$. By Theorem 1.8 of [2], the $\lambda$-ring $R(G)$ is equal to the sub-$\lambda$-ring of $K^0(BG)$ obtained by taking the differences of elements of finite $\lambda$-dimension. We claim that the real representation ring of $G$ is given by the formula

$$RO(G) = KO^0(BG) \cap R(G).$$

Granted this formula, we can complete our proof by invoking Theorem 1.3. For $G$ simply connected, the above equality is proved in [4, Corollary 2.4]. Moreover, the argument given by these authors is still valid when $G = H \times T$ with $H$ 1-connected and $T$ a torus. Now let $G$ be any compact connected Lie group; then there is a finite covering $\pi : H \times T \to G$ with $H$ 1-connected and $T$ a torus. The surjectivity of $\pi$ implies that, for any complex representation $\phi : G \to GL(V)$, $\phi$ is irreducible if and only if $\pi \circ \phi$ is irreducible. In addition, if $\pi \circ \phi$ is irreducible of real (respectively complex, quaternionic) type, then $\phi$ is also of real (respectively complex, quaternionic) type. The desired conclusion now follows from these observations and [1, Theorem 3.57].

As a second application, we give another proof of the following result of [15].

Theorem 5.2 (Jackowski et al. [15]). Let $G$ be a compact connected Lie group and let $Aut(BG)$ denote the group (of homotopy classes) of self-homotopy equivalences of the classifying space $BG$. Then $Aut(BG)$ is isomorphic to the group $Out(G)$ of outer automorphisms of $G$.

Our proof uses Theorem 4.3 of [14] and follows quite naturally from the developments of the previous sections. Recall that the group $Out(G)$ is the quotient $Aut(G)/Int(G)$, where $Int(G)$ denotes the group of inner automorphisms of $G$.

We begin by a well-known Lie theoretical description of the group $Out(G)$. For this purpose, we fix a maximal torus $T$ in $G$; the root diagram with respect to $T$ will be noted $\Delta = (X(T), X_0(T); \Phi)$, with $X(T) - R(T)(1) - Hom(T, S^1)$. $W$ will stand for the Weyl group of $\Delta$. We will also need the closed subgroup of $Aut(G)$ defined by

$$Aut(G, T) = \{ \alpha \in Aut(G); \alpha(T) = T \}.$$

It is easily seen that the quotient $Aut(G, T) / (Aut(G, T) \cap Int(G))$ is isomorphic to $Out(G)$; thus we can focus all our attention to the former.
Proposition 5.1. For any \( \varphi \in \text{Aut}(G, T) \), let \( X(\varphi) \) denote the corresponding automorphism of the root diagram \( \Delta \). The correspondence \( \varphi \mapsto X(\varphi^{-1}) \) induces a group isomorphism

\[
\xi : \text{Aut}(G, T)/\langle \text{Aut}(G, T) \cap \text{Int}(G) \rangle \cong \text{Aut}(\Delta)/W.
\]

Proof. See [7, p. 411]. \( \square \)

Recall now that the "classifying space functor" provides a group homomorphism

\[
\tilde{\beta} : \text{Aut}(G, T)/\langle \text{Aut}(G, T) \cap \text{Int}(G) \rangle \rightarrow \text{Aut}(BG), \quad [\varphi] \mapsto [B\varphi].
\]

This map goes "from algebra to topology". To go in the opposite direction, let us fix a class \( [f] \in \text{Aut}(BG) \). This class induces a map in \( K \)-theory which restricts to a \( \lambda \)-automorphism \( R(f) : R(G) \rightarrow R(G) \). Because of the relation (\( \Diamond \)) above, we can say that \( R(f) \) sends the subgroup \( RO(G) \subset R(G) \) into itself.

On the other hand, the inclusion of \( T \) into \( G \) induces a Galois \( \lambda \)-extension \( i^* : R(G) \rightarrow R(T) \), with Galois group \( W \). Therefore, the composite \( i^* \circ R(f) \) is also a Galois \( \lambda \)-extension; by Theorem 2.2 there is a \( \lambda \)-automorphism \( \tilde{\rho}(f) : R(T) \rightarrow R(T) \) satisfying the following properties:

(i) The diagram

\[
\begin{array}{ccc}
R(T) & \overset{\tilde{\rho}(f)}{\longrightarrow} & R(T) \\
\downarrow i^* & & \downarrow i^* \\
R(G) & \overset{R(f)}{\longrightarrow} & R(G)
\end{array}
\]

is commutative.

(ii) The map

\[
W \rightarrow W, \quad w \mapsto \tilde{\rho}(f)^{-1} \circ w \circ \tilde{\rho}(f)
\]

is well-defined, hence a group isomorphism.

Proposition 5.2. The restriction of \( \tilde{\rho}(f) \) to \( X(T) = R(T)(1) \) is an automorphism \( \rho(f) : X(T) \rightarrow X(T) \) of the root diagram \( \Delta \).

Proof. We have to check that \( \rho(f)(X_0(T)) \subset X_0(T) \) and \( \rho(f)(\Phi) \subset \Phi \). The first inclusion is a consequence of property (ii) above and the relation \( X_0(T) = X(T)^W \). For the other inclusion, we consider the set \( \Phi_{\text{min}} \) of "minimal vectors", as defined in Section 4.2. Let \( \alpha \) be a vector in \( \Phi_{\text{min}} \) and set \( \beta = \rho(f)(\alpha) \). Property (ii) above implies that \( \rho(f)^{-1} \circ s_\alpha \circ \rho(f) \) is a reflection in the Weyl group \( W \). Using the "minimality" of \( \alpha \) and Lemma 4.1, one can deduce that \( \beta = \rho(f)(\alpha) \) is also in \( \Phi_{\text{min}} \). As we have
seen before, the roots in \( \Phi \) are obtained from \( \Phi_{\text{min}} \) by stretching some irreducible components of the latter. In the proof of Theorem 4.3, we showed that this stretching process is controlled by \( RO(G) \). Since \( R(f) \) maps \( RO(G) \) into itself, \( \rho(f) \) must send a stretched vector onto another one; that is \( \rho(f)(\Phi) \subset \Phi \). \( \square \)

The next step is to observe that Theorem 2.2 also implies that \( \rho(f) \) is well-defined only in the quotient \( Aut(\Delta)/W \). It follows that our construction induces a group homomorphism

\[
\rho : Aut(BG) \longrightarrow Aut(\Delta)/W, \quad [f] \longmapsto [\rho(f)^{-1}].
\]

Notice that the presence of \( \rho(f)^{-1} \) in this definition is forced by the contravariance of \( K \)-theory. Finally if \( \varphi \) is an element of \( Aut(G, T) \), it is clear that we can take

\[
\tilde{\rho}(B\varphi) = (\varphi^{-1})^* = X(\varphi^{-1}),
\]

that is the map induced (in \( K \)-theory) by the restriction of \( \varphi^{-1} \) to \( T \); thus we have \( \tilde{\rho} \circ \tilde{\beta} = \tilde{\xi} \). Let us summarize our result as

**Proposition 5.3.** There is a homomorphism

\[
\rho : Aut(BG) \longrightarrow Aut(\Delta)/W
\]

such that \( \rho \circ \beta = \tilde{\xi} \).

**Proof of Theorem 5.2.** We want to prove that the map

\[
\tilde{\beta} : Aut(G, T)/(Aut(G, T) \cap \text{Int}(G)) \longrightarrow Aut(BG), \quad [\varphi] \longmapsto [B\varphi]
\]

is bijective. Its injectivity follows from Propositions 5.1 and 5.3. To show the surjectivity, let \([f] \in Aut(BG)\) and set \([\varphi] = \tilde{\xi}^{-1}(\tilde{\eta})[f]\). By construction \( f \) and \( B\varphi \) induces the same map in \( K \)-theory, that is \( B(\varphi^{-1}) \circ f \) induces the identity in \( K \)-theory. A Chern character argument shows that \( B(\varphi^{-1}) \circ f \) induces the identity in rational cohomology. This is where we have to invoke Theorem 4.3 of [14] to conclude that \( B\varphi \) is homotopic to \( f \). \( \square \)

**Acknowledgements**

This paper is a revised version of my Ph.D. Thesis at The University of Neuchatel (Switzerland). I am greatly indebted to my thesis advisor Ueli Suter for his guidance and encouragements. I would like to thank A. Jeanneret, B. Junod, D. Lines, G. Mislin and D. Notbohm for many stimulating discussions. I gratefully acknowledge the financial support of the Swiss National Science Foundation and the hospitality of the M.S.R.I (Berkeley) during the preparation of this paper.
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