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Uniqueness, Existence, and Optimality for Fourth-Order Lipschitz Equations

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1. INTRODUCTION

In this paper we will be concerned with solutions of boundary value problems for the fourth-order differential equation

$$y^{(4)}=f(t, y, y', y'', y'''), \tag{1}$$

where we assume throughout that

(A) f: $(a, b)\times R^4 \to R$ is continuous, and

(B) f satisfies the Lipschitz condition

$$|f(t, y_1, y_2, y_3, y_4)-f(t, z_1, z_2, z_3, z_4)| \leqslant \sum_{i=1}^{4} k_i \,|y_i - z_i|,$$

for each (t, y_1, y_2, y_3, y_4), $(t, z_1, z_2, z_3, z_4) \in (a, b)\times R^4$. In particular, we will characterize optimal length subintervals of (a, b), in terms of the Lipschitz coefficients k_i, $i=1, 2, 3, 4$, on which certain two, three, and four point boundary value problems for (1) have unique solutions. The techniques we employ here involve applications of the Pontryagin Maximum Principle [14, p. 314] in conjunction with uniqueness implies existence results for solutions of boundary value problems for (1). These techniques are motivated by works of Melentsova [15], and Melentsova and Mil'shtein [16, 17], and most notably by the two papers by Jackson [11, 12]. Furthermore, the results contained herein can be considered as extensions of a recent paper by Henderson [6] dealing with boundary value problems for third-order equations.

In relating the results of this paper to previous works, we will formulate the boundary value problems in terms of the nth-order differential equation

$$y^{(n)}=f(t, y, y', ..., y^{(n-1)}). \tag{2}$$

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Using notation introduced by Muldowney [18] and Peterson [22] and also used in [3, 7], we will refer to our problems as right $(m_1;...;m_l)$ focal boundary value problems.

DEFINITIONS. (a) Let $a<t_1\leqslant t_2\leqslant\cdots\leqslant t_n<b$ and let $\tau=(t_1,..., t_n)$. We say that $y(t)$ has *n zeros at* τ provided $y(t_i)=0$, $1\leqslant i\leqslant n$, and $y(t_i)=y'(t_i)=\cdots=y^{(m-1)}(t_i)=0$ if a point t_i occurs m times in τ. A *partition* $(\tau_1;...;\tau_l)$ of the ordered n-tuple τ is obtained by inserting $l-1$ semicolons within the expression for τ. Let $m_i=|\tau_i|$ be the number of components of τ_i, $(\sum_{i=1}^{l} m_i=n)$. (We *will not* allow $m_i=0$ in this paper.) We say that $(\tau_1;...;\tau_l)$ is an *increasing partition* of τ provided, if $t\in\tau_i$ and $s\in\tau_j$, with $i<j$, then either $t<s$, or $t=s$ and $i+m\leqslant j$, where m is the multiplicity of t in τ_i.

(b) We say that (2) is *right* $(m_1;...;m_l)$ *disfocal* on (a, b), $m_1+\cdots+m_l=n$, $m_j\geqslant 1$, provided there do not exist distinct solutions of (2) whose difference, $u(t)$, is such that $u^{(j-1)}(t)$ has m_j zeros at τ_j, $1\leqslant j\leqslant l$, where $(\tau_1;...;\tau_l)$ is an increasing partition of n points in (a, b).

(c) A *right* $(m_1;...;m_l)$ *focal boundary value problem* for (2) on (a, b) is one in which m_j values of the $(j-1)$st derivative of a solution $y(t)$ of (2) are specified at τ_j in the sense that, at each $t\in\tau_j$, $y^{(i)}(t)$ is specified, $j-1\leqslant i\leqslant j+r-2$, where r is the multiplicity of t in τ_j, $|\tau_j|=m_j$, $1\leqslant j\leqslant l$, and $(\tau_1;...;\tau_l)$ is an increasing partition of n points in (a, b).

In view of these definitions, we will be concerned with right $(m_1;...;m_l)$ disfocality and right $(m_1;...;m_l)$ focal boundary value problems for (1). More precisely, we have eight families of problems, the right (4), (3; 1), (2; 2), (2; 1; 1), (1; 3), (1; 2; 1), (1; 1; 2), and (1; 1; 1; 1) focal boundary value problems for (1). Right (4) focal boundary value problems are commonly referred to as conjugate problems, and right (1; 1; 1; 1) focal boundary value problems have been referred to as right focal "point" problems [8, 9]. Furthermore, since Eq. (1) is of fourth-order, the ordered tuple τ will consist of two, three, or four points. For clarity, we may refer to the appropriate problems in terms of right $(m_1;...;m_l)$-two, -three, or -four point boundary value problems for (1).

In the two papers by Jackson [11, 12], he determined optimal length intervals, in terms of the Lipschitz coefficients, on which solutions of right (n) focal boundary value problems (i.e., conjugate), for (2) and right $(1; 1;...; 1)$ focal boundary value problems (i.e., right focal "point"), for (2) have unique solutions. In each case, Jackson applied the Pontryagin Maximum Principle in determining optimal length intervals on which the relative two point boundary value problems for a certain class of nth order linear equations have unique solutions. Then, in each case, applying results such as those in [1, 5, 18, 19, 20, 21, 23, 26, 27] which relate existence of solutions of multipoint problems to existence of solutions of two point problems for linear equations, Jackson argued that solutions of the multipoint boundary value problems for (2) are unique, when they exist, on subintervals of length less than this optimal length for the two point problems. Then for existence of solutions of the conjugate boundary value problems or the right focal point boundary value problems, uniqueness implies existence results due to Hartman [4] and Klaasen [13] or Henderson [9] were applied, respectively. In [6], we showed that the techniques used by Jackson could be adapted to right $(m_1;...;m_l)$ focal boundary value problems for third-order ordinary differential equations.

For each family of the right $(m_1;...;m_l)$ focal boundary value problems for (1) (i.e., the right (4), right (3; 1), etc.), we extend these techniques further. The arguments for each of these families, while not identical, are similar. For that reason and because of the tediousness of some of the proofs, we present in this paper a complete sequence of theorems characterizing the optimal length subintervals of (a, b), in terms of the k_i, $i = 1, 2, 3, 4$, on which there exist unique solutions of only the right (2; 2) focal boundary value problems for (1). The discussion for right (2; 2) focal boundary value problems appears in Section 2 and is typical of that which would be used for the other families.

In particular, in Section 2A, we show that right (2; 2)-two point disfocality implies right (2; 2)-three and -four point disfocality for linear equations. This then provides the framework by which we apply in Section 2B the Pontryagin Maximum Principle in determining optimal length intervals on which solutions of right (2; 2) focal boundary value problems (-two, -three, or -four point) for (1) are unique, when they exist. In Section 2C, we sketch the proof of uniqueness implies existence for solutions of right (2; 2) focal boundary value problems for (1), and hence obtain the desired result.

In Section 3, we state an analogue of the "optimal, uniqueness, and existence" result given in Section 2C for the other families of right $(m_1;...;m_l)$ focal boundary value problems for (1). Section 3 is concluded with a remark concerning the limited use of the Pontryagin Maximum Principle in this type of optimal interval analysis for right $(m_1;...;m_l)$ focal boundary value problems for nth order Lipschitz equations (2), when $n > 4$.

With Section 4, we conclude the paper by including some numerical results in the case when $k_i = 1$, $i = 1, 2, 3, 4$. We determine optimal length intervals on which there exist unique solutions for each family of right $(m_1;...;m_l)$ focal boundary value problems for these Lipschitz conditions. The numerical results were obtained on the Cray-1 at the Institute for Defense Analysis/Communication Research Division using Richardson extrapolation to solve the initial value problems associated with finding the optimal intervals.

2. Right (2; 2) Focal Boundary Value Problems

In this section we will determine optimal length subintervals of (a, b) in terms of the Lipschitz coefficients k_i, $i = 1, 2, 3, 4$, on which there exist unique solutions of the right (2; 2) focal boundary value problems for (1). To be precise, we will be concerned with solutions of (1) satisfying the following right (2; 2)-two point, -three point, and -four point focal boundary conditions:

$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_3) = y_3, \quad y''(t_4) = y_4, \quad a < t_1 < t_2 = t_3 = t_4 < b, \tag{3}$$

$$y(t_1) = y_1, \quad y'(t_2) = y_2, \quad y'(t_3) = y_3, \quad y''(t_4) = y_4, \quad a < t_1 = t_2 < t_3 = t_4 < b, \tag{4}$$

$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_3) = y_3, \quad y''(t_4) = y_4, \quad a < t_1 < t_2 < t_3 = t_4 < b, \tag{5}$$

$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_3) = y_3, \quad y'(t_4) = y_4, \quad a < t_1 < t_2 = t_3 < t_4 < b, \tag{6}$$

$$y(t_1) = y_1, \quad y'(t_2) = y_2, \quad y'(t_3) = y_3, \quad y'(t_4) = y_4, \quad a < t_1 = t_2 < t_3 < t_4 < b, \tag{7}$$

$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_3) = y_3, \quad y'(t_4) = y_4, \quad a < t_1 < t_2 < t_3 < t_4 < b. \tag{8}$$

A. *Right* (2; 2)-*Two-Point Disfocality and Linear Equations*

In this subsection, we will show that right (2; 2)-two point disfocality implies right (2; 2) disfocality for the linear equation

$$y^{(4)} + \sum_{i=0}^{3} a_i(t)\, y^{(i)} = 0, \tag{9}$$

where we assume that the $a_i(t)$ are bounded Lebesgue measurable functions on (a, b). By a solution $y(t)$ of (9), we mean, in the usual sense, that $y(t) \in C^{(3)}(a, b)$, $y'''(t)$ is absolutely continuous on (a, b), and $y(t)$ satisfies (9) for almost all $t \in (a, b)$. Other studies of this type devoted to relationships between two point and multipoint problems for linear equations can be found in [1, 5, 18, 19, 20, 21, 23, 26, 27], to name a few.

THEOREM 1. *Assume that the linear equation* (9) *is right* (2; 2)-*two point disfocal on* (a, b). *Then* (9) *is right* (2; 2)-*three point and right* (2; 2)-*four point disfocal on* (a, b); *in particular*, (9) *is right* (2; 2) *disfocal on* (a, b).

Proof. Let $u_i(t)$, $i=1, 2, 3, 4$, be a fundamental set of solutions of (9). For the purpose of contradiction, assume the conclusion of the theorem to be false. Then there exists a nontrivial solution of (9) satisfying boundary conditions (5), (6), (7), or (8), with $y_i=0$, $i=1, 2, 3, 4$. Any such nontrivial solution for (5), (6), or (7) is essentially unique by the right (2; 2)-two point disfocality of (9).

(i) If there is a nontrivial solution $y(t)$ of (9) satisfying (7) with $y_i=0$, $i=1, 2, 3, 4$ (that is, $y(t_1)=y'(t_1)=y'(t_2)=y'(t_3)=0$, for some $a<t_1<t_2<t_3<b$), then define $\tau=\inf\{t>t_1 \mid$ there exists a nontrivial solution $z(t)$ of (9) satisfying $z(t_1)=z'(t_1)=z'(t_2)=z'(t)=0$, $t_1<t_2<t\}$. It follows that there exists a nontrivial solution $z(t)$ of (9) satisfying

$$z(t_1)=z'(t_1)=z'(t_2)=z'(\tau)=0,$$

$t_1<t_2<\tau$. Consequently,

$$\det X(u_1, u_2, u_3, u_4)(t_1, t_2, \tau)=0,$$

where X is the 4×4 matrix whose ith column, $i=1, 2, 3, 4$, is given by

$$[u_i(t_1), u_i'(t_1), u_i'(t_2), u_i'(\tau)]^{\mathrm{T}}.$$

We argue now that the minor of each entry in the third row of X

$$[u_1'(t_2), u_2'(t_2), u_3'(t_2), u_4'(t_2)] \tag{$*$}$$

is zero. If at least one of the minors is not zero, then by the essential uniqueness of the solution $z(t)$, we have, for some $c\neq 0$,

$$z(t)=c \det Z(u_1, u_2, u_3, u_4)(t_1, t, \tau),$$

where Z is the 4×4 matrix whose ith column, $i=1, 2, 3, 4$, is given by

$$[u_i(t_1), u_i'(t_1), u_i(t), u_i'(\tau)]^{\mathrm{T}}.$$

We remark here that by the right (2; 2)-two point disfocality of (9), $z''(t_2)\neq 0$.

Now define

$$Y(t, s) = c \det W(u_1, u_3, u_3, u_4)(t_1, t, s),$$

where W is the 4×4 matrix whose ith column, $i = 1, 2, 3, 4$, is given by

$$[u_i(t_1), u_i'(t_1), u_i'(t), u_i'(s)]^{\mathrm{T}}.$$

Then $Y(t_2, \tau) = z'(t_2) = 0$, whereas $(\partial Y/\partial t)\ (t_2, \tau) = z''(t_2) \neq 0$. It follows from the Implicit Function Theorem that there exist neighborhoods $O(\tau)$, $V(t_2)$, and a continuous mapping $T: O(\tau) \to V(t_2)$, such that $T(\tau) = t_2$ and $Y(T(s), s) = 0$, for all $s \in O(\tau)$. In particular, for $s < \tau$, but sufficiently near, there exists a nontrivial solution $w(t)$ of (9) satisfying

$$w(t_1) = w'(t_1) = w'(T(s)) = w'(s) = 0,$$

where $t_1 < T(s) < s < \tau$; this is a contradiction to the extremality of τ.

Therefore, the minors of the row $(*)$ are all zero. That being the case, if we replace the row $(*)$ in the matrix X by the row

$$[u_1''(\tau), u_2''(\tau), u_3''(\tau), u_4''(\tau)],$$

then

$$\det U(u_1, u_2, u_3, u_4)(t_1, \tau) = 0,$$

where U is the 4×4 matrix whose ith column, $i = 1, 2, 3, 4$, is given by

$$[u_i(t_1), u_i'(t_1), u_i''(\tau), u_i'(\tau)];$$

that is, there exists a nontrivial solution $v(t)$ of (9) satisfying

$$v(t_1) = v'(t_1) = v'(\tau) = v''(\tau) = 0,$$

which contradicts the right (2; 2)-two point disfocality of (9).

Therefore, the only solution of (9) satisfying (7), with $y_i = 0$, $i = 1, 2, 3, 4$, is $y(t) \equiv 0$.

(ii) For our second case, assume there is a nontrivial solution $y(t)$ of (9), (6), with $y_i = 0$, $i = 1, 2, 3, 4$ (that is, $y(t_1) = y(t_2) = y'(t_2) = y'(t_3) = 0$, for some $a < t_1 < t_2 < t_3 < b$). We proceed through steps very similar to those in (i). In this case, we define $\tau_2 = \inf\{t > t_1 \mid$ there exists a nontrivial solution $z(t)$ of (9) satisfying $z(t_1) = z(t_2) = z'(t_2) = z'(t) = 0,\ t_1 < t_2 < t\}$. Again, it follows that there is a nontrivial solution $z(t)$ of (9) satisfying

$$z(t_1) = z(t_2) = z'(t_2) = z'(\tau_2) = 0,$$

$t_1 < t_2 < \tau_2$, and so

$$\det X_2(u_1, u_2, u_3, u_4)(t_1, t_2, \tau_2) = 0,$$

where X_2 is the 4×4 matrix whose ith column, $i = 1, 2, 3, 4$, is given by

$$[u_i(t_1), u_i(t_2), u_i'(t_2), u_i'(\tau_2)]^{\mathrm{T}}.$$

Arguing as in case (i) and applying the Implicit Function Theorem again to an appropriate determinant, it can be shown that the minor of each entry in the third row of X_2,

$$[u_1'(t_2), u_2'(t_2), u_3'(t_2), u_4'(t_2)],$$

is zero. We can then replace that row by

$$[u_1'(t_1), u_2'(t_1), u_3'(t_1), u_4'(t_1)],$$

so that

$$\det U_2(u_1, u_2, u_3, u_4)(t_1, t_2, \tau_2) = 0,$$

where U_2 is the 4×4 matrix whose ith column, $i = 1, 2, 3, 4$, is given by

$$[u_i(t_1), u_i(t_2), u_i'(t_1), u_i'(\tau_2)]^{\mathrm{T}};$$

that is, there exists a nontrivial solution $v(t)$ of (9) satisfying

$$v(t_1) = v'(t_1) = v(t_2) = v'(\tau_2) = 0,$$

$t_1 < t_2 < \tau_2$. Applying Rolle's Theorem, there exists $t_1 < t_3 < t_2$ such that

$$v(t_1) = v'(t_1) = v'(t_3) = v'(\tau_2) = 0.$$

This is a contradiction to what was proven in case (i). Hence, the only solution of (9) satisfying (6), with $y_i = 0$, $i = 1, 2, 3, 4$, is $y(t) \equiv 0$.

(iii) Now, let us assume there is a nontrivial solution $y(t)$ of (9) satisfying (5), with $y_i = 0$, $i = 1, 2, 3, 4$ (that is, $y(t_1) = y(t_2) = y'(t_3) = y''(t_3) = 0$, some $a < t_1 < t_2 < t_3 < b$). Similar to the previous cases, if we define $\tau_3 = \inf\{t > t_1 \mid$ there exists a nontrivial solution $z(t)$ of (9) satisfying $z(t_1) = z(t_2) = z'(t) = z''(t) = 0,\ t_1 < t_2 < t\}$, then there is a nontrivial solution $z(t)$ of (9) satisfying

$$z(t_1) = z(t_2) = z'(\tau_3) = z''(\tau_3) = 0,$$

$t_1 < t_2 < \tau_3$, and so

$$\det X_3(u_1, u_2, u_3, u_4)(t_1, t_2, \tau_3) = 0,$$

where X_3 is the 4×4 matrix whose ith column, $i = 1, 2, 3, 4$, is given by

$$[u_i(t_1),\ u_i(t_2),\ u_i'(\tau_3),\ u_i''(\tau_3)]^{\mathrm{T}}.$$

Another analogous argument shows that the minor of each entry in the second row of X_3,

$$[u_1(t_2),\ u_2(t_2),\ u_3(t_2),\ u_4(t_2)]$$

is zero. In this case, if $U_3(u_1, u_2, u_3, u_4)(t_1, \tau_3)$ is the 4×4 matrix formed by replacing the second row of X_3 by, say

$$[u_1(\tau_3),\ u_2(\tau_3),\ u_3(\tau_3),\ u_4(\tau_3)],$$

then

$$\det U_3(u_1, u_2, u_3, u_4)(t_1, \tau_3) = 0,$$

and so there is a nontrivial solution of (9) satisfying (3), where $y_i = 0$, $i = 1, 2, 3, 4$; a contradiction to the right $(2; 2)$-two point disfocality of (9). Thus, the trivial solution is the only solution of (9) satisfying (5), with $y_i = 0$, $i = 1, 2, 3, 4$.

As a consequence of cases (i)–(iii), the linear equation (9) is right $(2; 2)$-three point disfocal on (a, b). To conclude the proof, we have one further case.

(iv) For this case, assume that (9) is not right $(2; 2)$-four point disfocal. Thus, there exists a nontrivial solution $y(t)$ of (9) satisfying $y(t_1) = y(t_2) = y'(t_3) = y'(t_4) = 0$, for some $a < t_1 < t_2 < t_3 < t_4 < b$. By the right $(2; 2)$-three point disfocality, any such $y(t)$ is essentially unique. Defining $\tau_4 = \inf\{t > t_1 \mid$ there exists a nontrivial solution $z(t)$ of (9) satisfying $z(t_1) = z(t_2) = z'(t_3) = z'(t) = 0,\ t_1 < t_2 < t_3 < t\}$, it follows that there is a nontrivial solution $z(t)$ of (9) such that

$$z(t_1) = z(t_2) = z'(t_3) = z'(\tau_4) = 0,$$

$t_1 < t_2 < t_3 < \tau_4$. Thus

$$\det X_4(u_1, u_2, u_3, u_4)(t_1, t_2, t_3, \tau_4) = 0,$$

where this time X_4 is the 4×4 matrix whose ith column, $i = 1, 2, 3, 4$, is given by

$$[u_i(t_1),\ u_i(t_2),\ u_i'(t_3),\ u_i'(\tau_4)]^{\mathrm{T}}.$$

This time, it can be shown that the minor of each entry in the third row of X_4 is zero. Replacing this third row of X_4 by

$$[u_1'(t_2), u_2'(t_2), u_3'(t_2), u_4'(t_2)],$$

and denoting the new 4×4 matrix by $U_4(u_1, u_2, u_3, u_4)(t_1, t_2, \tau_4)$, we have that

$$\det U_4(u_1, u_2, u_3, u_4)(t_1, t_2, \tau_4) = 0.$$

This implies that (9) has a non-trivial solution $v(t)$ satisfying

$$v(t_1) = v(t_2) = v'(t_2) = v'(\tau_4) = 0,$$

which contradicts case (ii) above.

Therefore, (9) is right (2; 2)-four point disfocal on (a, b), and the proof is complete.

Remarks. (a) The method of proof in Theorem 1 was motivated by arguments used by Muldowney [18, 19].

(b) It can be shown, in a manner similar to above, that if (9) is right (2; 2)-two point disfocal on (a, b), then the trivial solution is the only solution of (9) satisfying $y(t_1) = y'(t_1) = y'(t_2) = y''(t_3) = 0$, $a < t_1 < t_2 < t_3 < b$.

(c) For each of the other families of right $(m_1; \ldots; m_l)$ focal boundary value problems for (9), in analogy, it can be proved that right $(m_1; \ldots; m_l)$-two point disfocality implies right $(m_1; \ldots; m_l)$ disfocality.

B. *Optimal Length Intervals for Right* (2; 2) *Disfocality*

In this subsection, we will determine optimal length subintervals of (a, b) in terms of the Lipschitz constants k_i, $i = 1, 2, 3, 4$, on which (1) is right (2; 2) disfocal. In this analysis, we take the direction introduced in the papers by Melentsova [15] and Melentsova and Mil'shtein [16, 17] and then most notably applied by Jackson [11, 12], in that we apply the Pontryagin Maximum Principle in determining the optimal length interval on which a related family of linear equations is right (2; 2)-two point disfocal. It will follow from Theorem 1 that this family of linear equations is right (2; 2) disfocal on such a subinterval. More importantly, it will be the case that (1) is also right (2; 2) disfocal on such a subinterval.

In making this application of the Pontryagin Maximum Principle, define the control region

$U = \{(u_1(t),\ u_2(t),\ u_3(t),\ u_4(t)) \mid u_i(t)$ is Lebesgue measurable and $|u_i(t)| \leq k_i$, $i = 1, 2, 3, 4$, on $(a, b)\}$.

We will be concerned with solutions of right (2; 2) focal boundary value problems associated with the linear equations

$$x^{(4)} = \sum_{i=1}^{4} u_i(t)\, x^{(i-1)}, \tag{10}$$

where $u = (u_1(t), u_2(t), u_3(t), u_4(t)) \in U$.

Now, if (1) is not right (2; 2) disfocal on (a, b), then there exist distinct solutions $y(t)$ and $z(t)$ of (1) whose difference $x(t) \equiv y(t) - z(t)$ satisfies one of the conditions (3), (4), (5), (6), (7), or (8), with $y_i = 0$, $i = 1, 2, 3, 4$. If

$$h_1(t) = f(t, y(t), y'(t), y''(t), y'''(t)) - f(t, z(t), y'(t), y''(t), y'''(t)),$$

$$h_2(t) = f(t, z(t), y'(t), y''(t), y'''(t)) - f(t, z(t), z'(t), y''(t), y'''(t)),$$

$$h_3(t) = f(t, z(t), z'(t), y''(t), y'''(t)) - f(t, z(t), z'(t), z''(t), y'''(t)),$$

$$h_4(t) = f(t, z(t), z'(t), z''(t), y'''(t)) - f(t, z(t), z'(t), z''(t), z'''(t)),$$

and if $\tilde{u}_i(t)$, $i = 1, 2, 3, 4$, is defined by

$$\tilde{u}_i(t) = \begin{cases} \dfrac{h_i(t)}{y^{(i-1)}(t) - z^{(i-1)}(t)}, & \text{for } \; y^{(i-1)}(t) \neq z^{(i-1)}(t), \\ 0, & \text{for } \; y^{(i-1)}(t) = z^{(i-1)}(t), \end{cases}$$

then $\tilde{u}_i(t)$ is measurable and $|\tilde{u}_i(t)| \leqslant k_i$ on (a, b). Moreover, $x(t) \equiv y(t) - z(t)$ is a nontrivial solution of the linear equation (10), for $\tilde{u} = (\tilde{u}_1(t), \tilde{u}_2(t), \tilde{u}_3(t), \tilde{u}_4(t))$, satisfying one of the conditions (3), (4), (5), (6), (7), or (8) with $y_i = 0$, $i = 1, 2, 3, 4$. It follows from Theorem 1 that linear equation (10), for $\tilde{u}$, is not right (2; 2)-two point disfocal on (a, b); that is, there exists a nontrivial solution of (10) for $\tilde{u}$, satisfying either (3) or (4), with $y_i = 0$, $i = 1, 2, 3, 4$, for some two boundary points $a < \tau_1 < \tau_2 < b$.

Now, since there is a nontrivial solution of (10), (3) (or (10), (4)), with $y_i = 0$, $i = 1, 2, 3, 4$, for some $\tau_1 < \tau_2$ and $\tilde{u} \in U$, it follows that there is a boundary value problem in the collection which has a nontrivial time optimal solution (see Gamkrelidze [2, p. 147] or Lee and Markus [14, p. 259]); that is, there exists at least one non-trivial $u^* \in U$ and $\tau_1 \leqslant c < d \leqslant \tau_2$ such that

$$x^{(4)} = \sum_{i=1}^{4} u_i^*(t)\, x^{(i-1)}$$

$$x(c) = x(d) = x'(d) = x''(d) = 0,$$

$$(\text{or } x(c) = x'(c) = x'(d) = x''(d) = 0),$$

has a nontrivial solution $x(t)$, and $d-c$ is a minimum over all such solutions. For this time optimal solution, if $r(t)=(x(t), x'(t), x''(t), x'''(t))^{\mathrm{T}}$, then $r(t)$ is a solution of the first-order system

$$r'=A[u^*(t)]\,r.$$

By the Pontryagin Maximum Principle, the adjoint system

$$\psi'=-A^T[u^*(t)]\,\psi$$

has a nontrivial solution $\psi(t)=(\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t))^{\mathrm{T}}$ such that

(i) $\sum_{i=1}^{4} x^{(i)}(t)\,\psi_i(t) = \langle r'(t), \psi(t)\rangle = \mathrm{Max}_{u\in U}\ \{\langle A[u(t)]\ r(t), \psi(t)\rangle\}$, for almost all $t\in[c,d]$;

(ii) $\langle r'(t), \psi(t)\rangle$ is a nonnegative constant, for almost all $t\in[c,d]$; and

(iii) $\psi(t)$ satisfies the boundary conditions complementary to $x(t)$; that is, $\psi_2(c)=\psi_3(c)=\psi_4(c)=\psi_4(d)=0$, (or $\psi_3(c)=\psi_4(c)=\psi_1(d)=\psi_4(d)=0$).

The maximum condition in (i) can be rewritten as

$$\psi_4(t)\sum_{i=1}^{4} u_i^*(t)\,x^{(i-1)}(t)=\mathop{\mathrm{Max}}_{u\in U}\left\{\psi_4(t)\sum_{i=1}^{4} u_i(t)\,x^{(i-1)}(t)\right\}, \qquad (11)$$

for almost all $t\in[c,d]$, from whence it follows that, if $\psi_4(t)$ has no zeros on (c,d) and if $x(t)>0$ on (c,d), then (11) can be used to determine an optimal control $u^*(t)$, for almost all $t\in[c,d]$.

If $x(t)>0$ and $\psi_4(t)<0$ on (c,d), then the time optimal solution $x(t)$ is a solution of

$$x^{(4)}=-\left[k_1x+\sum_{i=2}^{4} k_i|x^{(i-1)}|\right] \qquad (12)$$

on $[c,d]$, whereas if $x(t)>0$ and $\psi_4(t)>0$ on (c,d), then the time optimal solution $x(t)$ is a solution of

$$x^{(4)}=k_1x+\sum_{i=2}^{4} k_i\,|x^{(i-1)}| \qquad (13)$$

on $[c,d]$.

Let us recall at this point that we assumed above that (1) is not right (2; 2) disfocal. As a consequence of the above discussion, if we can satisfy the appropriate sign conditions on the optimal solution $x(t)$ and the component $\psi_4(t)$ and thus determine intervals on which (12) and (13) are right (2; 2)-two point disfocal, then (1) will be right (2; 2) disfocal on such intervals.

Before showing that $x(t)$ and $\psi_4(t)$ indeed satisfy these sign conditions, the following remark concerning converse statements and the adjoint system will play a major role in subsequent arguments.

Remark. If $u \in U$ is such that the boundary value problem (10), (3) (or (10), (4)), with $y_i = 0$, $i = 1, 2, 3, 4$, some $\tau_1 < \tau_2$, has a nontrivial solution, then

$$\psi' = -A^{\mathrm{T}}[u(t)]\,\psi, \tag{14}$$

$$\psi_2(\tau_1) = \psi_3(\tau_1) = \psi_4(\tau_1) = \psi_4(\tau_2) = 0, \tag{15}$$

$$(\text{or } \psi_3(\tau_1) = \psi_4(\tau_1) = \psi_1(\tau_2) = \psi_4(\tau_2) = 0), \tag{16}$$

also has a nontrivial solution; thus the converse is also true. Hence the Pontryagin Maximum Principle associates with a time optimal solution of (10), (3), (or (10), (4)), with $y_i = 0$, $i = 1, 2, 3, 4$, a time optimal solution of (14), (15), (or (14), (16)), and conversely.

THEOREM 2. *If there is a vector $u \in U$ such that the corresponding linear equation* (10) *has a nontrivial solution satisfying*

$$y(t_1) = y(t_2) = y'(t_2) = y''(t_2) = 0, \; a < t_1 < t_2 < b,$$

and if $x(t)$ is a time optimal solution with

$$x(c) = x(d) = x'(d) = x''(d) = 0,$$

and with $d - c$ a minimum, then $x(t)$ is a solution of (12) *on* $[c, d]$.

Proof. By the time optimality, $x(t) \neq 0$ on (c, d), and so we may assume without loss of generality that $x(t) > 0$ on (c, d).

If $\psi(t)$ is a nontrivial optimal solution of the adjoint system associated with $x(t)$ by the Pontryagin Maximum Principle, then

$$\psi_2(c) = \psi_3(c) = \psi_4(c) = \psi_4(d) = 0.$$

By its own time optimality, $\psi_4(t) \neq 0$ on (c, d). Hence $x(t)$ is a solution of (12) or (13) on $[c, d]$. Thus, $x^{(4)}(t)$ is of constant sign on (c, d), so that $x'''(t)$ is strictly monotone on $[c, d]$, and then the boundary conditions satisfied by $x(t)$ imply that $x^{(4)}(t) < 0$ on (c, d). Therefore, $x(t)$ is a solution of (12) on $[c, d]$.

THEOREM 3. *Assume that for all vectors $u \in U$, the corresponding linear equation* (10) *has only the trivial solution satisfying*

$$x(t_1) = x(t_2) = x'(t_2) = x''(t_2) = 0, \; a < t_1 < t_2 < b.$$

If there is a control vector $u \in U$ such that the corresponding linear equation (10) *has a nontrivial solution satisfying*

$$y(t_1)=y'(t_1)=y'(t_2)=y''(t_2)=0,\ a<t_1<t_2<b,$$

and if $x(t)$ is a time optimal solution with

$$x(c)=x'(c)=x'(d)=x''(d)=0,$$

and with $d-c$ a minimum, then $x(t)$ is a solution of (13) *on* $[c, d]$.

Proof. We begin by observing that from the hypotheses and the minimality of $d-c$, equation (10) is right (2; 2)-two point disfocal (hence, right (2; 2) disfocal by Theorem 1), on any proper subinterval of $[c, d]$. Remark (b) after the proof of Theorem 1 implies that $x'(t)\neq 0$ on (c, d). (For if $x'(t_0)=0$, some $t_0 \in (c, d)$, then there exists $t_0<t_1<d$ such that, $x(c)=x'(c)=x'(t_0)=x''(t_1)=0$; a contradiction to Remark (b).) Thus, we may assume $x'(t)>0$ on (c, d), so that $x(t)>0$ on $(c, d]$.

Now if $\psi(t)$ is a nontrivial solution of the adjoint system associated with $x(t)$ by the Pontryagin Maximum Principle, then

$$\psi_3(c)=\psi_4(c)=\psi_1(d)=\psi_4(d)=0,$$

and it's also the case that $\psi_2(c)\neq 0$.

Our goal is to show that $\psi_4(t)\neq 0$ on (c, d). In that direction, let $y(t)=(y_1(t), y_2(t), y_3(t), y_4(t))^{\mathrm{T}}$, where $y_1(t)=\psi_4(t)$, $y_2(t)=\psi_3(t)$, $y_3(t)=\psi_2(t)$, $y_4(t)=\psi_1(t)$. Then $y(t)$ is a solution of

$$y' = B[u^*(t)]\,y, \tag{17}$$
$$y_1(c)=y_2(c)=y_1(d)=y_4(d)=0,$$

where

$$B[u(t)]=\begin{bmatrix} -u_4(t) & -1 & 0 & 0\\ -u_3(t) & 0 & -1 & 0\\ -u_2(t) & 0 & 0 & -1\\ -u_1(t) & 0 & 0 & 0\end{bmatrix}.$$

It suffices now to show that $y_1(t)=\psi_4(t)\neq 0$ on (c, d).

For $j=1, 2, 3, 4$, let

$$y^j(t)=(y_1^j(t), y_2^j(t), y_3^j(t), y_4^j(t))^{\mathrm{T}}$$

denote the solution of the initial value problem for (17) satisfying

$$y_i^j(c)=\delta_{ij}, \qquad i=1, 2, 3, 4.$$

Hence,

$$y(t) = C_3\, y^3(t) + C_4\, y^4(t),$$

for some C_3, C_4, and since $y_3(c) = \psi_2(c) \neq 0$, we have $C_3 \neq 0$.

Our next claim is that $y_1^4(t) \neq 0$ on (c, b). Assume to the contrary that there exists $\tau \in (c, b)$ such that $y_1^4(\tau) = 0$. Now $y^4(t)$ is a solution of (17), and so the adjoint system $\psi' = -A^{\mathrm{T}}[u^*(t)]\, \psi$ has a nontrivial solution $\eta(t)$ satisfying

$$\eta_4(c) = \eta_3(c) = \eta_2(c) = \eta_4(\tau) = 0.$$

Then, there is an optimal such solution to the adjoint equation for some $u^{**} \in U$, and thus by the Pontryagin Maximum Principle, there is a nontrivial optimal solution $v(t)$ of (10), for $u^{**} \in U$ satisfying

$$v(\tau_1) = v(\tau_2) = v'(\tau_2) = v''(\tau_2) = 0$$

where $c \leqslant \tau_1 < \tau_2 \leqslant \tau$; a contradiction to the hypotheses of the theorem. Therefore, $y_1^4(t) \neq 0$ on (c, b).

For the final part of the proof, assume $y_1(t_0) = 0$, for some $t_0 \in (c, d)$. Since

$$-W(y^4(t), y(t)) = [y_1^4(t)]^2\, (y_1(t)/y_1^4(t))',$$

(where $W(\cdot, \cdot)$ denotes the Wronskian), and since $y_1(d) = 0$, $y_1^4(d) \neq 0$, it follows from Rolle's Theorem that there exists $t_0 < t_1 < d$, such that $W(y^4(t_1), y(t_1)) = 0$. Now $W(y^4(t_1), y(t_1)) = C_3\, W(y^4(t_1), y^3(t_1))$, and since $C_3 \neq 0$, $W(y^4(t_1), y^3(t_1)) = 0$. In other words, there exist constants r_3, r_4 such that the solution

$$w(t) = r_3\, y^3(t) + r_4\, y^4(t)$$

of (17) satisfies $w_1(c) = w_2(c) = w_1(t_1) = w_2(t_1) = 0$. This in turn implies that the adjoint system has a solution $\beta(t)$ satisfying $\beta_3(c) = \beta_4(c) = \beta_3(t_1) = \beta_4(t_1) = 0$, $c < t_1 < d$. Finally, in turn, we have a nontrivial optimal solution $\gamma(t)$ of (10), for some $u^{**} \in U$, satisfying

$$\gamma(\tau_1) = \gamma'(\tau_1) = \gamma(\tau_2) = \gamma'(\tau_2) = 0, \qquad c \leqslant \tau_1 < \tau_2 \leqslant t_1 < d.$$

By Rolle's Theorem, there is $\tau_1 < \tau_3 < \tau_2$ such that

$$\gamma(\tau_1) = \gamma'(\tau_1) = \gamma'(\tau_3) = \gamma'(\tau_2) = 0,$$

where $\tau_2 - \tau_1 < d - c$. This contradicts the extremality of $d - c$, in that (10) is right (2; 2) disfocal, for all $u \in U$, by Theorem 1 on any proper subinterval of $[c, d]$.

Therefore, $y_1(t) = \psi_4(t) \neq 0$, and it follows that $x(t)$ is a solution of (12) or (13) on $[c, d]$. Again, from the constancy in sign of $x^{(4)}(t)$ on (c, d) and from the boundary conditions $x(c) = x'(c) = x'(d) = x''(d) = 0$, we have $x^{(4)}(t) > 0$ on (c, d), so that $x(t)$ is a solution of (13) on $[c, d]$, and the proof is complete.

In light of Theorems 2 and 3 and the discussion preceding them in this subsection, we can now state and prove a theorem establishing optimal length intervals in terms of the k_i, $i = 1, 2, 3, 4$, on which (1) is right (2; 2) disfocal.

THEOREM 4. *Let $h = h(k_1, k_2, k_3, k_4) = \min\{h_1, h_2\}$, where $h_1 > 0$ is the first positive number such that there exists a solution $x(t)$ of* (12) *satisfying*

$$x(0) = x(h_1) = x'(h_1) = x''(h_1) = 0,$$

with $x(t) > 0$ on $(0, h_1)$, or $h_1 = +\infty$ if no such solution exists, and where $h_2 > 0$ is the first positive number such that there exists a solution $y(t)$ of (13) *satisfying*

$$y(0) = y'(0) = y'(h_2) = y''(h_2) = 0,$$

with $y(t) > 0$ on $(0, h_2]$, or $h_2 = +\infty$ if no such solution exists. Then each of the boundary value problems for (1) *satisfying* (3), (4), (5), (6), (7), *or* (8) *has at most one solution, provided $t_4 - t_1 < h$ (i.e.* (1) *is right* (2; 2) *disfocal on subintervals of length less than h). Moreover, this result is best possible for the class of all fourth order differential equations satisfying the Lipschitz condition* (B).

Proof. We note first that since equations (12) and (13) are autonomous, in applying Theorems 2 and 3, rather than specifying boundary conditions at $a < c < d < b$, it suffices to consider conditions at 0, h_1, and h_2.

If there are distinct solutions of (1) satisfying one of (3), (4), (5), (6), (7), or (8) on some subinterval of (a, b) of length less than h, then from the first part of this subsection, Eq. (10), for some $u \in U$, has a nontrivial solution satisfying (3) or (4), with $y_i = 0$, $i = 1, 2, 3, 4$, on the same subinterval. By Theorems 2 and 3, this is a contradiction to the definition of h. Thus (1) is right (2; 2) disfocal on any subinterval of length less than h.

That this result is best possible follows from the fact that both Eqs. (12) and (13) satisfy the Lipschitz condition (B), and either $x(t)$ or $y(t)$ in the statement of the theorem is a nontrivial solution of (3) or (4), respectively, with $y_i = 0$, $i = 1, 2, 3, 4$, on $[0, h]$. In either case, each boundary value problem also has the zero solution.

C. *Uniqueness Implies Existence for Right* (2; 2) *Focal Boundary Value Problems*

Analogous to uniqueness implies existence results proven by Hartman [4], Klaasen [13], and Henderson [9] for boundary value problems for ordinary differential equations, we show in this subsection that the right (2; 2) disfocality of (1) on an interval (a, b) implies that each right (2; 2)-two point, -three point, or -four point boundary value problem for (1) has a solution on (a, b). The proof utilizes standard shooting methods such as those used by Peterson [24, 25] or Henderson [9, 10], and we will give only a sketch of the proof.

THEOREM 5. *Assume that* (1) *is right* (2; 2) *disfocal on* (a, b). *Then boundary value problems for* (1) *satisfying* (3), (4), (5), (6), (7), *or* (8) *all have unique solutions on* (a, b).

Proof. We remark first, from the right (2; 2) disfocality of (1) on (a, b) and Rolle's Theorem, that (1) is disconjugate on (a, b). That being the case, it follows that conjugate boundary value problems for (1) have unique solutions on (a, b); see [4, 13]. Thus the boundary value problem (1), (3) has a unique solution on (a, b), since it is of the conjugate type.

For the other right (2; 2) focal boundary value problems for (1), let $y_i \in R$, $i = 1, 2, 3, 4$, be given, and then we use the shooting method successively.

(1), (4): For this problem, let $a < t_1 < t_2 < b$ and let $y(t)$ be the solution of (1) satisfying conditions of type (3),

$$y(t_1) = y_1,\ y(t_2) = 0,\ y'(t_2) = y_3,\ y''(t_2) = y_4.$$

Define $S_1 \equiv \{z'(t_1) \mid z(t) \text{ is a solution of (1) and } z(t_1) = y(t_1),\ z'(t_2) = y'(t_2),\ z''(t_2) = y''(t_2)\}$. Using the right (2; 2) disfocality of (1) and the continuous dependence of solutions of (1), (4) on boundary conditions, it can be shown that $S_1 = R$; see [9]. Thus for $y_2 \in S_1$, the corresponding solution $z(t)$ of (1) satisfies (4).

(1), (5): For this three point problem, let $a < t_1 < t_2 < t_3 < b$ and let $y(t)$ be the solution of (1) satisfying conditions of type (4),

$$y(t_2) = y_2,\ y'(t_2) = 0,\ y'(t_3) = y_3,\ y''(t_3) = y_4.$$

Defining $S_2 \equiv \{z(t_1) \mid z(t) \text{ is a solution of (1) and } z(t_2) = y(t_2),\ z'(t_3) = y'(t_3),\ z''(t_3) = y''(t_3)\}$, it can be shown that $S_2 = R$. Choosing $y_1 \in S_2$, the corresponding solution $z(t)$ satisfies (1), (5).

(1), (6): Continuing the pattern, let $a < t_1 < t_2 < t_3 < b$ and let $y(t)$ be the solution of (1) satisfying conditions of type (5),

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y'(t_3) = y_4,\ y''(t_3) = 0.$$

In this case, we can argue that $S_3 \equiv \{z'(t_2) | z(t)$ is a solution of (1) and $z(t_1) = y(t_1)$, $z(t_2) = y(t_2)$, $z'(t_3) = y'(t_3)\}$ consists of the entire real line. Again, choosing $y_3 \in S_3$, the corresponding solution $z(t)$ satisfies (1), (6).

(1), (7): For our last three point problem, let $a < t_1 < t_2 < t_3 < b$ and let $y(t)$ be the solution of (1) satisfying conditions of type (6),

$$y(t_1) = y_1,\ y(t_2) = 0,\ y'(t_2) = y_3,\ y'(t_3) = y_4,$$

and define $S_4 \equiv \{z'(t_1) | z(t)$ is a solution of (1) and $z(t_1) = y(t_1)$, $z'(t_2) = y'(t_2)$, $z'(t_3) = y'(t_3)\}$. Here, we also have $S_4 = R$, and so for $y_2 \in S_4$, (1), (7) has a solution.

(1), (8): The pattern continues for the four point problem. Let $a < t_1 < t_2 < t_3 < t_4 < b$ and let $y(t)$ be the solution of (1) satisfying conditions (7),

$$y(t_2) = y_2,\ y'(t_2) = 0,\ y'(t_3) = y_3,\ y'(t_4) = y_4,$$

and set $S_5 \equiv \{z(t_1) | z(t)$ is a solution of (1) and $z(t_2) = y(t_2)$, $z'(t_3) = y'(t_3)$, $z'(t_4) = y'(t_4)\}$. As with the other cases, $S_5 = R$, and so choosing $y_1 \in S_5$, the boundary value problem (1), (8) has a solution.

In each case above, uniqueness of the solutions is by the right (2; 2) disfocality, and the proof is complete.

For completeness, we give the statement of the theorem establishing optimal length subintervals of (a, b), in terms of the Lipschitz coefficients k_i, $i = 1, 2, 3, 4$, on which each right (2; 2) focal boundary value problem for (1) has a unique solution.

THEOREM 6. *Let $h = h(k_1, k_2, k_3, k_4)$ be as defined in Theorem 4. Then each of the boundary value problems for* (1) *satisfying* (3), (4), (5), (6), (7), *or* (8) *has a unique solution, provided $t_4 - t_1 < h$. Furthermore, this result is best possible for the class of all fourth-order ordinary differential equations satisfying the Lipschitz condition* (B).

3. EXISTENCE, UNIQUENESS, AND OPTIMALITY FOR RIGHT $(m_1; \ldots; m_l)$ FOCAL BOUNDARY VALUE PROBLEMS

Through a sequence of theorems somewhat analogous to those for the right (2; 2) focal boundary value problems in Section 2, optimal length subintervals of (a, b) can be determined on which each of the other families of right $(m_1; \ldots; m_l)$ focal boundary value problems for (1) have unique solutions. In this section, we will state, without proof, an analogue of only Theorem 6 for each of these families of right $(m_1; \ldots; m_l)$ focal boundary value problems for (1). For notational convenience, we define the following positive numbers.

DEFINITIONS. Let $a_i > 0$, $1 \leqslant i \leqslant 8$, be the smallest positive numbers such that:

(i) There exists a solution $x(t)$ of (12) satisfying $x(0) = x'(0) = x''(0) = x(a_1) = 0$, with $x(t) > 0$ on $(0, a_1)$.

(ii) There exists a solution $x(t)$ of (13) satisfying $x(0) = x'(0) = x(a_2) = x'(a_2) = 0$, with $x(t) > 0$ on $(0, a_2)$.

(iii) There exists a solution $x(t)$ of (12) satisfying $x(0) = x'(0) = x''(0) = x'(a_3) = 0$, with $x(t) > 0$ on $(0, a_3)$.

(iv) There exists a solution $x(t)$ of (13) satisfying $x(0) = x'(0) = x'(a_4) = x''(a_4) = 0$, with $x(t) > 0$ on $(0, a_4)$.

(v) There exists a solution $x(t)$ of (12) satisfying $x(0) = x'(a_5) = x''(a_5) = x'''(a_5) = 0$, with $x(t) > 0$ on $(0, a_5)$.

(vi) There exists a solution $x(t)$ of (13) satisfying $x(0) = x'(0) = x''(0) = x''(a_6) = 0$, with $x(t) > 0$ on $(0, a_6)$.

(vii) There exists a solution $x(t)$ of (13) satisfying $x(0) = x'(0) = x''(a_7) = x'''(a_7) = 0$, with $x(t) > 0$ on $(0, a_7)$.

(viii) There exists a solution $x(t)$ of (13) satisfying $x(0) = x'(0) = x''(0)) = x'''(a_8) = 0$, with $x(t) > 0$ on $(0, a_8)$.

If in one of the cases (i)–(viii), no such solution $x(t)$ exists, then set $a_i = +\infty$ for that case.

THEOREM 7. *Let* $(c, d) \subseteq (a, b)$.

(i) *If* $d - c \leqslant \min\{a_1, a_2\}$, *then each right* (4)*-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

(ii) *If* $d - c \leqslant \min\{a_2, a_3\}$, *then each right* (3; 1)*-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

(iii) *If* $d - c \leqslant \min\{a_1, a_4\}$, *then each right* (2; 2) *-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

(iv) *If* $d - c \leqslant \min\{a_1, a_4\}$, *then each right* (2; 1; 1)*-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

(v) *If* $d - c \leqslant \min\{a_3, a_4, a_5\}$, *then each right* (1; 3)*-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

(vi) *If* $d - c \leqslant \min\{a_4, a_6\}$, *then each right* (1; 2; 1)*-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

(vii) *If* $d-c \leqslant \min\{a_5, a_6, a_7\}$, *then each right* $(1;1;2)$*-two, -three, and -four point focal boundary problem for* (1) *on* (c, d) *has a unique solution.*

(viii) *If* $d-c \leqslant \min\{a_5, a_7, a_8\}$, *then each right* $(1;1;1;1)$*-two, -three, and -four point focal boundary value problem for* (1) *on* (c, d) *has a unique solution.*

In each of the cases (i)–(viii), *this result is best possible for the class of all fourth-order differential equations satisfying the Lipschitz condition* (B).

Remark. In determining these optimal length intervals, application of the Pontryagin Maximum Principle depended upon our considering related two point boundary value problems. For Lipschitz equations of order $n > 4$, except for special partitions $m_1, ..., m_l$ of n, the applicability of this method is reduced; in particular, if $n > 4$, $l \geqslant 3$, and $m_1 + \cdots + m_l = n$ are such that, for some $1 \leqslant i < j < l$, m_i, $m_j \geqslant 2$, then there are no right $(m_1; ...; m_l)$-two point focal boundary value problems under our definition of such problems.

4. Optimal Interval Lengths for $k_i = 1$, $i = 1, 2, 3, 4$.

In this section, some numerical results are given in which we determine optimal length intervals, as given in Theorem 7, for generic problems for the several classes of boundary value problems we have studied here, in the case when the Lipschitz constants $k_i = 1$, $i = 1, 2, 3, 4$.

In all cases, Richardson's extrapolation method was used to solve the initial value problems associated with finding the solutions which determine the optimal interval lengths. The demanded error tolerances were 10^{-12} as were the criterion for meeting the boundary values. In all cases, because of the Lipschitz constants, one of the two equations was solved numerically:

$$x^{(4)} = -x - |x'| - |x''| - |x'''|, \tag{18}$$

$$x^{(4)} = x + |x'| + |x''| + |x'''|. \tag{19}$$

We used the Cray-1 facilities at the Institute for Defense Analysis/Communication Research Division to perform these numerical integrations.

The technique used to find the solutions to the appropriate boundary value problems was to employ Newton's method to search the parameter space of the free conditions on the initial value. The derivative of the solution with respect to the parameter was then done by finite difference methods. Figures 1–10 appended at the end of the paper depict graphs of the solutions obtained using these methods.

(4): Figures 1 and 2 are the graphs of the numerical solutions of the two point boundary value problems associated with (18) and (19) for this problem; i.e., Fig. 1 is the optimal solution of (18) satisfying $y(0)=y'(0)=y''(0)=y(h_1)=0$, whereas Fig. 2 shows the optimal solution of (19), (and its derivative), satisfying $y(0)=y'(0)=y(h_2)=y'(h_2)=0$. The optimal interval for this class of problems with all Lipschitz contants being 1 is then determined by the solution depicted in Fig. 2. Our numerical results give the interval as having length 3.06978181.

(3; 1): Figures 2 and 3 show the numerical solutions of the two point boundary value problems for (18) and (19) which determine the optimal length interval for this problem. Figure 2 is the optimal solution of (19) satisfying $y(0)=y'(0)=y(h_1)=y'(h_1)=0$, and Fig. 3 is the optimal solution of (18) satisfying $y(0)=y'(0)=y''(0)=y'(h_2)=0$. $h_2=2.54985052$ is the optimal interval length.

(2; 2): Here, Figs. 1, 4, and 5 show sketches of the numerical solutions which determine our optimal interval for existence of unique solutions of right (2; 2) focal problems. Figure 1 was discussed above, whereas Fig. 4 illustrates the solution of (19) satisfying $y(0)=y'(0)=y'(h)=y''(h)=0$; Fig. 5 shows the graphs of $y'(t)$ and $y''(t)$ of the solution $y(t)$ graphed in Fig. 4. Figure 5 gives us our optimal interval length $h=2.25850884$.

(2; 1; 1): From Theorem 7, the optimal interval length here is the same as the optimal length determined in the above case for right (2; 2) focal problems.

(1; 3): Figures 3, 4, 5, and 6 depict the graphs of the numerical solutions, (or their derivatives as in the case of Fig. 5), determining our optimal length intervals for this case. Figures 3, 4, and 5 have been discussed above. Figure 6 is the optimal solution of (18) satisfying $y'(0)=y''(0)=y'''(0)=y(h)=0$; this figure gives us the optimal length, 1.90372364.

(1; 2; 1): Optimal length intervals here are determined by the solutions shown in Figs. 4, 5, and 7. Figure 7, which illustrates the solution of (18) satisfying, $y(0)=y'(0)=y''(0)=y''(h)=0$, gives in this case the optimal interval length, 1.82216354.

(1; 1; 2): The optimal interval here is determined from the solutions of the two point boundary value problems shown in Figs. 6, 7, 8, and 9. We have already discussed Figs. 6 and 7. Figure 8 shows the numerical solution of (19) and $y(0)=y'(0)=y''(h)=y'''(h)=0$, and Fig. 9 shows the graphs of $y''(t)$ and $y'''(t)$ of the solution $y(t)$ in Fig. 8. The optimal interval is determined by this problem and has length 1.31170731.

(1; 1; 1; 1): For this family of problems, we need to introduce the graph of only one further solution. In particular, Figs. 6, 8, 9, and 10 show the numerical solutions (or their derivatives), of the two point boundary value problems associated with (18) and (19) which determine the optimal length interval for these problems. Figure 10 depicts the optimal solution of (18) with boundary values $y(0)=y'(0)=y''(0)=y'''(h)=0$. This solution in Fig. 10 gives the optimal length subinterval on which each right (1; 1; 1; 1) focal problem has a unique solution. The optimal interval length is 1.00718755.

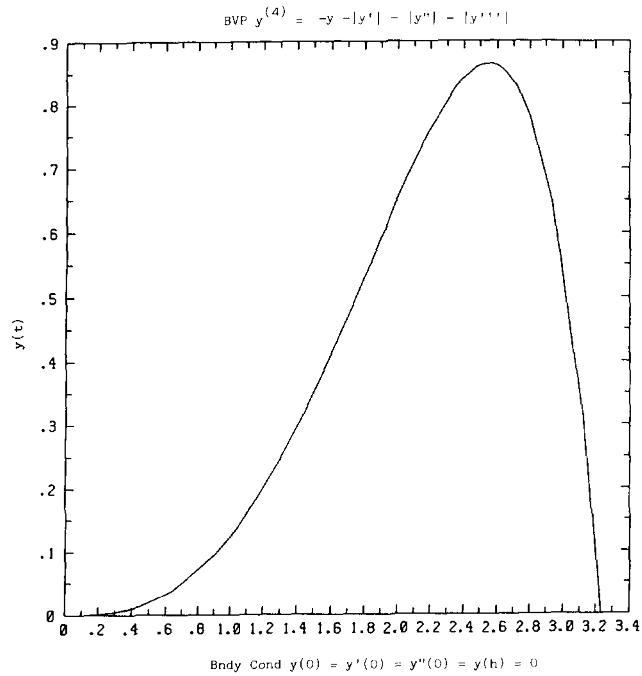


FIGURE 1

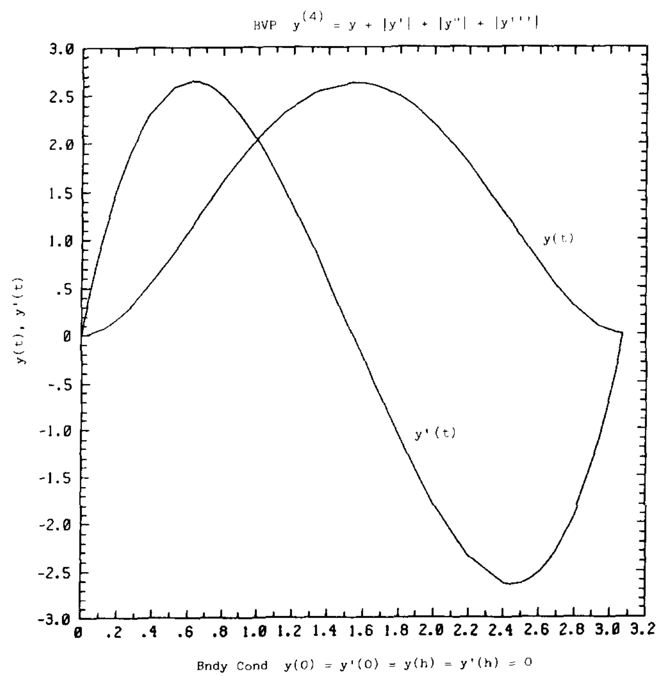


FIGURE 2

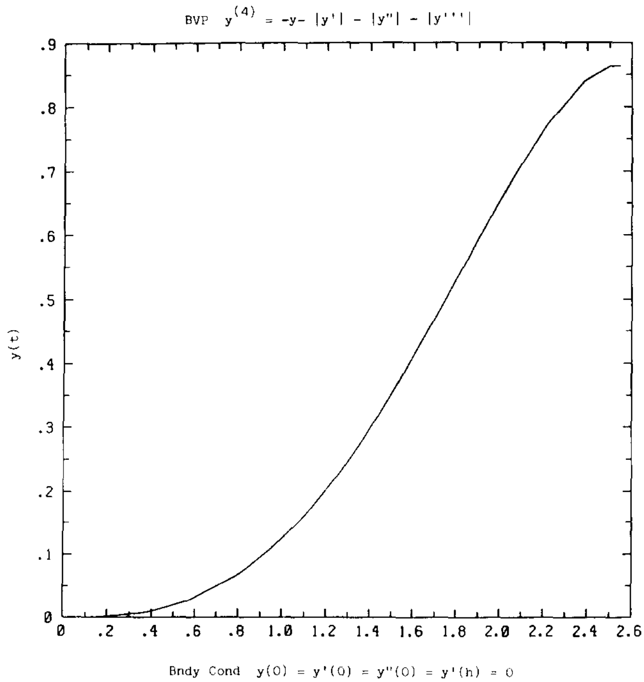


FIGURE 3

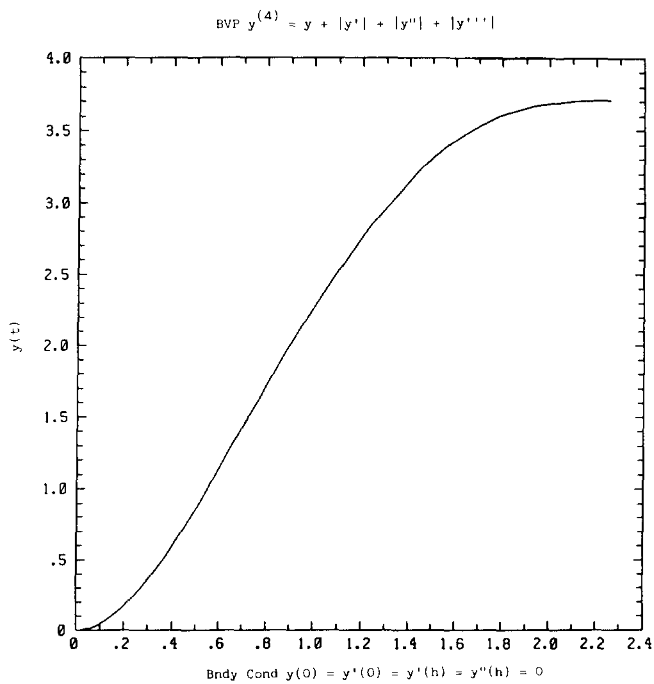


FIGURE 4

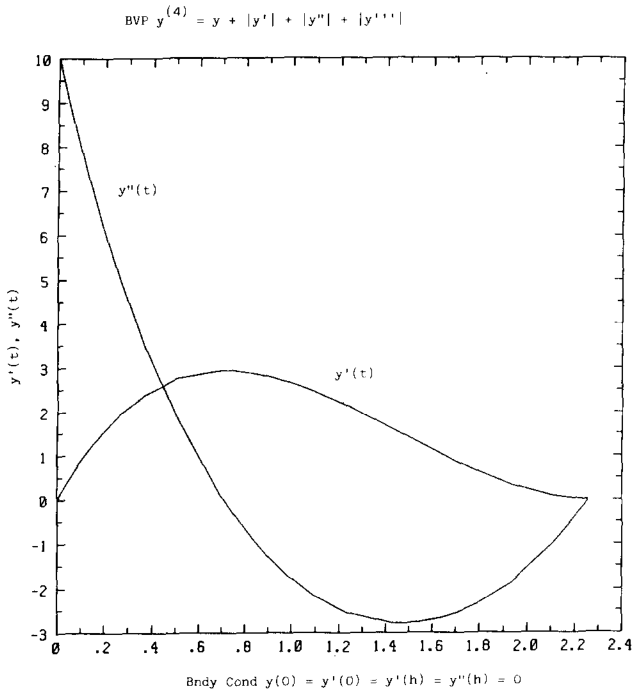


FIGURE 5

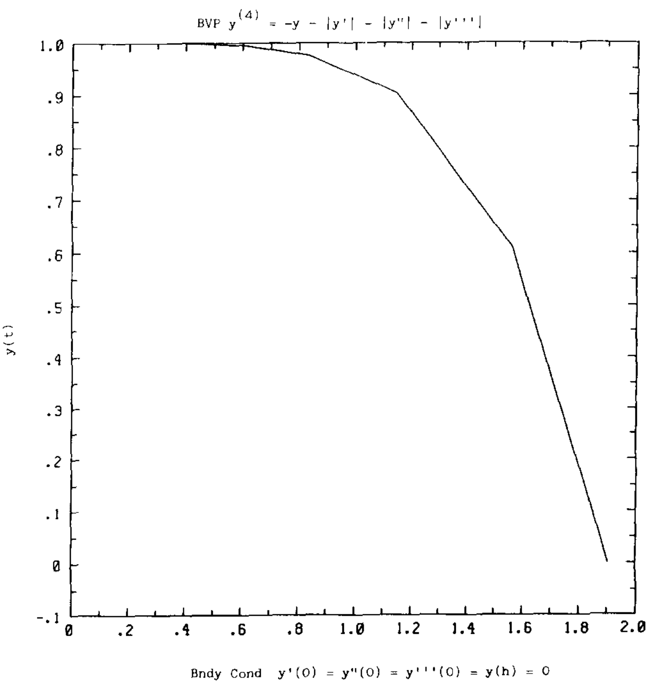
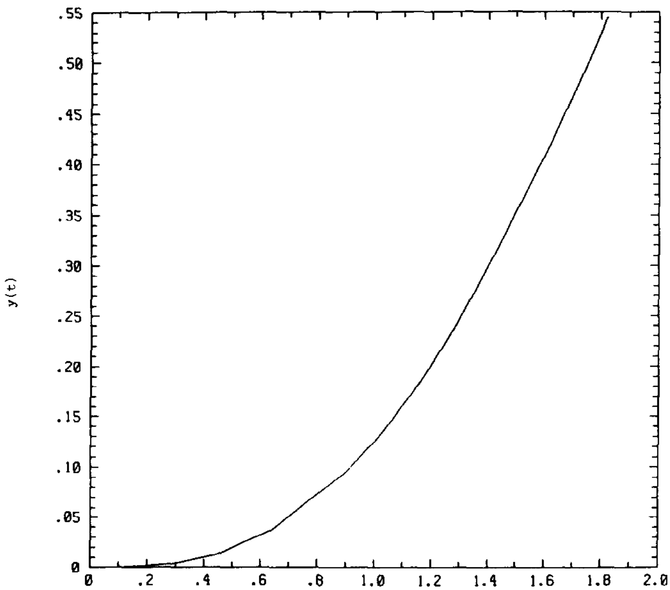


FIGURE 6

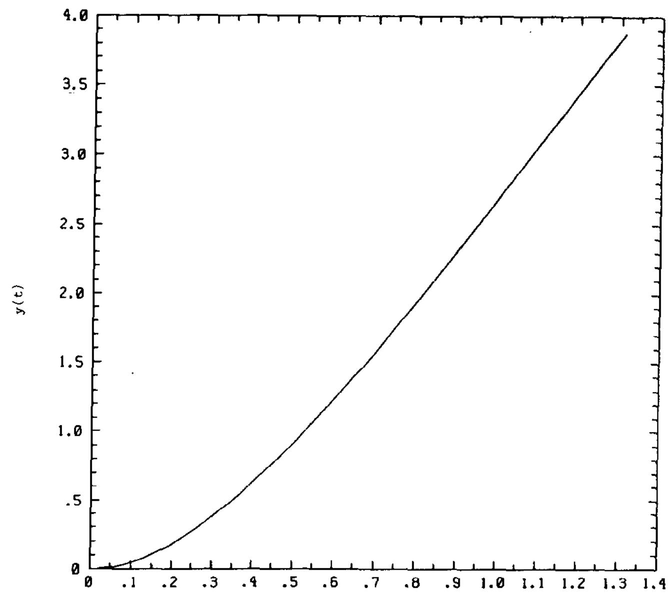
BVP $y^{(4)} = -y - |y'| - |y''| - |y'''|$



Bndy Cond $y(0) = y'(0) = y''(0) = y''(h) = 0$

FIGURE 7

BVP $y^{(4)} = y + |y'| + |y''| + |y'''|$



Bndy Cond $y(0) = y'(0) = y''(h) = y'''(h) = 0$

FIGURE 8

505/67/3-10

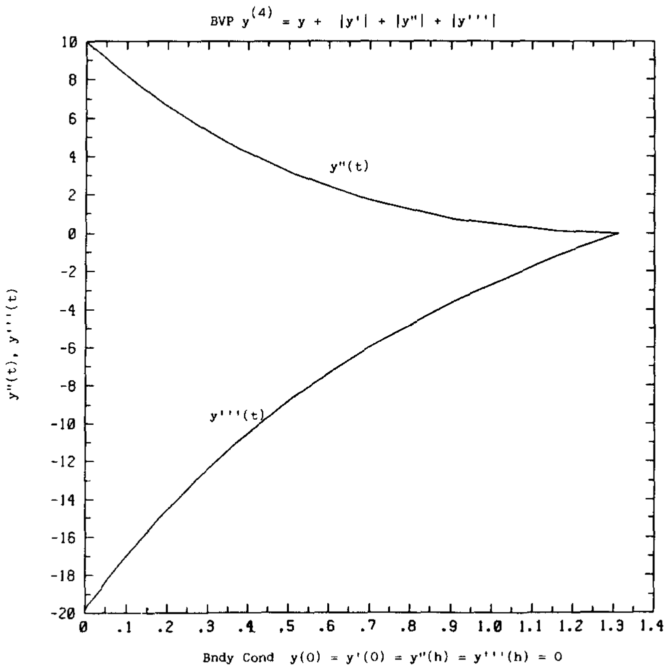


FIGURE 9

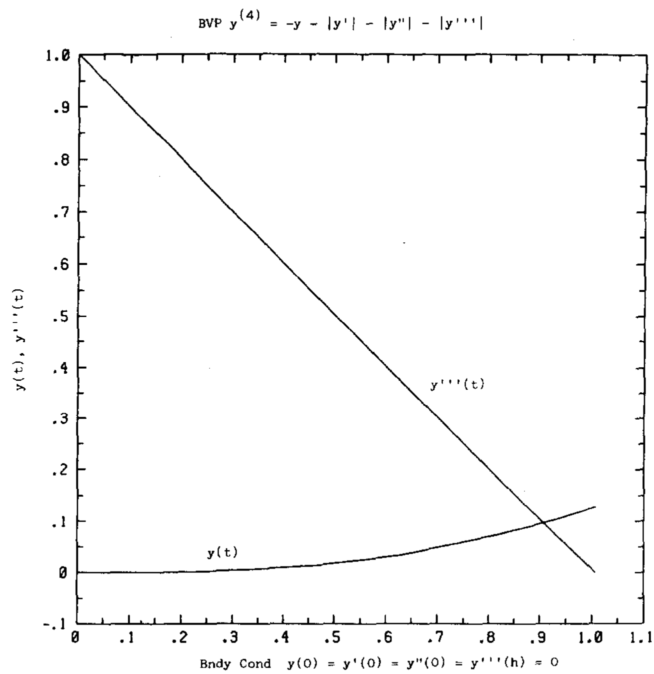


FIGURE 10

Acknowledgment

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