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On a Class of Similarity Solutions of the Porous Media Equation

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1. INTRODUCTION

Consider the one dimensional flow of a polytropic gas through a homogeneous porous medium. Then the density u of the gas satisfies the nonlinear diffusion equation

$$u_t = (u^m)_{xx} \tag{1}$$

whenever $u > 0$. Here, x denotes the space variable, t time and m a constant greater than 1 [16, 657].

Equation (1) is parabolic at any point (x, t) at which $u > 0$. However, at points where $u = 0$, it is degenerate parabolic. Because of this degeneracy (1) need not always have a classical solution. Classes of weak solutions for the Cauchy problem and the Cauchy–Dirichlet problem of Eq. (1) were introduced by Oleinik, Kalashnikov and Yui-Lin [11]. They proved existence and uniqueness of such solutions and in addition, they showed that if at some instant t_0 a weak solution $u(x, t_0)$ has compact support, then $u(x, t)$ has compact support for any $t \geqslant t_0$.

In this paper we shall study a class of similarity solutions of (1) in the domain $0 < x < \infty$, $0 < t \leqslant T$, where T is some positive constant. Let α and τ be real numbers. We shall seek solutions of the following three types:

$$\text{I.}\quad u_1(x, t) = (t + \tau)^\alpha f_1(\eta), \qquad \eta = x(t + \tau)^{-\frac{1}{2}\{1+(m-1)\alpha\}}$$

for $\tau > 0$;

$$\text{II.}\quad u_2(x, t) = (\tau - t)^\alpha f_2(\eta), \qquad \eta = x(\tau - t)^{-\frac{1}{2}\{1+(m-1)\alpha\}}$$

for $\tau > T$;

$$\text{III.}\quad u_3(x, t) = e^{\alpha(t+\tau)} f_3(\eta), \qquad \eta = x \exp\{-\tfrac{1}{2}\alpha(m - 1)(t + \tau)\}$$

for any τ.

Substitution of u_1, u_2 and u_3 into (1) leads to the following equations for the functions f_1, f_2 and f_3:

$$\text{I.}\quad (f_1^m)'' + \tfrac{1}{2}\{1 + (m-1)\alpha\}\,\eta f_1' = \alpha f_1 \qquad 0 < \eta < \infty \qquad \text{(2a)}$$

$$\text{II.}\quad (f_2^m)'' - \tfrac{1}{2}\{1 + (m-1)\alpha\}\,\eta f_2' = -\alpha f_2 \qquad 0 < \eta < \infty \qquad \text{(2b)}$$

$$\text{III.}\quad (f_3^m)'' + \tfrac{1}{2}\alpha(m-1)\,\eta f_3' = \alpha f \qquad 0 < \eta < \infty. \qquad \text{(2c)}$$

At the boundaries we impose the conditions

$$f_i(0) = U(\geqslant 0), \qquad f_i(\infty) = 0 \qquad i = 1, 2, 3.$$

Thus the solutions $u_i(x, t)$ satisfy the lateral boundary conditions

$$u_1(0, t) = (t + \tau)^\alpha U, \qquad u_2(0, t) = (\tau - t)^\alpha U, \qquad u_3(0, t) = e^{\alpha(t+\tau)} U$$

and

$$u_i(x, t) \to 0 \qquad \text{as} \qquad x \to \infty \qquad i = 1, 2, 3$$

for fixed $t \in [0, T]$.

It was Barenblatt [4], who first discussed the similarity solution u_1 ; he did this for $\alpha \geqslant 0$. In a subsequent paper [6] he also investigated the solution u_3 for $\alpha > 0$ and $m = 2$. Later Marshak [10] also discussed solution u_3 ; in addition he made a detailed, and partly numerical, study of solution u_1 for $\alpha = \frac{1}{5}$. For a number of values of α, explicit solutions were found by various authors [1, 4, 5, 7, 9, 11, 12, 15, 16].

The studies mentioned above are all to a greater or lesser extent of a heuristic nature, and it is only recently that a rigorous study of these similarity solutions was begun. This was done by Atkinson and Peletier [2, 3] and by Shampine [13, 14]. They considered the equation

$$(k(f)f')' + \tfrac{1}{2}\eta f' = 0 \qquad 0 < \eta < \infty \qquad (3)$$

in which $k(s)$ is defined, real and continuous for $s \geqslant 0$, with $k(0) \geqslant 0$ and $k(s) > 0$ if $s > 0$. Clearly, if we set $\alpha = 0$, equation (2a) becomes a special case of (3).

In the present paper we shall extend the analysis of [2] to the problem

$$(f^m)'' + p\eta f' = qf, \qquad 0 < \eta < \infty, \qquad (4)$$

$$f(0) = U, \qquad f(\infty) = 0, \qquad (5)$$

in which p and q are arbitrary real constants. Plainly, Eq. (4) incorporates Equations (2a)–(2c).

As in [2] it will be necessary to consider weak solutions of problem (4), (5). A function f will be said to be a weak solution of Eq. (4) if (a) f is bounded, continuous and nonnegative on $[0, \infty)$, (b) $(f^m)(\eta)$ has a continuous derivative with respect η on $(0, \infty)$, and (c) f satisfies the identity

$$\int_0^\infty \phi'\{(f^m)' + p\eta f\}\, d\eta + (p + q) \int_0^\infty \phi f\, d\eta = 0$$

for all $\phi \in C_0^1(0, \infty)$.

We shall establish the following results:

(i) Let $U > 0$. Then problem (4), (5) has a weak solution with compact support if and only if

$$p \geqslant 0 \qquad \text{and} \qquad 2p + q > 0.$$

This solution is unique.

(ii) Let $U = 0$. Then problem (4), (5) has a nontrivial weak solution with compact support if and only if

$$p > 0 \qquad \text{and} \qquad 2p + q = 0.$$

In this case there exists a one parameter family of such solutions.

2. The Method

Let f be a weak solution of problem (4), (5) with compact support in $[0, \infty)$. Then, as we shall see later, f is positive in a right neighborhood of $\eta = 0$. More specifically, there exists a number $a > 0$ such that

$$f > 0 \quad \text{on} \quad (0, a); \qquad f = 0 \quad \text{on} \quad [a, \infty).$$

It was shown in [2] that in a neighborhood of any point where $f > 0$, f is a classical solution of equation (4). Thus we shall be mainly concerned with proving the existence and uniqueness of a classical positive solution of (4) on an interval $(0, a)$ which satisfies the boundary conditions

$$f(0) = U, \tag{6}$$

$$f(a) = 0, \qquad (f^m)'(a) = 0. \tag{7}$$

The condition at $\eta = a$ follows from the requirement that f and $(f^m)'$ be continuous on $(0, \infty)$.

The existence proof is based on a shooting technique. Let a be an arbitrary positive number. Then we shall show that for suitable p and q, there exists a unique positive solution of problem (4), (7) in a left neighborhood of $\eta = a$, and that this solution can be continued back to $\eta = 0$. We then ask whether a can be chosen so that condition (6) is satisfied.

Before turning to the question of existence we obtain a preliminary non-existence result.

Lemma 1. *The existence of a nontrivial weak solution of Eq.* (4) *with compact support implies one of the following propositions.* (i) $p > 0$ *or* (ii) $p = 0$ *and* $q > 0$.

Proof. Suppose f is a nontrivial weak solution of equation (4) with compact support. Then there exists an $a > 0$ such that $f > 0$ in $(a - \epsilon, a)$ for some $\epsilon > 0$ and $f = 0$ in $[a, \infty)$. Thus, in $(a - \epsilon, a)$ f satisfies (4), and at $\eta = a$, f satisfies (7). Integration of (4) from $\eta \in (a - \epsilon, a)$ to a yields

$$-(f^m)'(\eta) = p\eta f(\eta) + (p + q) \int_\eta^a f(\xi)\, d\xi. \tag{8}$$

In view of the continuity of f and $(f^m)'$ it is possible to find an $\eta_0 \in (a - \epsilon, a)$ such that $f'(\eta_0) < 0$. Hence p and $p + q$ cannot both be less than zero. Thus, if $p = 0$, q must be positive.

Suppose now that $p < 0$. Then, by (8), $p + q > 0$ and hence $q > 0$. It follows from (4) that f cannot have a maximum in $(a - \epsilon, a)$ and hence, that $f' < 0$ on $(a - \epsilon, a)$. Therefore

$$-mf^{m-2}(\eta)f'(\eta) - p\eta \leqslant (p + q)(a - \eta)$$

for all $\eta \in (a - \epsilon, a)$. If we now let η tend to a, we obtain a contradiction.

3. Solutions Near $\eta = a$

Let a be an arbitrary positive number. It is clear from the proof of Lemma 1 that a necessary condition for the existence of a positive solution of problem (4), (7) in a left neighborhood of $\eta = a$ is that either $p > 0$ or $p = 0$ and $q > 0$. The object of this section is to show that this condition is also sufficient.

We begin by assuming that $p = 0$ and $q > 0$. Then we can solve problem (4), (6), (7) uniquely. For it follows after an elementary computation that the function

$$f(\eta; a) = \left\{ \frac{q(m-1)^2}{2m(m+1)} (a - \eta)^2 \right\}^{1/(m-1)} \qquad 0 < \eta < a \tag{9}$$

is the unique solution of problem (4), (7). Because $f(0; a)$ is a continuous, monotonically increasing function of a, such that $f(0; 0) = 0$ and $f(0; \infty) = \infty$, the equation $f(0; a) = U$ is uniquely solvable for every $U \geqslant 0$. Let $a(U)$ be its solution. Then $f = f(\eta; a(U))$ is the unique solution of problem (4), (6), (7).

Next, we turn to the case $p > 0$. We first prove a preparatory lemma.

Lemma 2. *Let $b \in (0, a)$, and let f be a positive solution of problem* (4), (7) *on* $[b, a)$.

(i) *If* $p + q \geqslant 0$ *then* $f'(\eta) < 0$ *on* $[b, a)$.

(ii) *If* $p + q < 0$, *and there exists an* $\eta_0 \in [b, a)$ *such that* $f'(\eta_0) = 0$, *then f has a maximum at* η_0, *and* $\eta_0 < \{(p + q)/q\}a$.

If f is a positive solution of problem (4), (7) *on* $[0, a)$, *then if* $p + q > 0$, $f'(0) < 0$; *if* $p + q = 0, f'(0) = 0$; *and if* $p + q < 0, f'(0) > 0$.

Proof. Integration of (4) from $\eta \in [b, a)$ to a yields, as before, equation (8). If $p + q \geqslant 0$, this implies that $(f^m)'(\eta) < 0$ and hence that $f'(\eta) < 0$ on $[b, a)$.

If $p + q < 0$, we note that $q < 0$ and hence $f'(\eta_0) = 0$ implies that $f''(\eta_0) < 0$. It follows that f has a maximum at $\eta = \eta_0$, and $f'(\eta) < 0$ on (η_0, a). To estimate η_0, we set $\eta = \eta_0$ in (8). Using the fact that $f'(\eta) < 0$ on (η_0, a) we obtain

$$0 = p\eta_0 f(\eta_0) + (p + q) \int_{\eta_0}^{a} f(\xi)\, d\xi > p\eta_0 f(\eta_0) + (p + q) \int_{\eta_0}^{a} f(\eta_0)\, d\xi.$$

Hence

$$p\eta_0 + (p + q)(a - \eta_0) < 0$$

or

$$(p + q)\, a - q\eta_0 < 0.$$

Recalling that $q < 0$, we obtain the desired upper bound for η_0.

Finally, if $b = 0$, (8) yields the relation

$$-(f^m)'(0) = (p + q) \int_0^a f(\xi)\, d\xi$$

from which the sign of $f'(0)$ follows.

We now turn to the question of existence.

LEMMA 3. *Let* $p > 0$ *and let q be arbitrary. Then given any* $a > 0$, *there exists an* $\epsilon > 0$ *such that in* $(a - \epsilon, a)$ *problem* (4), (7) *has a unique positive solution.*

Proof. As in [2] we reduce the problem to that of establishing the local existence of a solution of an equivalent integral equation. To derive this equation we assume that f is a positive solution in an interval $(a - \epsilon, a)$ for some $\epsilon > 0$. By Lemma 2, it is possible to choose ϵ such that $f' < 0$ in $(a - \epsilon, a)$. This allows us to formulate the problem in terms of the inverse function $\eta = \sigma(f)$.

We write (8) in the form

$$(f^m)'(\eta) = q\eta f(\eta) + (p + q) \int_\eta^a \xi f'(\xi)\, d\xi.$$

Hence the function $\sigma(f)$ satisfies the integrodifferential equation

$$\frac{d\sigma}{df} = \frac{mf^{m-1}}{qf\sigma(f) - (p+q)\int_0^f \sigma(\varphi)\,d\varphi}.$$

Integration from 0 to f yields

$$\sigma(f) - a = m\int_0^f \frac{\varphi^{m-1}\,d\varphi}{q\varphi\sigma(\varphi) - (p+q)\int_0^\varphi \sigma(\psi)\,d\psi},$$

or, when we write

$$\tau(f) = 1 - a^{-1}\sigma(f),$$

$$\tau(f) = \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}\,d\varphi}{p\varphi + q\varphi\tau(\varphi) - (p+q)\int_0^\varphi \tau(\psi)\,d\psi}. \tag{10}$$

The next step is to prove that (10) has a unique positive solution in a right neighborhood of $f = 0$. Let $\gamma > 0$, and let X be the set of bounded functions $\tau(f)$ defined on $[0, \gamma]$ such that

$$0 \leqslant \tau(f) \leqslant \rho = \frac{p}{2(|q| + |p+q|)}.$$

We denote by $\|\cdot\|$ the supremum norm on X. Then X is a complete metric space. On X we define the operator

$$M(\tau)(f) = \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}\,d\varphi}{p\varphi + q\varphi\tau(\varphi) - (p+q)\int_0^\varphi \tau(\psi)\,d\psi}.$$

Suppose $\tau \in X$. Then

$$p\varphi + q\varphi\tau(\varphi) - (p+q)\int_0^\varphi \tau(\psi)\,d\psi \geqslant \{p - (|q| + |p+q|)\|\tau\|\}\,\varphi \geqslant \tfrac{1}{2}p\varphi.$$

Hence

$$M(\tau)(f) \leqslant \frac{2m}{pa^2}\int_0^f \varphi^{m-2}\,d\varphi \leqslant \frac{2m}{(m-1)\,pa^2}\gamma^{m-1}.$$

Thus, $M(\tau)$ is well defined on the whole of X. Clearly, if $\tau \in X$, $M(\tau)$: $[0, \gamma] \to R$ is nonnegative and continuous; moreover there exists a $\gamma_0 > 0$ such that if $\gamma \leqslant \gamma_0$ and $\tau \in X$, $\|M(\tau)\| \leqslant \rho$. Thus, if $\gamma \leqslant \gamma_0$, M maps X into X.

Let τ_1, $\tau_2 \in X$, and let $\gamma \leqslant \gamma_0$. Then

$$\| M(\tau_1) - M(\tau_2)\| \leqslant \frac{4m}{a^2p^2}\int_0^f \varphi^{m-3}\left(|q|\,\varphi\,\|\tau_1-\tau_2\| + |p+q|\int_0^\varphi \|\tau_1-\tau_2\|\,d\psi\right)d\varphi \leqslant \frac{4m}{(m-1)\,p^2a^2}\,(|q| + |p+q|)\,\gamma^{m-1}\,\|\tau_1-\tau_2\|.$$

Hence, there exists a $\gamma_1 \in (0, \gamma_0]$ such that if $\gamma \leqslant \gamma_1$ M is a contraction on X. Thus by the Banach–Cacciopoli contraction mapping principle ([8, 404]) M has a unique fixed point in X, and equation (10) has a unique solution.

It follows from a routine computation that this result implies the existence and uniqueness of a positive solution of problem (4), (7) in a left neighborhood of $\eta = a$.

4. Backward Continuation

Let $a > 0$, and let $f(\eta)$ be the solution of (4), (7) we constructed in the previous section. Then f is defined and positive in a left neighborhood of $\eta = a$. We now continue f backwards as a function of η. By the standard theory [8] this can be done uniquely so long as f remains positive and bounded. There are now three possibilities:

(A) $f(\eta) \to \infty$ as $\eta \downarrow \eta_1$ for some $\eta_1 \in [0, a)$;

(B) $f(\eta)$ can be continued back to $\eta = 0$;

(C) $f(\eta) \to 0$ as $\eta \downarrow \eta_2$ for some $\eta_2 \in (0, a)$.

We begin by ruling out possibility (A).

LEMMA 4. *Let $b \in [0, a)$, and let f be a positive solution of problem* (4), (7) *on (b, a). Then, if $p > 0$,*

$$\sup_{(b,a)} f(\eta) \leqslant [((m-1)/2m)\,a^2 \max\{p, 2p+q\}]^{1/(m-1)}.$$

Proof. (i) Assume $p + q \geqslant 0$. Then, by Lemma 2, $f' < 0$ on (b, a). Using this in (8) we obtain:

$$-m(f^m)'(\eta) \leqslant p\eta f(\eta) + (p+q)\,f(\eta)(a-\eta)$$

or

$$-mf^{m-2}(\eta)\,f'(\eta) \leqslant (p+q)\,a - q\eta, \qquad b \leqslant \eta < a.$$

Integration from η to a yields

$$[m/(m-1)]f^{m-1}(\eta) \leqslant \{pa + \tfrac{1}{2}q(a-\eta)\}\,(a-\eta) \quad b \leqslant \eta \leqslant a \tag{11}$$

and hence

$$\sup_{(b,a)}[m/(m-1)]\, f^{m-1}(\eta) \leqslant \tfrac{1}{2}(2p+q)\, a^2. \tag{12}$$

(ii) Assume $p+q<0$. Then it follows from (8) that

$$-(f^m)'(\eta) \leqslant p\eta f(\eta).$$

If we divide by f and integrate from η to a we obtain the inequality:

$$[m/(m-1)]f^{m-1}(\eta) \leqslant \tfrac{1}{2}p(a^2-\eta^2), \qquad b \leqslant \eta \leqslant a. \tag{13}$$

Thus

$$\sup_{(b,a)}[m/(m-1)]\, f^{m-1}(\eta) \leqslant \tfrac{1}{2}pa^2. \tag{14}$$

Because the bound of Lemma 4 is uniform in b, $f(\eta)$ can never become unbounded as η decreases.

The estimates (11) and (13) are of some interest in their own right in that they provide upper bounds for $f(\eta)$ which also tend to zero as $\eta \to a$. Lower bounds can be derived in exactly the same way; one finds:

(i) if $p+q \geqslant 0$,

$$[m/(m-1)]\, f^{m-1}(\eta) \geqslant \tfrac{1}{2}p(a^2-\eta^2) \qquad b \leqslant \eta \leqslant a; \tag{15}$$

(ii) if $p+q<0$,

$$\begin{aligned}[m/(m-1)]\, f^{m-1}(\eta) &\geqslant \{pa + \tfrac{1}{2}q(a-\eta)\}\,(a-\eta), \qquad \max\{b,\eta_0\} \leqslant \eta \leqslant a\\ &\geqslant \tfrac{1}{2}(2p+q)\,(a^2-\eta^2).\end{aligned} \tag{16}$$

The following Lemma distinguishes between the possibilities (B) and (C).

LEMMA 5. *Let f be the positive solution of problem* (4), (7) *in a left neighborhood of $\eta = a$. Assume that $p>0$. Then:*

(i) *if $2p+q>0$, $f(\eta)>0$ on $[0,a)$;*

(ii) *if $2p+q=0$, $f(\eta)>0$ on $(0,a)$ and $f(0)=0$;*

(iii) *if $2p+q<0$, there exists an $\eta^* \in (0,a)$ such that $f(\eta)>0$ on (η^*,a) and $f(\eta^*)=0$.*

Proof. Integration of (8) from η to a yields the following integral equation for f:

$$f^m(\eta) = p\eta \int_\eta^a f(\xi)\, d\xi + (2p+q)\int_\eta^a (\xi-\eta)\, f(\xi)\, d\xi. \tag{17}$$

Lemma 5 now follows at once.

Suppose $2p + q > 0$. Then, by the previous Lemma we may continue $f(\eta)$ back to $\eta = 0$, and $f(0) > 0$. However, using the bounds for f we obtained earlier, we can actually give upper and lower bounds for $f(0)$. This will be done in the following Proposition. It will be convenient to define the quantities:

$$\lambda = (2p + q)/p, \qquad \mu = 1 - [(p + q)/q]^2, \qquad A = \{[(m - 1)/2m]\, pa^2\}^{1/(m-1)}.$$

PROPOSITION 1. *Let $p > 0$ and $2p + q > 0$. Then:*

(i) *if $p + q \geqslant 0$ ($\lambda \geqslant 1$)*

$$\lambda^{1/m} A \leqslant f(0) \leqslant \lambda^{1/(m-1)} A;$$

(ii) *if $p + q \leqslant 0$ ($0 < \lambda \leqslant 1$)*

$$(\mu\lambda)^{1/(m-1)} A \leqslant f(0) \leqslant \lambda^{1/m} A.$$

Both estimates are sharp for $p + q = 0$.

Proof. (i) The upper bound follows at once from (11). To obtain the lower bound we use (15) in (17),

$$f^m(0) = (2p + q) \int_0^a \xi f(\xi)\, d\xi. \tag{18}$$

The result follows after an elementary computation.

(ii) In this case we only have a bound for f on the interval $[\eta_0, a)$, where η_0 is the value of η for which f reaches its maximum value. By (13) and (16)

$$\lambda^{1/(m-1)} A\{1 - (\eta/a)^2\}^{1/(m-1)} \leqslant f(\eta) \leqslant A\{1 - (\eta/a)^2\}^{1/(m-1)} \quad \eta_0 \leqslant \eta \leqslant a. \tag{19}$$

However, $f(\eta) \leqslant f(\eta_0)$ on $[0, \eta_0]$ and therefore (19) holds for $0 \leqslant \eta \leqslant a$. Using this estimate in (18) we obtain the desired upper bound.

To obtain the lower bound, we note that by (18)

$$f^m(0) \geqslant (2p + q) \int_{a^*}^a \xi f(\xi)\, d\xi, \tag{20}$$

where $a^* = \{(p + q)/q\}a$. Because, by Lemma 2, $\eta_0 \leqslant a^*$ we can use (19) in (20) to estimate $f(0)$.

We conclude this section with a result about the dependence of f on the choice of a.

PROPOSITION 2. *Let $p > 0$ and $2p + q \geqslant 0$. Suppose $f(\eta; a_1)$ and $f(\eta; a_2)$ are solutions of problem* (4), (7) *on, respectively,* $(0, a_1)$ *and* $(0, a_2)$. *Then, if $a_1 > a_2$, $f(\eta; a_1) > f(\eta; a_2)$ everywhere on* $(0, a_2)$.

Proof. Denote $f(\eta; a_i)$ by $f_i(\eta)$ for $i = 1, 2$. Suppose the Proposition is not true. Then there exists an $\bar{\eta} \in (0, a_2)$ such that $f_1(\bar{\eta}) = f_2(\bar{\eta})$ and $f_1(\eta) > f_2(\eta)$ on $(\bar{\eta}, a_2)$. It follows from (17) that for $i = 1, 2$,

$$f_i^m(\bar{\eta}) = p\bar{\eta} \int_{\bar{\eta}}^{a_i} f_i(\xi)\, d\xi + (2p + q) \int_{\bar{\eta}}^{a_i} (\xi - \bar{\eta}) f_i(\xi)\, d\xi.$$

Hence

$$p\bar{\eta} \int_{\bar{\eta}}^{a_2} (f_1 - f_2)\, d\xi + (2p + q) \int_{\bar{\eta}}^{a_2} (\xi - \bar{\eta})(f_1 - f_2)\, d\xi$$
$$+ p\bar{\eta} \int_{a_2}^{a_1} f_1\, d\xi + (2p + q) \int_{a_2}^{a_1} (\xi - \bar{\eta}) f_1\, d\xi = 0.$$

The second and the fourth term of this expression are nonnegative, whilst the other two are positive. We therefore have a contradiction.

5. The Main Result

We begin by proving the existence and uniqueness of a solution of problem (4), (6), (7) which is positive on $(0, a)$. By Lemma 1 a necessary condition for the existence of such a solution is that $p \geqslant 0$.

Let $p > 0$. Then, by Lemma 3, for each $a > 0$ there exists a unique positive solution $f(\eta; a)$ of (4), (7) in a left neighborhood of $\eta = a$. By Lemma 5 this solution can be continued back to $\eta = 0$ if and only if $2p + q \geqslant 0$. Thus the boundary condition at $\eta = 0$ is satisfied if we can find an $a > 0$ such that

$$f(0; a) = U. \tag{21}$$

If only one such an a exists, the solution is unique.

We distinguish two cases:

(i) $U = 0$. Then by Lemma 5, Eq. (21) can only be satisfied if $2p + q = 0$. Moreover (21) is then satisfied for any $a > 0$.

(ii) $U > 0$. It follows from Lemma 5 that now a necessary condition for (21) to have a solution is that $2p + q > 0$. To prove that it is also sufficient we use an observation due to Barenblatt [4].

Let $f(\eta; a)$ be a solution of problem (4), (7) on $(0, a)$. Then for any $\mu > 0$ the function $\mu^{-2/(m-1)} f(\mu\eta; \mu a)$ is a solution of problem (4), (7) on $(0, \mu a)$. Thus, choosing $\mu = a^{-1}$:

$$f(0; a) = a^{2/(m-1)} f(0; 1).$$

Equation (21) can therefore be written as

$$a^{2/(m-1)} f(0; 1) = U. \tag{22}$$

Because $2p + q > 0$, $f(0; 1) > 0$. It follows that for each $U > 0$, Eq. (22) has a unique solution $a(U)$. The function $f(\eta; a(U))$ now satisfies (4), (6) and (7). Moreover, in view of the uniqueness of $a(U)$ it is the only function which does so. Remembering the solution we constructed for $p = 0$, we have proved the following result.

THEOREM 1. (i) *Let $U > 0$. Then there exists a unique $a > 0$ and a unique solution of problem* (4), (6), (7) *which is positive on* $(0, a)$ *if and only if $p \geqslant 0$ and $2p + q > 0$.*

(ii) *Let $U = 0$. Then for every $a > 0$ there exists a solution of problem* (4), (6), (7) *which is positive on* $(0, a)$ *if and only if $p > 0$ and $2p + q = 0$.*

It is easy to see that the function

$$f(\eta) = \begin{cases} f(\eta; a), & 0 \leqslant \eta < a, \\ 0, & a \leqslant \eta < \infty, \end{cases}$$

is a weak solution of Eq. (4) which satisfies the boundary conditions (5). Hence, it remains to show that if $U > 0$, this is the only solution of problem (4), (5) with compact support, and that if $U = 0$, this is the only family of nontrivial solutions of problem (4), (5) with compact support.

Let $f(\eta)$ be a weak solution of problem (4), (5) with compact support. It follows from Lemma 5 that if $U > 0$, problem (4), (5) only has such a solution if $2p + q > 0$, and that this solution is of the form:

$$f(\eta) > 0 \quad \text{on} \quad [0, a),$$
$$f(\eta) = 0 \quad \text{on} \quad [a, \infty),$$

for some $a > 0$. That is, f must be of the type discussed above, and by Theorem 1 there exists only one such solution.

If $U = 0$, one might expect, besides the family of solutions discussed above, nontrivial solutions which are zero on a disconnected subset of $(0, \infty)$. However, we shall show that such solutions cannot exist.

Let f be a weak solution such that $f > 0$ on (a_1, a_2), where $0 < a_1 < a_2 < \infty$ and $f = 0$ at $\eta = a_1$ and at $\eta = a_2$. Then, for f to be a weak solution of (4), we must require

$$f(a_i) = 0, \qquad (f^m)'(a_i) = 0 \qquad i = 1, 2.$$

On (a_1, a_2) f is a classical solution of (4), and hence we obtain by integrating (4) from a_1 to a_2:

$$0 = (p+q)\int_{a_1}^{a_2} f(\xi)\,d\xi.$$

Because $p + q = (2p + q) - p < 0$ and $f > 0$ on (a_1, a_2) we have arrived at a contradiction.

It follows that if $U = 0$, any weak solution of problem (4), (5) with compact support must belong to the family of solutions discussed above.

THEOREM 2. (i) *Let* $U > 0$. *Then there exists a unique weak solution with compact support of problem* (4), (5) *if and only if* $p \geqslant 0$ *and* $2p + q > 0$.

(ii) *Let* $U = 0$. *Then there exists a nontrivial weak solution with compact support of problem* (4), (5) *if and only if* $p > 0$ *and* $2p + q = 0$. *For every* $a > 0$ *there exists one such solution* f *with the property* $f > 0$ *on* $(0, a)$ *and* $f = 0$ *on* $[a, \infty)$.

6. EXAMPLES

We conclude with a discussion of the implications of Theorems 1 and 2 for the similarity solutions of Eq. (1)

(a) *Similarity Solutions of Type I.*

In this case, $p = \frac{1}{2}\{1 + (m - 1)\alpha\}$ and $q = \alpha$. Hence

$$p + q = \tfrac{1}{2}\{1 + (m+1)\alpha\}, \qquad 2p + q = 1 + m\alpha.$$

It follows from Theorem 2 that there exists a non trivial similarity solution with compact support if and only if $\alpha \geqslant -1/m$.

Below we tabulate those solutions which may be derived explicitly from Eq. (2a)

(i) $\alpha = -1/m$. This yields the so called "dipole type solution" [7, 16]. Because $2p + q = 0$ in this case, equation (17) becomes

$$f^m(\eta) = \frac{1}{2m}\,\eta \int_\eta^a f(\xi)\,d\xi.$$

Putting $g(\eta) = \int_\eta^a f(\xi)\,d\xi$, we can write this as

$$g'(\eta) = -\left(\frac{1}{2m}\,\eta g\right)^{1/m}, \qquad g(a) = 0.$$

After a routine computation this yields for f:

$$f(\eta) = \eta^{1/m} \left\{ \left(\frac{m-1}{2m(m+1)} \right)^m (a^{(m+1)/m} - \eta^{(m+1)/m}) \right\}^{1/(m-1)} \qquad 0 \leqslant \eta \leqslant a.$$

Returning to the variables x and t we obtain

$$u_1(x, t) = \begin{cases} (t+\tau)^{-1/m} f(x(t+\tau)^{-1/(2m)}), & 0 \leqslant x \leqslant a(t+\tau)^{1/(2m)}, \\ 0, & a(t+\tau)^{1/(2m)} < x < \infty. \end{cases}$$

(ii) $\alpha = -1/(m+1)$. This solution is often called the "instantaneous point source solution" [1, 4, 12, 15, 16]. Eq. (2a) can now easily be integrated to yield

$$f(\eta) = \left\{ \frac{m-1}{2m(m+1)} (a^2 - \eta^2) \right\}^{1/(m-1)} \qquad 0 \leqslant \eta \leqslant a,$$

and hence

$$u_1(x, t) = \begin{cases} (t+\tau)^{-1/(m+1)} f(x(t+\tau)^{-1/(m+1)}), & 0 \leqslant x \leqslant a(t+\tau)^{1/(m+1)}, \\ 0, & a(t+\tau)^{1/(m+1)} < x < \infty. \end{cases}$$

(iii) $\alpha = 1/(m-1)$. This yields the wave solution [5, 11]. The function

$$f(\eta) = \{[(m-1)/m]\, a(a-\eta)\}^{1/(m-1)} \qquad 0 \leqslant \eta \leqslant a$$

satisfies (4) and (7), and gives

$$u_1(x, t) = \begin{cases} \{[(m-1)/m]\, a[a(t+\tau) - x]\}^{1/(m-1)}, & 0 \leqslant x \leqslant a(t+\tau), \\ 0, & a(t+\tau) < x < \infty. \end{cases}$$

(b) *Similarity Solutions of Type II*

An application of Theorem 2 to Eq. (2b) yields the existence of a nontrivial similarity solution of type u_2 with compact support if and only if $\alpha \leqslant -1/(m-1)$. In all cases $u_2(0, t)$ is positive on $(0, T)$.

When $\alpha = -1/(m-1)$, Eq. (2b) is the special case of (4) with $p = 0$ and $q = 1/(m-1)$. It therefore has an exact solution given by expression (9) with $q = 1/(m-1)$. We derive that

$$u_2(x, t) = \begin{cases} \left\{ \dfrac{m-1}{2m(m+1)} \dfrac{(a^2 - x^2)}{(\tau - t)} \right\}^{1/(m-1)} & 0 < x < a \\ 0, & a \leqslant x < \infty \end{cases}$$

(cf. [9]).

(c) *Similarity Solutions of Type III*

By Theorem 2 now applied to Eq. (2c), there are nontrivial similarity solutions of type u_3 with compact support if and only if $\alpha > 0$. In every case the lateral boundary data must be positive.

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