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Some Counterexamples in the Partition Calculus*

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We show that the pairs (2-element subsets; edges of the complete graph) of a set of cardinality $\aleph_1$ can be colored with 4 colors so that every uncountable subset contains pairs of every color, and that the pairs of real numbers can be colored with $\aleph_0$ colors so that every set of reals of cardinality $2^{\aleph_0}$ contains pairs of every color. These results are counterexamples to certain transfinite analogs of Ramsey's theorem. Results of this kind were obtained previously by Sierpiński and by Erdös, Hajnal, and Rado. The Erdös-Hajnal-Rado result is much stronger than ours, but they used the continuum hypothesis and we do not. As by-products, we get an uncountable tournament with no uncountable transitive subtournament, and an uncountable partially ordered set such that every uncountable subset contains an infinite antichain and a chain isomorphic to the rationals. The tournament was pointed out to us by R. Laver, and is included with his permission.

1. INTRODUCTION

The cardinal number of a set S is $|S|$. An ordinal is identified with the set of all smaller ordinals, and a cardinal is identified with its initial ordinal. If S is a set and r is a cardinal,

$$[S]^r = \{X \subseteq S : |X| = r\}.$$

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For cardinals a, b, r, k $(k > 0)$, the symbol $a \to [b]^r_k$ denotes the following statement, and $a \nrightarrow [b]^r_k$ denotes its negation: if $|A| = a$ and $|I| = k$, then, for any family of pairwise disjoint sets $K_i \subseteq [A]^r$ $(i \in I)$, there exist $i_0 \in I$ and $B \subseteq A$ such that $|B| = b$ and $[B]^r \cap K_{i_0} = \varnothing$. Thus, $a \nrightarrow [b]^2_k$ holds iff the edges of the complete graph with a vertices can be colored with k colors so that every complete subgraph with b vertices has edges of every color. This notation is due to Erdös, Hajnal, and Rado [3, p. 144]. In this notation, Ramsey's theorem [10, Theorem A] says that $\aleph_0 \to [\aleph_0]^r_2$ for all $r < \aleph_0$.

The first negative result is due to Sierpiński, who proved [11] in effect that $2^{\aleph_0} \nrightarrow [\aleph_1]^2_2$. (We remark that an easy generalization of Sierpiński's proof shows that $2^{\aleph_0} \nrightarrow [\aleph_1]^r_{r!(r-1)!}$ for every positive integer r.) Erdös, Hajnal, and Rado proved [3, Theorem 17, p. 145] that, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, then $\aleph_{\alpha+1} \nrightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$. Erdös and Hajnal asked [2, Problem 15, p. 25] if any of the statements $2^{\aleph_0} \nrightarrow [\aleph_1]^2_3$, $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_3$, $\aleph_1 \nrightarrow [\aleph_1]^2_3$ can be proved without the continuum hypothesis. Clearly, the first statement is the strongest of the three; as we have not been able to prove it, we conjecture that the positive relation $2^{\aleph_0} \to [\aleph_1]^2_3$ is consistent with ZFC (Zermelo-Fraenkel set theory including the axiom of choice). The other two statements are provable in ZFC; indeed we prove $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$ (Theorem 1) and $\aleph_1 \nrightarrow [\aleph_1]^2_4$ (Theorem 2). The weaker results $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_n$ (for $n < \aleph_0$) and $\aleph_1 \nrightarrow [\aleph_1]^2_3$ were proved independently by R. Laver (private communication). Our results were announced in [6].

Theorem 1 seems best possible, in view of a result of R. Solovay [9, Theorem 1.13, p. 11] which says, in particular, that, if $2^{\aleph_0}$ is real-measurable, then $2^{\aleph_0} \to [2^{\aleph_0}]^r_{\aleph_1}$ for every $r < \aleph_0$. As for Theorem 2, to the best of our knowledge everything from $\aleph_1 \nrightarrow [\aleph_1]^2_5$ to $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ is open with regard to its provability in ZFC. However, we know that $\aleph_1 \nrightarrow [\aleph_1]^3_{\aleph_1}$; in fact, $\aleph_n \nrightarrow [\aleph_n]^{n+2}_{\aleph_n}$ for every positive integer n. This was concluded by Shelah from a result of Galvin; the proof will appear elsewhere.

2. $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$

A set $B \subseteq \omega_\alpha$ is *cofinal* in ω_α if, for every $\mu \in \omega_\alpha$, there is $\nu \in B$ such that $\mu \leqslant \nu$. For an infinite cardinal $\aleph_\alpha$, $cf\, \aleph_\alpha$ denotes the least cardinal b such that ω_α has a cofinal subset of cardinality b. It is well known that $\mathrm{cf}\, 2^{\aleph_0} > \aleph_0$. Let R be the real line.

LEMMA 1. *Let* $\aleph_\alpha = 2^{\aleph_0}$, $n \in \omega$. *Let* $f_0, \dots, f_n : \omega_\alpha \to R$ *be* 1-*to*-1 *mappings, and suppose that for any cofinal* $B \subseteq \omega_\alpha$ *there exist* $\mu, \nu \in B$ *such that* $f_i(\mu) < f_i(\nu)$ *for all* $i \leqslant n$. *Then there is a* 1-*to*-1 *mapping* $g : \omega_\alpha \to R$ *such that, for any cofinal* $B \subseteq \omega_\alpha$:

(1) *there exist* μ, $\nu \in B$ *such that* $f_i(\mu) < f_i(\nu)$ *for all* $i \leqslant n$, *and* $g(\mu) > g(\nu)$;

(2) *there exist* μ, $\nu \in B$ *such that* $f_i(\mu) < f_i(\nu)$ *for all* $i \leqslant n$, *and* $g(\mu) < g(\nu)$.

Proof. We write $f(\mu) < f(\nu)$ to mean that $f_i(\mu) < f_i(\nu)$ for all $i \leqslant n$. Since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, we can write $[\omega_\alpha]^{\aleph_0} = \{A_\mu : \mu < \omega_\alpha\}$.

Let $\nu \in \omega_\alpha$, and suppose that $g(\mu)$ has been defined for all $\mu < \nu$. Now choose $g(\nu) \in R$ so that:

(3) $g(\nu) \neq g(\mu)$ for all $\mu < \nu$;

(4) if $\mu < \nu$ and $A_\mu \subseteq \nu$, then $g(\nu)$ is neither the sup nor the inf of $\{ g(\lambda) : \lambda \in A_\mu$ and $f(\lambda) < f(\nu)\}$. So $g : \omega_\alpha \to R$ is defined and 1-to-1.

Consider any cofinal $B \subseteq \omega_\alpha$, and suppose that (1) fails; this means that, for μ, $\nu \in B$, $f(\mu) < f(\nu) \Rightarrow g(\mu) < g(\nu)$. Define $h : \omega_\alpha \to R^{n+2}$ so that $h(\nu) = (f_0(\nu),..., f_n(\nu), g(\nu))$. Since R^{n+2} is a separable metric space, we can choose $\mu \in \omega_\alpha$ so that $A_\mu \subseteq B$ and $h(A_\mu)$ is dense in $h(B)$. Let such a μ be fixed. Recall that $cf\, 2^{\aleph_0} > \aleph_0$; consequently, if we partition a cofinal subset of ω_α into countably many parts, at least one of the parts will be cofinal. Let $B' = \{\nu \in B : \mu < \nu$ and $A_\mu \subseteq \nu\}$; then B' is cofinal and, for every $\nu \in B'$, $g(\nu) > \sup\{ g(\lambda) : \lambda \in A_\mu, f(\lambda) < f(\nu)\}$. Choose $\epsilon > 0$ and $B'' \subseteq B'$ so that B'' is cofinal and, for every $\nu \in B''$,

$$g(\nu) - \epsilon > \sup\{ g(\lambda) : \lambda \in A_\mu, f(\lambda) < f(\nu)\}.$$

Choose a cofinal $B''' \subseteq B''$ so that $| g(\nu_1) - g(\nu_2)| < \epsilon/2$ for all $\nu_1, \nu_2 \in B'''$. Choose $\lambda_0, \nu \in B'''$ so that $f(\lambda_0) < f(\nu)$. Since $h(A_\mu)$ is dense in $h(B)$, we can choose $\lambda \in A_\mu$ so that $| g(\lambda) - g(\lambda_0)| < \epsilon/2$ and

$$| f_i(\lambda) - f_i(\lambda_0)| < f_i(\nu) - f_i(\lambda_0) \qquad \text{for all} \quad i \leqslant n.$$

Then $f(\lambda) < f(\nu)$ and $g(\lambda) > g(\nu) - \epsilon$; but this is a contradiction, since $\nu \in B''$. This proves that (1) holds; (2) follows by symmetry. This completes the proof of Lemma 1.

LEMMA 2. *Let* $\aleph_\alpha = 2^{\aleph_0}$. *There are* 1-*to*-1 *mappings* $f_n : \omega_\alpha \to R (n \in \omega)$ *such that, for any cofinal* $B \subseteq \omega_\alpha$ *and* $n \in \omega$, *there exist* μ, $\nu \in B$ *such that* $f_i(\mu) < f_i(\nu)$ *for all* $i \leqslant n$ *but* $f_{n+1}(\mu) > f_{n+1}(\nu)$.

Proof. Start with any 1-to-1 mapping $f_0 : \omega_\alpha \to R$, and then obtain $f_1, f_2, ...$ by repeated application of Lemma 1.

Clearly, the method used to prove Lemma 1 would also suffice to prove the following generalization of Lemma 2. We do not need it for the proof of Theorem 1, but we state it anyway in case it will have some future use.

LEMMA 3. *Let $\aleph_\alpha = 2^{\aleph_0}$. There are 1-to-1 mappings $f_n : \omega_\alpha \to R(n \in \omega)$ such that, for any cofinal $B \subseteq \omega_\alpha$ and any disjoint finite $I, J \subseteq \omega$, there exist $\mu, \nu \in B$ such that $f_i(\mu) < f_i(\nu)$ for all $i \in I$ and $f_j(\mu) > f_j(\nu)$ for all $j \in J$.*

LEMMA 4. *Let $\aleph_\alpha = 2^{\aleph_0}$. There are pairwise disjoint sets*

$$K_n \subseteq [\omega_\alpha]^2 \ (n \in \omega)$$

such that $[B]^2 \cap K_n \neq \varnothing$ for every cofinal $B \subseteq \omega_\alpha$ and every $n \in \omega$.

Proof. Let the mappings $f_n : \omega_\alpha \to R$ be as in Lemma 2, and let K_n consist of all pairs $\{\mu, \nu\} \in [\omega_\alpha]^2$ such that $f_i(\mu) < f_i(\nu)$ for all $i \leqslant n$ but $f_{n+1}(\mu) > f_{n+1}(\nu)$.

THEOREM 1. $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$.

Theorem 1 follows immediately from Lemma 4. Note that Lemma 4 says more than Theorem 1 if $2^{\aleph_0}$ is singular. For example, Lemma 4 implies $cf\, 2^{\aleph_0} \nrightarrow [\mathrm{cf}\, 2^{\aleph_0}]^2_{\aleph_0}$. Thus, if $cf\, 2^{\aleph_0} = \aleph_1$, then $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0}$; compare this with Theorem 2 and with the Erdös-Hajnal-Rado theorem mentioned in the introduction.

3. $\aleph_1 \nrightarrow [\aleph_1]^2_4$

The order type of a (totally) ordered set $(S, <)$ is denoted by tp $(S, <)$, or simply tp S if there is no danger of confusion; if $\varphi = \mathrm{tp}\, S$, we write $|\varphi|$ for $|S|$. If φ is an order type, φ^* is the converse type; i.e., $\varphi^* = \mathrm{tp}(S, >)$ if $\varphi = \mathrm{tp}(S, <)$. If φ, ψ are order types, $\varphi \geqslant \psi$ means that an ordered set of type φ has a subset of type ψ. The order types of the real numbers and the rational numbers are λ and η, respectively.

An order type φ is a *Specker type* if $|\varphi| = \aleph_1$, $\varphi \ngeqslant \omega_1$, $\varphi \ngeqslant \omega_1{}^*$, and there is no uncountable order type ψ such that $\psi \leqslant \varphi$ and $\psi \leqslant \lambda$. The existence of such types was shown by E. Specker [4, p. 443, n. 7]. As Specker's construction does not appear in print, we mention that it is like the construction of an Aronszajn tree, which can be found in [8, p. 2]; the only difference is that the elements (certain increasing transfinite sequences of rationals), instead of being partially ordered by inclusion, are totally ordered lexicographically.

LEMMA 5. *Let* tp S *be a Specker type, and let* $f: S \to [0, 1]$ *be* 1-to-1. *Then there exists* $x_0 \in S$ *such that* $\{y \in S : y > x_0$ *and* $f(y) > f(x_0)\}$ *is uncountable.*

Proof. For $x \in S$, let $g(x) =$ the least $t \in [0, 1]$ such that $f(y) < t$ for all but countably many $y > x$. Since g is a decreasing real-valued function on S, g can only have countably many different values; otherwise there would be an uncountable type $\psi \leqslant \operatorname{tp} S$, $\psi \leqslant \lambda$. Hence g is constant on some uncountable $S_0 \subseteq S$. We can choose $x \in S_0$ so that $\{y \in S_0 : y > x\}$ is uncountable; otherwise S_0 would have a subset of type $\omega_1{}^*$. Now we can choose $x_0 \in S_0$ so that $x_0 > x$ and $f(x_0) < g(x)$. Since $g(x_0) = g(x) > f(x_0)$, there are uncountably many $y > x_0$ such that that $f(y) > f(x_0)$. This completes the proof of Lemma 5.

LEMMA 6. *If* $\operatorname{tp}(S, <_1)$ *is a Specker type and* $\operatorname{tp}(S, <_2) \leqslant \lambda$, *then there is an* $x \in S$ *such that* $\{y \in S : x <_1 y$ *and* $x <_2 y\}$ *is uncountable.*

Proof. This is just a restatement of Lemma 5.

THEOREM 2. $\aleph_1 \nrightarrow [\aleph_1]^2_4$.

Proof. Let $|S| = \aleph_1$. Choose orderings $<_0, <_1, <_2$ so that $\operatorname{tp}(S, <_0) = \omega_1$, $\operatorname{tp}(S, <_1)$ is a Specker type, and $\operatorname{tp}(S, <_2) \leqslant \lambda$. Analogously to Sierpiński's proof of $2^{\aleph_0} \nrightarrow [\aleph_1]^2_2$, we define a partition

$$[S]^2 = K_1 \cup K_2 \cup K_3 \cup K_4,$$

where

$$\begin{aligned}
K_1 &= \{\{x, y\} : x <_0 y,\ x <_1 y,\ x <_2 y\},\\
K_2 &= \{\{x, y\} : x <_0 y,\ x <_1 y,\ y <_2 x\},\\
K_3 &= \{\{x, y\} : x <_0 y,\ y <_1 x,\ x <_2 y\},\\
K_4 &= \{\{x, y\} : x <_0 y,\ y <_1 x,\ y <_2 x\}.
\end{aligned}$$

Consider any uncountable $S' \subseteq S$. Since $\operatorname{tp}(S', <_1)$ is still a Specker type, and $\operatorname{tp}(S', <_2) \leqslant \lambda$, by Lemma 6 there is an $x \in S'$ such that

$$\{y \in S' : x <_1 y, x <_2 y\}$$

is uncountable. Since $\operatorname{tp}(S', <_0) = \omega_1$, we can choose $y \in S'$ so that $x <_1 y, x <_2 y$, and $x <_0 y$. This shows that $[S']^2 \cap K_1 \neq \varnothing$. Since $\operatorname{tp}(S', >_1)$ is a Specker type and $\operatorname{tp}(S', >_2) \leqslant \lambda$, it follows by symmetry that $[S']^2 \cap K_i \neq \varnothing$ for $i = 2, 3, 4$. This completes the proof of Theorem 2.

4. Tournaments and Partially Ordered Sets

In this section we give some further applications of the ideas used in Theorem 2. First we need to improve Lemma 6. If A, B are subsets of an ordered set $(S, <)$, then $A < B$ means that $a < b$ for all $a \in A$ and $b \in B$.

LEMMA 7. *If* $\operatorname{tp}(S, <_1)$ *is a Specker type and* $\operatorname{tp}(S, <_2) \leqslant \lambda$, *then there are uncountable sets* $A, B \subseteq S$ *such that* $A <_1 B$ *and* $A <_2 B$.

Proof. Let $U =$ the set of all $x \in S$ such that $\{y \in S : x <_1 y, x <_2 y\}$ is countable; let $L =$ the set of all $x \in S$ such that $\{y \in S : y <_1 x, y <_2 x\}$ is countable. Then U is countable, or else we would get a contradiction by applying Lemma 6 to U; symmetrically, L is countable. Choose

$$x \in S - (U \cup L),$$

and let $A = \{y \in S : y <_1 x, y <_2 x\}$ and $B = \{y \in S : x <_1 y, x <_2 y\}$.

LEMMA 8. *Let* $\operatorname{tp} Q = \eta$. *If* $\operatorname{tp}(S, <_1)$ *is a Specker type and*

$$\operatorname{tp}(S, <_2) \leqslant \lambda,$$

then there are uncountable sets $A_t \subseteq S$ $(t \in Q)$ *such that* $s < t \Rightarrow A_s < A_t$.

Proof. Iterate Lemma 7. This is similar to a proof of Erdös and Rado [4, Lemma 1, pp. 446-447].

A *tournament* is an oriented complete graph; in other words, it consists of a set S and a binary relation R such that, for any $x, y \in S$, exactly one of the alternatives $x = y$, xRy, yRx holds. We call the elements of S *players*, and instead of xRy we write: x *beats* y. *Transitive* tournament and *subtournament* of a tournament are defined in the obvious way.

If a and b are cardinals, the symbol $T(a, b)$ denotes the statement: every tournament with a players has a transitive subtournament with b players. Stearns [12] showed that $T(2^n, n+1)$ holds for every finite n. Erdös and Rado [5, Theorem 4, p. 632] considered the transfinite case; they showed, for example, that $T(\aleph_0, \aleph_0)$ and $T((2^{\aleph_0})^+, \aleph_1)$, where a^+ denotes the least cardinal $> a$. R. Laver observed that the idea we used to prove Theorem 2 can also be used to show that $T(\aleph_1, \aleph_1)$ is false.

THEOREM 3 (Laver). *There is an uncountable tournament which has no uncountable transitive subtournament.*

Proof. Let $|S| = \aleph_1$. Choose orderings $<_0, <_1, <_2$ as in the proof of Theorem 2. The elements of S are the players, and x beats y iff

$|\{i : x <_i y\}| \geqslant 2$. Consider any uncountable $S' \subseteq S$. By Lemma 7, there are uncountable sets $B, D \subseteq S'$ such that $B <_1 D$ and $B >_2 D$. By Lemma 7 again, there are uncountable sets $C, A \subseteq D$ such that $C <_1 A$ and $C <_2 A$. Choose $a \in A, b \in B, c \in C$ so that $a <_0 b <_0 c$. Then $b <_1 c <_1 a$ and $c <_2 a <_2 b$. So a beats b, b beats c, and c beats a; i.e., S' is not transitive.

Our next theorem was also known to Erdös and Hajnal (private communication). It shows that, in some sense, $T(\aleph_0, \aleph_0)$ is weaker than Ramsey's theorem.

THEOREM 4. *If* $\aleph_\alpha \to [\aleph_\alpha]^2_3$, *then* $T(\aleph_\alpha, \aleph_\alpha)$.

Proof. The conclusion is trivial if $\aleph_\alpha \to [\aleph_\alpha]^2_2$, so we can assume that $\aleph_\alpha \not\to [\aleph_\alpha]^2_2$. Then by a theorem of Hanf [7] there is an order type φ such that $|\varphi| = \aleph_\alpha$, $\varphi \not\geqslant \omega_\alpha$, $\varphi \not\geqslant \omega_\alpha{}^*$. Let S be a tournament, $|S| = \aleph_\alpha$. Choose orderings $<_0$ and $<_1$ so that $\operatorname{tp}(S, <_0) = \omega_\alpha$ and $\operatorname{tp}(S, <_1) = \varphi$. Define a partition $[S]^2 = K_1 \cup K_2 \cup K_3 \cup K_4$, where:

$$\begin{aligned} K_1 &= \{\{x, y\} : x <_0 y,\ x <_1 y,\ x \text{ beats } y\}, \\ K_2 &= \{\{x, y\} : x <_0 y,\ x <_1 y,\ y \text{ beats } x\}, \\ K_3 &= \{\{x, y\} : x <_0 y,\ y <_1 x,\ x \text{ beats } y\}, \\ K_4 &= \{\{x, y\} : x <_0 y,\ y <_1 x,\ y \text{ beats } x\}. \end{aligned}$$

Choose $S' \subseteq S$ so that $|S'| = \aleph_\alpha$ and $[S']^2$ meets as few as possible of the classes K_1, K_2, K_3, K_4. Considering that $\aleph_\alpha \to [\aleph_\alpha]^2_3$, $\varphi \not\geqslant \omega_\alpha$, and $\varphi \not\geqslant \omega_\alpha{}^*$, we must have either $[S']^2 \subseteq K_1 \cup K_3$ or $[S']^2 \subseteq K_1 \cup K_4$ or $[S']^2 \subseteq K_2 \cup K_3$ or $[S']^2 \subseteq K_2 \cup K_4$. Checking each case, we see that S' is transitive.

We do not know if $T(2^{\aleph_0}, \aleph_1)$ is consistent with ZFC. The symbol $2^{\aleph_0} \to [\aleph_1]^2_{4,2}$ of Erdös, Hajnal, and Rado [3, p. 144] denotes the following strengthening of the relation $2^{\aleph_0} \to [\aleph_1]^2_3$ considered in Section 1: for any coloring of the pairs of real numbers with 4 colors, there is an uncountable set of reals which contains pairs of at most 2 colors. A similar argument to the proof of Theorem 4 shows that, if $2^{\aleph_0} \to [\aleph_1]^2_{4,2}$, then $T(2^{\aleph_0}, \aleph_1)$.

A subset X of a partially ordered set is a *chain* if the elements of X are pairwise comparable; an *antichain* if they are pairwise incomparable. Sierpiński [11] constructed an uncountable partially ordered set S having no uncountable chain and no uncountable antichain. In fact, every uncountable subset of S contains an infinite antichain and a chain of type α for every $\alpha < \omega_1$, but no chain of type ω^*. Sierpiński's partial order was obtained by intersecting the usual ordering of the real numbers with a well-ordering. By using a Specker ordering instead of a well-ordering, we get a partially ordered set with somewhat different properties.

THEOREM 5. *There is an uncountable partially ordered set S such that every uncountable subset of S contains an infinite antichain and a chain of type η.*

Proof. Let $\operatorname{tp}(S, <_1)$ be a Specker type and $\operatorname{tp}(S, <_2) \leqslant \lambda$. Define a partial ordering $<$ so that $x < y$ iff $x <_1 y$ and $x <_2 y$. Clearly there is no uncountable chain; i.e., every uncountable subset of S contains a pair of incomparable elements. By a theorem of Dushnik and Miller [1, Theorem 5.25, p. 608], it follows that every uncountable subset of S contains an infinite chain. (See [4, Theorem 44, p. 475] for another proof of the Dushnik-Miller theorem.) On the other hand, it follows from Lemma 8 that every uncountable subset of S contains a chain of type η.

(*Note added June* 12, 1972. Let $|S| = \aleph_1$. Erdös and Hajnal have asked [13, Problem II] if there is a partition $[S]^2 = I_0 \cup I_1, I_0 \cap I_1 = \varnothing$, such that, for any $i \in \{0, 1\}$ and any uncountable set $Z \subseteq S$, there are uncountable sets $A, B \subseteq Z$ such that $\{a, b\} \in I_i$ for all $a \in A. b \in B$. An affirmative answer to this question follows immediately from Lemma 7; let $I_0 = K_1 \cup K_4$ and $I_1 = K_2 \cup K_3$ in the notation of the proof of Theorem 2.)

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