JOURNAL OF FUNCTIONAL ANALYSIS 4, 88–94 (1969)

On Certain Functionals Arising in the Theory of Interpolation Spaces

TORD HOLMSTEDT AND JAAK PEETRE

Department of Mathematics, Lund Institute of Technology, Lund, Sweden

Communicated by J. L. Lions

Received July 4, 1968

1. FORMULATION OF RESULTS AND DISCUSSION

Let $\vec{A} = \{A_0, A_1\}$ be a Banach couple (i.e. A_0 and A_1 are two Banach spaces both continuously embedded in one and the same topological vector space $\mathscr{A}$). If $a \in A_0 + A_1$ (which is also a Banach space continuously embedded in $\mathscr{A}$,

$$a \in A_0 + A_1 \Leftrightarrow \exists a_0 \in A_0,\ a_1 \in A_1 : a = a_0 + a_1,$$

and if $0 < t < \infty$, we set

$$K = K(t) = K(t, a) = K(t, a; \vec{A}) = \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0, a_1 \in A_1}} (\| a_0 \|_{A_0} + t \| a_1 \|_{A_1}). \quad (1)$$

The functional $K(t)$ plays a major role in the theory of *interpolation spaces* (see e.g. [7], [8], [5], [6], [4]). However in some cases (see e.g. [9], [3]) it is more natural to study

$$\begin{aligned} K_p &= K_p(t) = K_p(t, a) = K_p(t, a; \vec{A}) \\ &= \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0, a_1 \in A_1}} (\| a_0 \|_{A_0}^p + t^p \| a_1 \|_{A_1}^p)^{1/p}, \end{aligned} \quad (2)$$

$1 \leqslant p \leqslant \infty$ (with the usual interpretation if $p = \infty$), where p is adjusted to the special problem in question. The purpose of this note is to study more closely the connection between various K_p for different values of p. Clearly $K_1 = K$. On the other hand K_∞ is in a way the most natural one because $a \to K_\infty(t, a)$ is the norm corresponding to the *Minkowsky sum* of the unit balls of the spaces A_0 and tA_1. We shall prove

THEOREM 1. *Let* $1 \leqslant p \leqslant q \leqslant \infty$. *Then*

$$K_p(t) = \inf_s (1 + (t/s)^\pi)^{1/\pi} K_q(s), \qquad 1/\pi = (1/p) - (1/q). \quad (3)$$

THEOREM 2. *Let* $1 \leqslant q \leqslant p \leqslant \infty$. *Then*

$$K_p(t) = \sup_s (1 + (s/t)^\pi)^{-1/\pi} K_q(s), \qquad 1/\pi = (1/q) - (1/p). \tag{4}$$

Note in particular that ($p = q$).

$$K_p(t) = \inf_s \max(1, t/s) K_p(s) = \sup_s \min(1, t/s) K_p(s) \tag{5}$$

which can also easily be shown directly. We may disregard this case in what follows.

Before proceeding to the proofs (see Section 2) let us give one important application of these results. We introduce for each p a binary relation $\mathscr{R}_p$ in $A_0 + A_1$ (i.e. a subset of the Cartesian product $(A_0 + A_1) \times (A_0 + A_1)$) as follows:

$$(a, a') \in \mathscr{R}_p \Leftrightarrow K_p(t, a) \leqslant K_p(t, a').$$

Write $\mathscr{R} = \mathscr{R}_1$. By the same reason as above however $\mathscr{R}_\infty$ is the most natural one. It has already been studied by Gagliardo (cf. e.g. [2]). To see the connection let us prove the following lemma which is of interest in itself.

LEMMA. $(a, a') \in \mathscr{R}_\infty \Leftrightarrow$ *For every* $\epsilon > 0$ *and every decomposition* $a' = a_0' + a_1'$ *of* a', *with* $a_0' \in A_0$, $a_1' \in A_1$, *there exists a decomposition* $a = a_0 + a_1$ *of* a, *with* $a_0 \in A_0$, $a_1 \in A_1$, *such that*

$$\| a_0 \|_{A_0} \leqslant \| a_0' \|_{A_0} + \epsilon, \qquad \| a_1 \|_{A_1} \leqslant \| a_1' \|_{A_1} + \epsilon. \tag{6}$$

Proof. From (6) follows

$$K_\infty(t, a) \leqslant \max(\| a_0 \|_{A_0}, t\| a_1 \|_{A_1}) \leqslant \max(\| a_0' \|_{A_0}, t\| a_1' \|_{A_1}) + \epsilon \max(1, t)$$

Making a_0', a_1' vary we get

$$K_\infty(t, a) \leqslant K_\infty(t, a') + \epsilon \max(1, t)$$

and making ϵ vary

$$K_\infty(t, a) \leqslant K_\infty(t, a'). \tag{7}$$

This settles the implication $\Leftarrow$. For the converse $\Rightarrow$ we choose, given a_0', a_1', t in such a way that

$$a_0' \|_{A_0} + \epsilon/2 = t(\| a_1' \|_{A_1} + \epsilon/2).$$

By definition we can now, given ϵ, find a_0, a_1 so that

$$\max(\| a_0 \|_{A_0}, t\| a_1 \|_{A_1}) \leqslant K_\infty(t, a) + \epsilon/2 \min(1, t),$$

and we also have

$$K_\infty(t, a') \leqslant \max(\| a_0' \|_{A_0}, t\| a_1' \|_{A_1}).$$

From (7) now follows

$$\| a_0 \|_{A_0} \leqslant \max(\| a_0 \|_{A_0}, t\| a_1 \|_{A_1}) \leqslant K(t, a) + \epsilon/2 \min(1, t)$$

$$\leqslant K_\infty(t, a') + \epsilon/2 \leqslant \max(\| a_0' \|_{A_0}, t\| a_1' \|_{A_1}) + \epsilon/2 = \| a_0' \|_{A_0} + \epsilon.$$

In the same way we find

$$\| a_1 \|_{A_1} \leqslant \| a_1' \|_{A_1} + \epsilon.$$

The proof is complete.

Returning to the general case we find at once

COROLLARY (*of Theorem* 1 *and Theorem* 2). $\mathscr{R}_p$ *is independent of* p, i.e. $\mathscr{R}_p = \mathscr{R}$.

Remark. Let us call a K_p-space a Banach space A continuously embedded in $\mathscr{A}$ such that (cf. [2])

$$a' \in A, \qquad (a, a') \in \mathscr{R}_p \Leftrightarrow a \in A, \qquad \| a \|_A \leqslant \| a' \|_A .$$

(These are particular interpolation spaces.) Let us say K-space instead of K_1-space. Then we may formulate the corollary also as follows. *Every* K_p*-space is a* K*-space and vice versa.*

EXAMPLE. Choose $A_0 = L_p = L_p(R^n)$, $A_1 = \dot{W}_p = \dot{W}_p(R^n)$ (i.e. $a \in \dot{W}_p$ if and only if grad $a \in L_p$). We say that a is a *contraction* of a' if there exists a real valued function f satisfying

$$|f(u) - f(v)| \leqslant | u - v |$$

such that $a = f \circ a'$. Contractions have been much studied by Beurling (cf. e.g. [1]). We claim that if a is a contraction of a' then $(a, a') \in \mathscr{R}$. Indeed let

$$a' = a_0' + a_1', \qquad a_0' \in L_p, \qquad a_1' \in \dot{W}_p .$$

Taking

$$a_0 = f \circ a' - f \circ a'_1, \qquad a_1 = f \circ a'_1$$

it follows easily that

$$| a_0(x)| \leqslant | a'_0(x)|, \qquad | \operatorname{grad} a_1(x)| \leqslant | \operatorname{grad} a'_1(x)| \qquad (x \in R^n)$$

so that

$$\| a_0 \|_{L_p} \leqslant \| a'_0 \|_{L_p}, \qquad \| a_1 \|_{W_p} \leqslant \| a'_1 \|_{W_p}.$$

Therefore $(a, a') \in \mathscr{R}_\infty$ by the lemma and so $(a, a') \in \mathscr{R}$ by the corollary. By the remark we see that in this case, $\vec{A} = \{L_p, \dot{W}_p\}$, *every K-space is contraction invariant.* It is conceivable that the converse is not true. It follows also that $K_p(t, a)$ only depends on $| a(x) - a(y)|$, i.e. we have

$$K_p(t, a) = \Phi_p[| a(x) - a(y)|]$$

for a certain functional Φ_p (depending on t too). This should be confronted with other expressions for $K_p(t, a)$ (cf. e.g. [*8*]). There arises the question of the explicit determination of Φ_p. If $n = 1$ and $p = 2$ it follows from [*9*] and from Parseval's formula that

$$K_2(t, a) = \left(1/t \iint | a(x) - a(y)|^2 e^{-|x-y|/t} \, dx \, dy\right)^{1/2}.$$

2. Proofs

In the proof of Theorem 1 and Theorem 2 we shall use the "Gagliardo indicator" (cf. [*2*]):

$$\Gamma = \Gamma(a) = \{\vec{x} = (x_0, x_1) \mid \exists a_0 \in A_0, \ a_1 \in A_1 :$$
$$a = a_0 + a_1, \ \| a_0 \|_{A_0} \leqslant x_0, \ \| a_1 \|_{A_1} \leqslant x_1\}.$$

This is thus a convex plane set. (Note that the lemma can be reformulated so that

$$(a, a') \in \mathscr{R}_\infty \Leftrightarrow \overline{\Gamma(a)} \supset \overline{\Gamma(a')} \cdot)$$

Let $\partial\Gamma$ denote the boundary. From (2) we at once obtain

$$K_p(t) = \inf_{\vec{x} \in \Gamma} (x_0{}^p + t^p x_1{}^p)^{1/p} = \inf_{\vec{x} \in \partial\Gamma} (x_0{}^p + t^p x_1{}^p)^{1/p}. \tag{8}$$

Proof of Theorem 1. In view of Hölder's inequality we have

$$(x_0{}^p + t^p x_1{}^p)^{1/p} = \inf_s ((1 + (t/s)^\pi)^{1/\pi}(x_0{}^q + s^q x_1{}^q)^{1/q}$$

Therefore by (8) we get

$$\begin{aligned} K_p(t) &= \inf_{\vec{x}\in\Gamma} \inf_s (1 + (t/s)^\pi)^{1/\pi}(x_0{}^q + s^q x_1{}^q)^{1/q} \\ &= \inf_s (1 + (t/s)^\pi)^{1/\pi} \inf_{\vec{x}\in\Gamma} (x_0{}^q + s^q x_1{}^q)^{1/q} \\ &= \inf_s (1 + (t/s)^\pi)^{1/\pi} K_q(s), \end{aligned}$$

which thus establishes (3).

Proof of Theorem 2. The inequality $\geqslant$ in (4) follows from Theorem 1 (or directly). It remains to prove the inequality $\leqslant$. To this end we choose $x_0 = x_0(s)$ and $x_1 = x_1(s)$ so that the inf is assumed in (8), with p replaced by q and t by s. In other words we have

$$K_q(s) = ((x_0(s))^q + s^q(x_1(s))^q)^{1/q}. \tag{9}$$

We distinguish two cases: a) $1 < q < \infty$, b) $q = 1$. (We may exclude $q = \infty$, because of (5).)

a) $1 < q < \infty$. In this case $x_0(s)$ and $x_1(s)$ are uniquely determined. This follows from the fact that

$$\vec{x} \to (x_0{}^q + s^q x_1{}^q)^{1/q}$$

is strictly convex. From the uniqueness follows easily that $x_0(s)$ and $x_1(s)$, as functions of s, are continuous. Also $x_0(s)$ is increasing and $x_1(s)$ decreasing. To see this we rewrite (8) as

$$K_q(s) = \inf_{x_0} (x_0{}^q + s^q(\phi(x_0))^q)^{1/q},$$

where $x_1 = \phi(x_0)$ is the equation for $\partial\Gamma$, and note that, since $\phi(x_0)$ is convex and decreasing, and $q > 1$, the function $(\phi(x_0))^q$ will have the same properties. From Hölder's inequality and (8) follows that

$$K_q(s) \leqslant ((x_0(s))^p + t^p(x_1(s))^p)^{1/p}(1 + (s/t)^\pi)^{1/\pi}$$

with equality if and only if

$$[(x_0(s))^\pi]/1 = [t^p(x_1(s))^p]/[(s/t)^\pi],$$

or

$$[x_0(s)]/[x_1(s)] = t^{1+\pi/p}s^{-\pi/p}.$$

Now $f(s) = [x_0(s)]/[x_1(s)]$ is increasing and $g(s) = t^{1+\pi/p}s^{-\pi/p}$ is decreasing, with $\lim_{s\to 0} g(s) = \infty$ and $\lim_{s\to\infty} g(s) = 0$. Since both are continuous it follows that there exists a number σ (depending thus on t) so that $f(\sigma) = g(\sigma)$, i.e.

$$((x_0(\sigma))^p + t^p(x_1(\sigma))^p)^{1/p} = (1 + (\sigma/t)^\pi)^{-1/\pi}K_q(\sigma).$$

But by (2)

$$K_p(t) \leqslant ((x_0(\sigma))^p + t^p(x_1(\sigma))^p)^{1/p}.$$

Therefore we get

$$K_p(t) \leqslant (1 + (\sigma/t)^\pi)^{-1/\pi}K_q(\sigma) \leqslant \sup_s (1 + (s/t)^\pi)^{-1/\pi})\, K_q(s)$$

which completes the proof in this case.

b) $q = 1$, Now it may happen that $x_0(s)$ and $x_1(s)$ are not unique. Let $s = \sigma$ be a value of non-uniqueness. Then $(x_0(\sigma), x_1(\sigma))$ may be any point on a straight line segment contained in $\partial\Gamma$, along which thus the inf is assumed. Moreover $x_0(s)$ and $x_1(s)$ have both jumps at $s = \sigma$. The function $f(s)$ defined as above is not continuous. Therefore at the start we cannot always find σ so that $f(\sigma) = g(\sigma)$ but at any rate we can achieve

$$\lim_{s\to\sigma-0} f(s) \leqslant g(\sigma) \leqslant \lim_{s\to\sigma+0} f(s).$$

But by the above we can then redefine $x_0(\sigma)$ and $x_1(\sigma)$ so that still $f(\sigma) = g(\sigma)$ can be obtained. The rest of thep roof runs as in case (a).

References

1. Beurling, A. and Deny, J., Dirichlet spaces. *Proc. Nat. Acad. Sci. U.S.* **45** (1959), 208–215.

2. Gagliardo, E., Una stuttura unitaria in diverse famiglie de spazi funzionali. *I. Ricerche Mat.* **10** (1961), 245–281.

3. Goulaouic, C., Prolongements de foncteurs d'interpolation et applications. *Ann. Inst. Fourier* **18** (1968), 1–98.

4. Holmstedt, T., Interpolation d'espaces quasi-normés. *C. R. Acad. Sci. Paris* **264** (1967), 242–244.
5. Krée, P., Interpolation d'espaces qui ne sont ni normés, ni complets. Applications. *Ann. Inst. Fourier* **17** (1968), 137–174.
6. Oklander, E. T., L_{pq} interpolators and the theorem of Marcinkiewicz. *Bull. Amer. Math. Soc.* **72** (1966), 49–53.
7. Peetre, J., Nouvelles propriétés d'espaces d'interpolation. *C. R. Acad. Sci. Paris* **256** (1963), 1424–1426.
8. Peetre, J., Espaces d'interpolation, généralisations, applications. *Rend. Sem. Mat. Fis. Milano* **34** (1964), 133–164.
9. Peetre, J., On an interpolation theorem of Foias and Lions. *Acta Sci. Math.* (Szeged) **25** (1964), 255–261.