Numerical Treatment of a Generalized Vandermonde System of Equations

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ABSTRACT

A stable method is proposed for the numerical solution of a linear system of equations having a generalized Vandermonde matrix. The method is based on Gaussian elimination and establishes explicit expressions for the elements of the resulting upper triangular matrix. These elements can be computed by means of sums of exclusively positive terms. In an important special case these sums can be reduced to simple recursions. Finally the method is retraced for the case of a confluent type of generalized Vandermonde matrix.

1. INTRODUCTION

It is well known [1] that the system of exponentials

$$\{e^{\lambda_1 x}, e^{\lambda_2 x},\dots,e^{\lambda_{n+1} x}\}$$

satisfies the Haar condition on an arbitrary interval, i.e.

$$\det(e^{\lambda_i x_k})\neq 0$$

for every possible choice of $n+1$ points x_k. As a consequence, this is also true for the system

$$\{x^{j_1},x^{j_2},\dots,x^{j_{n+1}}\} \tag{1.1}$$

(where the powers are nonnegative integers) on each positive interval. Hence a function of the form

$$\Phi(x)=\sum_{i=1}^{n+1} c_i x^{j_i} \tag{1.2}$$

is the unique interpolant of a given function $f(x)$ when subjected to the conditions

$$\sum_{i=1}^{n+1} c_i x_k^{j_i}=f(x_k), \qquad x_k>0, \qquad k=1,2,\ldots,n+1, \tag{1.3}$$

or is the unique minimax approximation to a given function on a positive interval. When in the latter case $\Phi(x)$ is computed by means of Remez's algorithm (see [2], for example), one has to solve in each iteration step a system of the type (1.3), where the x_k denote now approximate extremal points of the function $f(x)-\Phi(x)$.

Thus, in both problems arises a linear system

$$Mc=g,$$

where the matrix M has the form

$$M\equiv\begin{bmatrix} x_1^{j_1} & x_1^{j_2} & \cdots & x_1^{j_{n+1}} \\ \cdot & \cdot & \cdots & \cdot \\ x_{n+1}^{j_1} & x_{n+1}^{j_2} & \cdots & x_{n+1}^{j_{n+1}} \end{bmatrix}.$$

With $i_k=j_{k+1}-j_1$, $k=1, 2,\ldots,n$, this system can be reduced to

$$V_{i_1,i_2,\ldots,i_n}(x_1,x_2,\ldots,x_{n+1})\cdot c=f, \tag{1.4}$$

where

$$V_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})\equiv\begin{bmatrix} 1 & x_1^{i_1} & \cdots & x_1^{i_n} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_{n+1}^{i_1} & \cdots & x_{n+1}^{i_n} \end{bmatrix} \tag{1.5}$$

is a generalized Vandermonde matrix. Determinants of such matrices are special cases of so-called alternants, an extensive treatment of which can be found in Muir's book [3]. This type of determinants is also mentioned in Pólya and Szegö [4, pp. 45, 229]; it is shown there that they always assume a positive value if the x_k are real and satisfy

$$0<x_1<\cdots<x_{n+1}. \tag{1.6}$$

Since interpolation or approximation with the set of powers (1.1) is performed on a positive interval, we always may arrange the system (1.4) so that (1.6) holds true.

The purpose of this paper is to investigate the numerical treatment of the generalized Vandermonde system (1.4), with a view towards methods of solution which are a match for instability occurring in the Gaussian elimination process, owing to possible ill-conditioning of the matrix of the system. We mention here similar investigations for ordinary Vandermonde and confluent Vandermonde systems by Ballester and Pereyra [5], Björck and Pereyra [6], and Galimberti and Pereyra [7].

2. AUXILIARY FORMULAS

We will treat matrices and determinants of the type (1.5) with the help of multivariate homogeneous polynomials $P_k(x_1,\dots,x_r)$ which we define recursively by

$$P_k(x_1,\dots,x_r)=x_rP_{k-1}(x_1,\dots,x_r)+P_k(x_1,\dots,x_{r-1}); \tag{2.1.a}$$

$$P_k(x)=x^k; \qquad P_0(x_1,\dots,x_r)\equiv 1; \qquad P_{-s}(x_1,\dots,x_r)\equiv 0, \quad s>0. \tag{2.1.b}$$

It is shown in the appendix that these polynomials have the following properties:

$$P_p(x_1,\dots,x_r)=\sum_{k=0}^{q} x_r^kP_{p-k}(x_1,\dots,x_{r-1})+x_r^{q+1}P_{p-q-1}(x_1,\dots,x_r),$$

$$q \text{ arbitrary}; \tag{A1}$$

$$P_p(x_1,\dots,x_q,x_i)-P_p(x_1,\dots,x_q,x_{q+1})=(x_i-x_{q+1})P_{p-1}(x_1,\dots,x_q,x_{q+1},x_i); \tag{A2}$$

$$P_p(x_1,\dots,x_r)-x_1^{p-q}P_q(x_1,\dots,x_r)=\sum_{i=0}^{r-2} x_{r-i}P_q(x_{r-i},\dots,x_r)P_{p-q-1}(x_1,\dots,x_{r-i}),$$

$$r\geqslant 2, \quad p\geqslant q\geqslant 0. \tag{A3}$$

Repeated use of Eqs. (A2) and (A3) yields the formula

$$\det\left[V_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})\right] = D_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})\cdot\det\left[V(x_1,\ldots,x_{n+1})\right], \tag{2.2}$$

where the second factor in the right member denotes the ordinary Vandermonde determinant, and the first factor is an $n\times n$ determinant defined by (and denoted by its kth row)

$$D_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1}) = \left|P_{i_k-1}(x_1,x_2)\; P_{i_k-2}(x_1,x_2,x_3)\cdots P_{i_k-n}(x_1,\ldots,x_{n+1})\right|. \tag{2.3}$$

Equations (2.2), (2.3) are the equivalent of a formula by Muir [3, Sec. 338]. If the principal minor of degree $n-1$ in $\det[V_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})]$ is Vandermonde, i.e., $i_k=k$, $k=1,2,\ldots,n-1$, and $i_n=n+p$, $p\geqslant 1$, then (2.3) clearly reduces to

$$D_{1,2,\ldots,n-1,n+p}(x_1,\ldots,x_{n+1}) = P_p(x_1,\ldots,x_{n+1}). \tag{2.4}$$

In the sequel we will need the following property of D-determinants:

$$\begin{vmatrix} D_{i_1,\ldots,i_{p-1},i_q}(x_1,\ldots,x_p,x_{q+1}) & D_{i_1,\ldots,i_{p-1},i_q}(x_1,\ldots,x_p,x_{p+1}) \\ D_{i_1,\ldots,i_{p-1},i_p}(x_1,\ldots,x_p,x_{q+1}) & D_{i_1,\ldots,i_{p-1},i_p}(x_1,\ldots,x_p,x_{p+1}) \end{vmatrix}$$
$$= (x_{q+1}-x_{p+1})D_{i_1,\ldots,i_{p-1}}(x_1,\ldots,x_p)D_{i_1,\ldots,i_p,i_q}(x_1,\ldots,x_p,x_{p+1},x_{q+1}), \qquad q>p. \tag{A4}$$

The proof of (A4) is again given in the appendix.

3. CONDITION AND NUMERICAL SOLUTION OF A GENERALIZED VANDERMONDE SYSTEM

D-determinants enable us to compare the conditioning of the numerical problem (1.4) with that of an ordinary Vandermonde system

$$V(x_1,\ldots,x_{n+1})\cdot c = f. \tag{3.1}$$

Indeed, using Eq. (2.2) the elements v_{jk} of $V^{-1}(x_1,\dots,x_{n+1})$ can easily be calculated, to give

$$v_{j1}=(-1)^{j+1}\frac{|V(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+1})|}{|V(x_1,\dots,x_{n+1})|}$$

$$\times\frac{(x_1x_2\cdots x_{n+1})^{i_1}D_{i_2-i_1,\dots,i_n-i_1}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+1})}{D_{i_1,\dots,i_n}(x_1,\dots,x_{n+1})},$$

$$j=1,\dots,n+1; \qquad (3.2.a)$$

$$v_{jk}=(-1)^{j+k}\frac{|V(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+1})|}{|V(x_1,\dots,x_{n+1})|}$$

$$\times\frac{D_{i_1,\dots,i_{k-2},i_k,\dots,i_n}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+1})}{D_{i_1,\dots,i_n}(x_1,\dots,x_{n+1})},$$

$$j=1,\dots,n+1,\quad k=2,\dots,n+1. \qquad (3.2.b)$$

The first factor in both formulas represents the element w_{jk} of $V^{-1}(x_1,\dots,x_{n+1})$. For this element the following expression also holds [8]:

$$w_{jk}=(-1)^{k-1}\frac{\sigma_{n+1-k}^{(j)}}{\prod_{m\neq j}^{1,n+1}(x_m-x_j)}, \qquad (3.3)$$

where $\sigma_p^{(q)}$ is the pth elementary symmetric function in the arguments $x_1,\dots,x_{q-1},\ x_{q+1},\dots,x_{n+1}$. When we use the uniform matrix norm, i.e.,

$$\|A\|=\max_{1<p<r}\sum_{q=1}^{r}|a_{pq}|,\quad A\equiv(a_{pq}),$$

Eqs. (3.2), (3.3) reveal that a generalized Vandermonde matrix (as well as an ordinary one—see Gautschi [9]) may be ill-conditioned.

The ill-conditioning of the matrix of a linear system will manifest itself in the Gaussian elimination process for solving it numerically. For example, even the sophisticated algorithms of Ballester and Pereyra [5] for solving ordinary Vandermonde systems remained fully susceptible to this instability, though it could be removed nicely through Björck and Pereyra's [6] radical modification of the elimination process.

We now investigate how the possible instability of the Gaussian elimination process can be removed (at least partially) when applied to the generalized Vandermonde system (1.4). Thus we search for matrices L and U, of lower and upper triangular form respectively, such that $L\cdot U = V_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})$. We then have

$$Uc = L^{-1}f = g, \qquad g = (g_1,\ldots,g_{n+1})^T.$$

Let $V^{(k+1)} \equiv (a_{pq}^{(k+1)})$ be the matrix obtained after elimination of c_k from the last $n+1-k$ equations, and $f^{(k+1)} \equiv (f_1^{(k+1)},\ldots,f_{n+1}^{(k+1)})^T$ the corresponding new right member, $k=1,2,\ldots,n$. Then $V^{(n+1)} = U$ and $f^{(n+1)} = g$. Application of the appropriate elimination formulas (see Isaacson and Keller [10], for example) with the help of Eq. (A4) leads to

$$a_{pq}^{(2)} = D_{i_{q-1}}(x_1,x_p), \qquad p,q = 2,3,\ldots,n+1;$$

$$f_p^{(2)} = \frac{f_p - f_1}{x_p - x_1}, \qquad p = 2,3,\ldots,n+1;$$

$$a_{pq}^{(k)} = D_{i_1,\ldots,i_{k-2},i_{q-1}}(x_1,\ldots,x_{k-1},x_p), \quad p,q = k,k+1,\ldots,n+1, \quad k = 3,4,\ldots,n+1;$$

$$f_p^{(k)} = \frac{f_p^{(k-1)} D_{i_1,\ldots,i_{k-2}}(x_1,\ldots,x_{k-1}) - f_{k-1}^{(k-1)} D_{i_1,\ldots,i_{k-2}}(x_1,\ldots,x_{k-2},x_p)}{(x_p - x_{k-1}) D_{i_1,\ldots,i_{k-3}}(x_1,\ldots,x_{k-2})},$$

$$p = k,k+1,\ldots,n+1, \quad k = 3,4,\ldots,n+1. \tag{3.4}$$

For U we then have

$$U = \begin{bmatrix} 1 & x_1^{i_1} & x_1^{i_2} & \cdots & x_1^{i_n} \\ & D_{i_1}(x_1,x_2) & D_{i_2}(x_1,x_2) & \cdots & D_{i_n}(x_1,x_2) \\ & & D_{i_1,i_2}(x_1,x_2,x_3) & \cdots & D_{i_1,i_n}(x_1,x_2,x_3) \\ & \text{O} & & \ddots & \vdots \\ & & & & D_{i_1,i_2,\ldots,i_n}(x_1,\ldots,x_{n+1}) \end{bmatrix}. \tag{3.5}$$

The elements of L can also be expressed in terms of D-determinants, but they are not needed for the computation of c.

In the next section, which is of a rather technical nature, we establish a few properties of D-determinants allowing for a very accurate calculation of the elements of U. We state here the main result:

THEOREM 1. *Each D-determinant can be expressed as a sum of exclusively positive terms, provided the arguments* $x_1,\dots,x_{n+1}$ *are positive.*

So ill-conditioning may manifest itself only in the computation of the quantities $f_p^{(k)}$ and in the backward substitution. The Björck-Pereyra modification is essentially based on Newton's formula for polynomial interpolation, and hence is not applicable, since we work here with a set of powers $i_1,\dots,i_{n+1}$ [see (1.1)] which are not "in line".

As a test example we took

$$i_k=k,\qquad k=1,2,\dots,n-1,\quad i_n=n+p;$$

$$f_k=x_k^n,\qquad k=1,2,\dots,n+1.$$

It then easily follows from Eq. (2.4) that

$$c_1=(-1)^{n-1}\frac{P_{p-1}(x_1,\dots,x_{n+1})}{P_p(x_1,\dots,x_{n+1})}\prod_{m=1}^{n+1}x_m, \tag{3.6.a}$$

$$c_{n+1}=\frac{1}{P_p(x_1,\dots,x_{n+1})}, \tag{3.6.b}$$

$$c_k=(-1)^{n-k}\frac{D_{1,\dots,k-2,k,\dots,n,n+p}(x_1,\dots,x_{n+1})}{P_p(x_1,\dots,x_{n+1})},\qquad k=2,3,\dots,n. \tag{3.6.c}$$

By choosing further

$$x_k=\frac{1}{n+p-k},\qquad k=1,2,\dots,n+1,$$

we turn $V_{i_1,\dots,i_n}(x_1,\dots,x_{n+1})$ into an ill-conditioned Hilbert-like matrix.

The unknowns $c_1,c_2,\dots,c_{n+1}$ have been computed on an IBM 370/158 computer in double precision (16 digits) by the method of D-determinants,

TABLE 1

	Number of Exact Figures	
n	Gauss	New Method
3	14	15
5	12	13
7	9	12
9	7	12
11	5	11
13	0	10
15	0	9

by Gauss's method with complete pivoting, and by the formulas (3.6) (the results of the latter being assumed as exact in at least 15 digits) for $p=3$ and $n=3(2)15$. Table 1 gives the number of exact figures in the result for c_1, obtained by the former two methods, as compared with the values out of from the formula (3.6.a).

4. EVALUATION OF D-DETERMINANTS

In this section we first show that Theorem 1 holds true for the following generalized type of D-determinant:

$$D^* = \left|P_{q_k - r_1}(x_1, \ldots, x_{r_1+1}) \cdots P_{q_k - r_m}(x_1, \ldots, x_{r_m+1})\right|, \tag{4.1}$$

with $q_1 > r_1$, $r_1 < r_2 < \cdots < r_m$. Next we show how determinants of this kind are related to D-determinants as defined by Eq. (2.3). Thereupon we give some hints for their evaluation in practice.

THEOREM 2. *The $m \times m$ determinant D^* defined by* (4.1) *can be reduced to a sum of $(m-1)\times(m-1)$ determinants of the same type, each having a positive coefficient.*

We give the proof in the form of a procedure which realizes the promised reduction.

Step 1. By virtue of Eq. (A1) we have

$$P_{q_k - r_m}(x_1,\dots,x_{r_m+1}) = \sum_{j=0}^{q_{m-1}-r_m} x_{r_m+1}^j P_{q_k - r_m - j}(x_1,\dots,x_{r_m})$$

$$+ x_{r_m+1}^{q_{m-1}-r_m+1} P_{q_k - q_{m-1}-1}(x_1,\dots,x_{r_m+1}), \qquad k=1,2,\dots,m.$$

Applying this formula to the last column of D^* yields

$$D^* = x_{r_m+1}^{q_m - r_{m-1}+1} P_{q_m - q_{m-1}-1}(x_1,\dots,x_{r_m+1}) \big| P_{q_k - r_1}(x_1,\dots,x_{r_1+1}) \cdots$$

$$P_{q_k - r_{m-1}}(x_1,\dots,x_{r_{m-1}+1}) \big|$$

$$+ \sum_{j=0}^{q_{m-1}-r_m} x_{r_m+1}^j \big| P_{q_k - r_1}(x_1,\dots,x_{r_1+1}) \cdots$$

$$P_{q_k - r_{m-1}}(x_1,\dots,x_{r_{m-1}+1}) P_{q_k - r_m - j}(x_1,\dots,x_{r_m}) \big| .$$

The first term in this development contains only a $(m-1)\times(m-1)$ determinant (with a positive coefficient); the remaining $m \times n$ determinants are characterized by their last column P-polynomials having one argument less. We submit these determinants several times to the same kind of reduction, until the following type is left:

$$D_1 \equiv \big| P_{q_k - r_1}(x_1,\dots,x_{r_1+1}) \cdots P_{q_k - r_{m-1}}(x_1,\dots,x_{r_{m-1}+1}) P_{q_k - r_m'}(x_1,\dots,x_{r_{m-1}+1}) \big|, \tag{4.2}$$

with $r_{m-1} < r_m'$, i.e., until the P-polynomials in the last two columns have the same number of arguments.

Step 2. Again from Eq. (A1) we have

$$P_{q_k - r_{m-1}}(x_1,\dots,x_{r_{m-1}+1}) = \sum_{j=0}^{r_m' - r_{m-1}-1} x_{r_{m-1}+1}^j P_{q_k - r_{m-1}-j}(x_1,\dots,x_{r_{m-1}})$$

$$+ x_{r_{m-1}+1}^{r_m' - r_{m-1}} P_{q_k - r_m'}(x_1,\dots,x_{r_{m-1}+1}), \qquad k=1,2,\dots,m.$$

The last term in the right member, however, produces a determinant with two identical columns; hence

$$D_1=\sum_{j=0}^{r'_m-r_{m-1}-1} x^j_{r_{m-1}+1}\big|P_{q_k-r_1}(x_1,\dots,x_{r_1+1})\cdots P_{q_k-r_{m-2}}(x_1,\dots,x_{r_{m-2}+1})$$

$$P_{q_k-r_{m-1}-j}(x_1,\dots,x_{r_{m-1}})\;P_{q_k-r'_m}(x_1,\dots,x_{r_{m-1}+1})\big|.$$

The determinants thus obtained are of type (4.1), and hence can be reduced to type (4.2) by means of step 1.

Step 3. When combining and applying steps 1 and 2 several times, we arrive at determinants of the type

$$D_2\equiv\big|P_{q_k-r_1}(x_1,\dots,x_{r_1+1})\cdots$$

$$P_{q_k-r_{m-2}}(x_1,\dots,x_{r_{m-2}+1})\;P_{q_k-r'_{m-1}}(x_1,\dots,x_{r_{m-2}+1})\;P_{q_k-r''_m}(x_1,\dots,x_{r_{m-2}+2})\big|.$$

In a similar way, we can handle the last three columns by means of steps 1 and 2, until the numbers of arguments in their P-polynomials are reduced to $r_{m-3}+1$, $r_{m-3}+1$, and $r_{m-3}+2$, respectively. Subjecting at last all columns to this procedure, we finally obtain a determinant of the type

$$D_3\equiv\big|P_{q_k-s_1}(x_1,\dots,x_{r_1})\;P_{q_k-s_2}(x_1,\dots,x_{r_1+1})\cdots P_{q_k-s_m}(x_1,\dots,x_{r_1+m-1})\big|,$$

with $s_1<s_2<\cdots<s_m$, $q_1\geqslant s_1$. Being only a special case of type (4.1), it can be transformed by applying steps 1 and 2 several times until we obtain a determinant of the type

$$D_4=\big|P_{q_k-s'_1}(x_1,x_2)\;P_{q_k-s'_2}(x_1,x_2,x_3)\cdots P_{q_k-s'_m}(x_1,\dots,x_{m+1})\big|,\qquad(4.3)$$

with $s'_1<s'_2<\cdots<s'_m$, $q_1\geqslant s'_1$. The net result of steps 1, 2, and 3 now is that we have reduced D^* to a sum of $(m-1)\times(m-1)$ determinants of the type (4.1), and $m\times m$ determinants of the type (4.3); it is clear that all determinants in this sum have positive coefficients.

Step 4. We transform D_4 as follows. Let t be an arbitrary natural number. Repeated application of Eq. (A1) (with $q=t$) to the last column, to the last but first,..., to the first, respectively, yields

$$D_4 = x_{m+1}^{t+1}\left|P_{q_k-s_1'}(x_1,x_2)\cdots P_{q_k-s_{m-1}'}(x_1,\ldots,x_m)\ P_{q_k-s_m'-t-1}(x_1,\ldots,x_{m+1})\right|$$

$$+\sum_{j_1=0}^{t} x_{m+1}^{j_1}x_m^{j_1+1}\left|P_{q_k-s_1'}(x_1,x_2)\cdots P_{q_k-s_{m-2}'}(x_1,\ldots,x_{m-1})\right.$$

$$\left. P_{q_k-s_{m-1}'-j_1-1}(x,\ldots,x_m)\ P_{q_k-s_m'-j_1}(x_1,\ldots,x_m)\right|$$

$$+\cdots$$

$$+\sum_{j_1=0}^{t} x_{m+1}^{j_1}\sum_{j_2=0}^{j_1} x_m^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-1}} x_4^{j_{m-2}}x_3^{j_{m-2}+1}$$

$$\times\left|P_{q_k-s_1'}(x_1,x_2)P_{q_k-s_2'-j_{m-2}}(x_1,x_2,x_3)P_{q_k-s_1'-j_{m-2}}(x_1,x_2,x_3)\cdots\right.$$

$$\left. P_{q_k-s_m'-j_1}(x_1,\ldots,x_m)\right|$$

$$+\sum_{j_1=0}^{t} x_{m+1}^{j_1}\sum_{j_2=0}^{j_1} x_m^{j_2}\cdots\sum_{j_{m-1}=0}^{j_{m-2}} x_3^{j_{m-1}}\sum_{j_m=0}^{j_{m-1}} x_2^{j_m}$$

$$\times\left|P_{q_k-s_1'-j_m}(x_1)\ P_{q_k-s_2'-j_{m-1}}(x_1,x_2)\cdots P_{q_k-s_m'-j_1}(x_1,\ldots,x_m)\right|. \tag{4.4}$$

In this development the second, third,..., last but first sums are equal to zero whenever $s_m'-s_{m-1}'=0$, $s_{m-1}'-s_{m-2}'-1=0,\ldots,s_2'-s_1'-1=0$, respectively; all determinants occurring in these sums contain somewhere two columns with P-polynomials having the same number of arguments, and hence can be further transformed following steps 2 and 1. Concerning the first and last terms in (4.4), we claim that $j_m \leqslant q_1-s_1'$, and hence $t \leqslant q_1-s_1'$. For in that case we can replace $P_{q_k-s_1'-j_m}(x_1)$ by $x_1^{q_k-s_1'-j_m}$, and remove the factor $x_1^{q_1-s_1'-j_m}$ from the determinant, leaving $x_1^{q_k-q_1}$, $k=1,2,\ldots,m$, as elements of its first column. If, however, $q_{m-1}-s_m' \leqslant q_1-s_1'$, it suffices to take $t=q_{m-1}-s_m'$ in order to reduce the first term in (4.4) to a $(m-1)$th degree determinant. Otherwise this determinant is again of the type (4.3) and has to undergo the same transformation as D_4 itself. Since by such transformations the degrees of P-polynomials in the last column decrease, repeated application will at last lead to the case that $q_{m-1}-s_m' \leqslant q_1-s_1'$. Thus, in Eq. (4.4) we choose $t=\min(q_1-s_1', q_{m-1}-s_m')$, and we only concern ourselves with the last term in the right member, the first column of its determinant being $x_1^{q_k-q_1}$, $k=1,2,\ldots,m$. The second column can be transformed as follows. If $q_1-s_2'-j_{m-1}\geq 0$, we multiply the first column by $P_{q_1-s_2'-j_{m-1}}(x_1,x_2)$ and subtract it from the second. By virtue of Eq. (A3) we then obtain

$$x_2^{q_1-s_2'+1-j_{m-1}}P_{q_k-q_1-1}(x_1,x_2)$$

for its kth element, whereby the first factor can be removed from the determinant. If $q_1-s_2'-j_{m-1}=-l_{m-1}<0$, then the kth element of the second column can be written as $P_{q_k-q_1-l_{m-1}}(x_1,x_2)$; this consequently represents in general the kth element of the new second column, with $l_{m-1}\geq 1$. If $q_1-s_3'-j_{m-2}\geq 0$, we can transform the third column in an analogous way, giving

$$x_3^{q_1-s_3'+1-j_{m-3}}P_{q_k-q_1-1}(x_1,x_2,x_3)$$

for the kth element. [The other term in (A3) gives rise to a determinant with two identical columns.] The power of x_3 can again be removed, and the general expression for the kth element becomes $P_{q_k-q_1-l_{m-2}}(x_1,x_2,x_3)$, including the case that $q_1-s_3'-j_{m-2}=-l_{m-2}<0$. Likewise we can transform the other columns. Since now the first row consists of the elements $1,0,\ldots,0$, we ultimately arrive at the following $(m-1)\times(m-1)$ determinant:

$$\left|P_{q_k-q_1-l_{m-1}}(x_1,x_2)\cdots P_{q_k-q_1-l_1}(x_1,\ldots,x_m)\right|,$$

which is of the type (4.3). This concludes the proof.

When applying the procedure of Theorem 2 to a 2×2 determinant of the type (4.1), it is reduced to a sum of P-polynomials. This proves

THEOREM 3. *Each $m\times m$ determinant D^*, defined by* (4.1), *can be expressed as a sum of P-polynomials each having a positive coefficient.*

Theorem 1 is a corollary of Theorem 3; both theorems state a property of D- or D^*-determinants which is useful for accurate numerical computation.

We have not attempted to obtain explicit expressions for the terms in the sums mentioned in the two theorems, because this would involve a cumbersome calculation. Moreover, it is not possible, in general, to represent these sums in closed form.

Let us now show how determinants of the type (4.1) may intervene in the evaluation of D-determinants. These can be reduced to lower order determinants by means of compact formulas, as is shown by

THEOREM 4. *If* $i_{n-1}-n>i_1-1$, *then*

$$D_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})$$

$$=P_{i_1-1}\left(\prod_{k\neq n+1}^{1,n+1}x_k,\ldots,\prod_{k\neq 1}^{1,n+1}x_k\right)\cdot D_{i_2-i_1,\ldots,i_n-i_1}(x_1,\ldots,x_n)$$

$$+\sum_{l=2}^{n}(x_{l+1}\cdots x_{n+1})^{i_1}P_{i_1-1}\left(\prod_{k\neq l}^{1,l}x_k,\ldots,\prod_{k\neq 1}^{1,l}x_k\right)$$

$$\times\left|P_{i_k-i_1-1}(x_1,x_2)\cdots P_{i_k-i_1-l+2}(x_1,\ldots,x_{l-1})\,P_{i_k-i_1-l}(x_1,\ldots,x_{l+1})\right.$$

$$\left.\cdots P_{i_k-i_1-n}(x_1,\ldots,x_{n+1})\right|; \quad (4.5.a)$$

if $i_{n-1}-n\leqslant i_1-1$, *then*

$$D_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})$$

$$=(x_1\cdots x_n)^{i_1-i_{n-1}+n-1}P_{i_{n-1}-n}\left(\prod_{k\neq n+1}^{1,n+1}x_k,\ldots,\prod_{k\neq 1}^{1,n+1}x_k\right)\cdot D_{i_2-i_1,\ldots,i_n-i_1}(x_1,\ldots,x_n)$$

$$+x_{n+1}^{i_{n-1}-n+1}P_{i_n-i_{n-1}-1}(x_1,\ldots,x_{n+1})\cdot D_{i_1,\ldots,i_{n-1}}(x_1,\ldots,x_n). \quad (4.5.b)$$

Proof. Take t as in step 4 of the procedure of Theorem 2. Since here $s_j'=j$, $j=1,2,\ldots,n$, the formula (4.4) simplifies, yielding

$$D_{i_1,\ldots,i_n}(x_1,\ldots,x_{n+1})$$

$$=x_{n+1}^{t+1}\left|P_{i_k-1}(x_1,x_2)\cdots P_{i_k-n+1}(x_1,\ldots,x_n)\,P_{i_k-n-t-1}(x_1,\ldots,x_{n+1})\right|$$

$$+\sum_{j_1=0}^{t}x_{n+1}^{j_1}\sum_{j_2=0}^{j_1}x_n^{j_2}\cdots\sum_{j_n=0}^{j_{n-1}}x_2^{j_n}\cdot\left|P_{i_k-1-j_n}(x_1)\,P_{i_k-2-j_{n-1}}(x_1,x_2)\cdots\right.$$

$$\left.P_{i_k-n-j_1}(x_1,\ldots,x_n)\right|. \quad (4.6)$$

The multiple sum term in the right member can be transformed as described in the fourth step of the procedure of Theorem 2, to give

$$x_1^{i_1-1}x_2^{i_1-2}\cdots x_n^{i_1-n}\left|P_{i_k-i_1}(x_1,x_2)P_{i_k-i_1-1}(x_1,x_2,x_3)\cdots P_{i_k-i_1-1}(x_1,\ldots,x_n)\right|$$
$$\times\sum_{j_1=0}^{t}\left(\frac{x_{n+1}}{x_n}\right)^{j_1}\sum_{j_2=0}^{j_1}\left(\frac{x_n}{x_{n-1}}\right)^{j_2}\cdots\sum_{j_n=0}^{j_{n-1}}\left(\frac{x_2}{x_1}\right)^{j_n}. \qquad (4.7)$$

Now it follows from Eq. (A1) that

$$\sum_{j_n=0}^{j_{n-1}}\left(\frac{x_2}{x_1}\right)^{j_n}=P_{j_{n-1}}\left(1,\frac{x_2}{x_1}\right),$$

and further that

$$\sum_{j_{n-1}=0}^{j_{n-2}}\left(\frac{x_3}{x_2}\right)^{j_{n-1}}P_{j_{n-1}}\left(1,\frac{x_2}{x_1}\right)=P_{j_{n-2}}\left(1,\frac{x_3}{x_2},\frac{x_3}{x_1}\right),$$

$$\cdots$$

$$\sum_{j_1=0}^{t}\left(\frac{x_{n+1}}{x_n}\right)^{j_1}P_{j_1}\left(1,\frac{x_n}{x_{n-1}},\ldots,\frac{x_n}{x_1}\right)=P_t\left(1,\frac{x_{n+1}}{x_n},\ldots,\frac{x_{n+1}}{x_1}\right).$$

A slight transformation of the determinant in (4.7), based on Eq. (2.1.a), finally gives for this expression

$$(x_1x_2\cdots x_n)^{i_1-1-t}P_t\left(\prod_{k\neq n+1}^{1,n+1}x_k,\ldots,\prod_{k\neq 1}^{1,n+1}x_k\right)\cdot D_{i_2-i_1,\ldots,i_n-i_1}(x_1,\ldots,x_n). \qquad (4.8)$$

In case $t=i_{n-1}-n$ the first term in the right member of Eq. (4.6) reduces to the second term in Eq. (4.5.b), while (4.8) reduces to the first term in Eq. (4.5.b). Consider now the case $t=i_1-1$. Then (4.8) reduces to the first term in Eq. (4.5.a). For calculating the first term in Eq. (4.6), T say, we proceed as follows. The determinant is of the type (4.3), with $s_j'=j$, $j=1,2,\ldots,n-1$, and

$s_n' = i_1 + n$. Application of the development (4.4) then gives

$$T = (x_n x_{n+1})^{i_1} |P_{i_k-1}(x_1,x_2) \cdots$$
$$P_{i_k-n+2}(x_1,\dots,x_{n-1})\, P_{i_k-i_1-n+1}(x_1,\dots,x_n)\, P_{i_k-i_1-n}(x_1,\dots,x_{n+1})|$$
$$+ x_{n+1}^{i_1} \sum_{j_2=0}^{i_1-1} x_n^{j_2} |P_{i_k-1}(x_1,x_2) \cdots P_{i_k-n+2}(x_1,\dots,x_{n-1})$$
$$P_{i_k-n+1-j_2}(x_1,\dots,x_{n-1})\, P_{i_k-i_1-n}(x_1,\dots,x_{n+1})|$$
$$= T_1 + T_2 \quad \text{(say)}.$$

Using the same calculation technique as in the first part of the proof, we obtain for the latter term

$$T_2 = x_{n+1}^{i_1} \sum_{j_2=0}^{i_1-1} x_n^{j_2} \sum_{j_3=0}^{j_2} x_{n-1}^{j_3} \cdots \sum_{j_n=0}^{j_{n-1}} x_2^{j_n}$$
$$\times |P_{i_k-1-j_n}(x_1) \cdots P_{i_k-n+1-j_2}(x_1,\dots,x_{n-1})\, P_{i_k-i_1-n}(x_1,\dots,x_{n+1})|$$
$$= x_{n+1}^{i_1} P_{i_1-1}\left(\prod_{\substack{k\neq n}}^{1,n} x_k, \dots, \prod_{\substack{k\neq 1}}^{1,n} x_k \right) |P_{i_k-i_1-1}(x_1,x_2) \cdots$$
$$P_{i_k-i_1-n+2}(x_1,\dots,x_{n-1})\, P_{i_k-i_1-n}(x_1,\dots,x_{n+1})|.$$

The term T_1 can again be split up into two terms T_3, T_4 (say), the latter of which can be calculated in the same way as T_2; on the former we repeat the splitting procedure. Finally we obtain a term T_{2n-3} which can be expressed as

$$T_{2n-3} = (x_3 x_4 \cdots x_{n+1})^{i_1} |X P_{i_k-1}(x_1,x_2)\, P_{i_k-i_1-2}(x_1,x_2,x_3) \cdots$$
$$P_{i_k-i_1-n}(x_1,\dots,x_{n+1})|$$
$$+ (x_4 x_5 \cdots x_{n+1})^{i_1} \sum_{j_{n-1}=0}^{i_1-1} x_3^{j_{n-1}} |P_{i_k-1}(x_1,x_2)\, P_{i_k-2-j_{n-1}}(x_1,x_2)$$
$$P_{i_k-i_1-3}(x_1,\dots,x_4) \cdots P_{i_k-i_1-n}(x_1,\dots,x_{n+1})|$$
$$= T_{2n-1} + T_{2n} \quad \text{(say)}.$$

For T_{2n} we obtain successively

$$T_{2n}=(x_4x_5\cdots x_{n+1})^{i_1}\sum_{j_{n-1}=0}^{i_1-1}x_3^{j_{n-1}}\sum_{j_n=0}^{j_{n-1}}x_2^{j_n}$$

$$\times\left|P_{i_k-1-j_n}(x_1)\,P_{i_k-2-j_{n-1}}(x_1,x_2)\,P_{i_k-i_1-3}(x_1,\ldots,x_4)\cdots\right.$$

$$\left.P_{i_k-i_1-n}(x_1,\ldots,x_{n+1})\right|$$

$$=(x_4x_5\cdots x_{n+1})^{i_1}P_{i_1-1}(x_1x_2,x_1x_3,x_2x_3)$$

$$\times\left|P_{i_k-i_1-1}(x_1,x_2)\,P_{i_k-i_1-3}(x_1,\ldots,x_4)\cdots P_{i_k-i_1-n}(x_1,\ldots,x_{n+1})\right|,$$

while T_{2n-1} is easily found to be

$$T_{2n-1}=(x_3x_4\cdots x_{n+1})^{i_1}P_{i_1-1}(x_1,x_2)$$

$$\times\left|P_{i_k-i_1-2}(x_1,x_2,x_3)\cdots P_{i_k-i_1-n}(x_1,\ldots,x_{n+1})\right|.$$

Thus,

$$T=\sum_{i=1}^{n}T_{2i}+T_{2n-1},$$

and this sum of T-terms is precisely the sum term in the right member of Eq. (4.5.a). This completes the proof. ∎

Consider the case that $i_{n-1}-n>i_1-1$ (the special case $i_1-1\leqslant i_{n-1}-n$ will be discussed in the next section). Observe that the determinants in the formula (4.5.a) have one missing column and hence already are of the general type (4.1). They can be further reduced to determinants with two missing columns, etc. Again the calculations involved are so elaborate that we omit them here; moreover they would result in expressions which are too complicated to be of practical interest. Instead, we show by means of an example or two that the procedure of Theorem 2, though difficult to automatize, is easy to apply in each concrete case separately, and that is it useful for the simultaneous evaluation of all D-determinants appearing in the matrix U of Eq. (3.5).

EXAMPLE 1. Take $n=4$, $i_1=6$, $i_2=8$, $i_3=11$, $i_4=15$. The formula (4.5a) then gives for the corresponding D-determinant

$$\begin{aligned} D_{6,8,11,15}(x_1,\ldots,x_5) = {}& P_5(x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4x_5, x_1x_3x_4x_5, x_2x_3x_4x_5) \\ & \times D_{2,5,9}(x_1,\ldots,x_4) \\ & + (x_3x_4x_5)^6 P_5(x_1,x_2)\cdot D_1 + (x_4x_5)^6 P_5(x_1x_2,x_1x_3,x_2x_3)\cdot D_2 \\ & + x_5^6 P_5(x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)\cdot D_3, \end{aligned}$$

with

$$D_1 = \begin{vmatrix} P_2(x_1,\ldots,x_4) & P_1(x_1,\ldots,x_5) \\ P_6(x_1,\ldots,x_4) & P_5(x_1,\ldots,x_5) \end{vmatrix}; \qquad D_2 = P_1(x_1,x_2)\cdot D_1;$$

$$D_3 = \begin{vmatrix} P_1(x_1,x_2) & 1 & 0 \\ P_4(x_1,x_2) & P_3(x_1,x_2,x_3) & P_1(x_1,\ldots,x_5) \\ P_8(x_1,x_2) & P_7(x_1,x_2,x_3) & P_5(x_1,\ldots,x_5) \end{vmatrix}.$$

The transformation of D_1 runs along the following lines:

$$\begin{aligned} D_1 = {}& \begin{vmatrix} P_2(x_1,x_2,x_3) & P_1(x_1,\ldots,x_4) \\ P_6(x_1,x_2,x_3) & P_5(x_1,\ldots,x_4) \end{vmatrix} + x_5 \begin{vmatrix} P_2(x_1,\ldots,x_4) & 1 \\ P_6(x_1,\ldots,x_4) & P_4(x_1,\ldots,x_5) \end{vmatrix} \\ = {}& D_{3,7}(x_1,x_2,x_3) + x_4 \begin{vmatrix} P_2(x_1,x_2,x_3) & 1 \\ P_6(x_1,x_2,x_3) & P_4(x_1,\ldots,x_5) \end{vmatrix} \\ & + x_5^2 P_3(x_1,\ldots,x_5) P_2(x_1,\ldots,x_4) \\ & + x_5 \left\{ \begin{vmatrix} P_2(x_1,x_2,x_3) & 1 \\ P_6(x_1,x_2,x_3) & P_4(x_1,\ldots,x_4) \end{vmatrix} + x_4 \begin{vmatrix} P_1(x_1,\ldots,x_4) & 1 \\ P_5(x_1,\ldots,x_4) & P_4(x_1,\ldots,x_4) \end{vmatrix} \right\} \\ = {}& D_{3,7}(x_1,x_2,x_3) + x_5^2 P_3(x_1,\ldots,x_5) P_2(x_1,\ldots,x_4) \\ & + P_1(x_4,x_5)\left[P_3(x_1,\ldots,x_4) P_2(x_1,\ldots,x_3) + \overline{D} + x_3 D_{2,6}(x_1,x_2,x_3) \right] \\ & + x_4x_5\left[x_4 P_3(x_1,\ldots,x_4) P_1(x_1,x_1,x_3) + D_{2,6}(x_1,x_2,x_3) \right], \end{aligned}$$

with

$$\bar{D}=\begin{vmatrix} P_2(x_1,x_2) & 1 \\ P_6(x_1,x_2) & P_4(x_1,x_2,x_3) \end{vmatrix}.$$

It is easily verified that the reduction of D_3 results in

$$D_3=D_{2,5}(x_1,x_2,x_3)\left[x_5P_3(x_1,\ldots,x_5)+x_4P_3(x_1,\ldots,x_4)\right]$$
$$+x_3P_1(x_1,x_2)\left[\bar{D}+x_3D_{2,6}(x_1,x_2,x_3)\right]+x_1x_2\bar{D}.$$

Finally one has

$$D_{2,5,9}(x_1,\ldots,x_4)=P_1(x_1x_2x_3,x_1x_2x_4,x_2x_3x_4)\cdot D_{3,7}(x_1,x_2,x_3)$$
$$+(x_3x_4)^2P_1(x_1,x_2)\cdot D_4+x_4^2P_1(x_1x_2,x_1x_3,x_2x_3)\cdot D_5,$$

with

$$D_4=\begin{vmatrix} P_1(x_1,x_2,x_3) & 1 \\ P_5(x_1,x_2,x_3) & P_4(x_1,\ldots,x_4) \end{vmatrix}$$
$$=x_4P_3(x_1,\ldots,x_4)P_1(x_1,x_2,x_3)+D_{2,6}(x_1,x_2,x_3),$$

and

$$D_5=\begin{vmatrix} P_2(x_1,x_2) & 1 \\ P_6(x_1,x_2) & P_4(x_1,\ldots,x_4) \end{vmatrix}=x_4P_3(x_1,\ldots,x_4)P_2(x_1,x_2)+\bar{D}.$$

The two D-determinants occurring in the resulting expression for $D_{6,8,11,15}(x_1,\ldots,x_5)$ are of second degree and hence can always be evaluated by means of the formula (4.5.b). The other 2×2 determinant, $\bar{D}$, is of the type (4.3); the reduction of this type to a sum of P-polynomials is quite easy in general for the case $m=3$, as we show in

Example 2. Consider

$$M\equiv\begin{vmatrix} P_p(x_1,x_2) & P_q(x_1,x_2,x_3) \\ P_{p+r}(x_1,x_2) & P_{q+r}(x_1,x_2,x_3) \end{vmatrix},\qquad p>q.$$

With $t=q$, application of the formula (4.4) yields

$$M=x_3^{q+1}P_p(x_1,x_2)P_{r-1}(x_1,x_2,x_3)+\sum_{j_1=0}^{q} x_3^{j_1}\begin{vmatrix} P_p(x_1,x_2) & P_{q-j_1}(x_1,x_2) \\ P_{p+r}(x_1,x_2) & P_{q+r-j_1}(x_1,x_2)\end{vmatrix}.$$

For the second term in the right member, T say, we obtain further

$$T=\sum_{j_1=0}^{q} x_3^{j_1}\sum_{j_2=0}^{p-q-1+j_1} x_2^{j_1}\begin{vmatrix} P_{p-j_2}(x_1) & P_{q-j_1}(x_1,x_2) \\ P_{p+r-j_2}(x_1) & P_{q+r-j_1}(x_1,x_2)\end{vmatrix}$$

$$=x_1^p x_2^{q+1}P_{r-1}(x_1,x_2)\sum_{j_1=0}^{q}\left(\frac{x_3}{x_2}\right)^{j_1}\sum_{j_2=0}^{p-q-1+j_1}\left(\frac{x_2}{x_1}\right)^{j_2}.$$

The second sum in this expression equals $P_{p-q-1+j_1}(1,x_2/x_1)$. The first sum then becomes successively:

$$\left(\frac{x_2}{x_3}\right)^{p-q-1}\sum_{j_1=0}^{q} P_{p-q-1+j_1}\left(\frac{x_3}{x_2},\frac{x_3}{x_1}\right)$$

$$=\left(\frac{x_2}{x_3}\right)^{p-q-1}\left[P_{p-1}\left(1,\frac{x_3}{x_2},\frac{x_3}{x_1}\right)-P_{p-q-2}\left(1,\frac{x_3}{x_2},\frac{x_3}{x_1}\right)\right]$$

$$=\left(\frac{x_2}{x_3}\right)^{p-q-1}\left[\left(\frac{x_3}{x_1}\right)^{p-q-1}P_q\left(1,\frac{x_3}{x_2},\frac{x_3}{x_1}\right)+\frac{x_3}{x_2}P_{p-q-2}\left(\frac{x_3}{x_2},\frac{x_3}{x_1}\right)P_q\left(1,\frac{x_3}{x_2}\right)\right]$$

[by Eq. (A3)]. After rearrangment of factors we obtain

$$T=P_{r-1}(x_1,x_2)\left[x_1x_2^{p-q}P_q(x_1x_2,x_1x_3,x_2x_3)+x_1^{q+2}x_2P_{p-q-2}(x_1,x_2)P_q(x_2,x_3)\right].$$

5. SOME IMPORTANT SPECIAL CASES

I

The procedure of Theorem 1 need not be applied if the condition $i_{n-1}-n\leqslant i_1-1$ is fulfilled, since then the formula (4.5.b) holds. This condition implies that $i_1-i_{n-1}+n-1$ has the value 0 or 1, and further that only i_1

and i_n can be chosen freely. For the intermediate powers we have either

$$i_k = i_{k-1} + 1, \qquad k = 2,3,\dots,n-1, \tag{5.1.a}$$

or

$$i_2 = i_1 + 1, \dots, i_{k-1} = i_{k-2} + 1,$$

$$i_k = i_{k-1} + 2, \qquad i_{k+1} = i_k + 1, \dots, i_{n-1} = i_{n-2} + 1, \qquad 2 \leqslant k \leqslant n-1 \tag{5.1.b}$$

—i.e., at most three powers are out of line. However, when choosing an interpolant or approximant of the type (1.2), i_1 and i_n are the more important powers, since they determine the behavior of $\Phi(x)$ for small and for large values of x, respectively. So, in the author's opinion, the restrictions (5.1) are not serious for practical purposes.

The formula (4.5.b) permits an accurate computation of all elements of U in a recursive way; the method of D-determinants for the numerical solution of a generalized Vandermonde system can then be fully automatized and programmed compactly. (In fact, we have used this method for the numerical example of Sec. 3.)

II

Another special case, for which the formula (4.5.b) does not hold, arises from choosing

$$i_1 = 1, \dots, i_k = k, \qquad i_{k+1} = k + p + 1, \dots, i_n = n + p, \quad p > 1; \qquad 1 \leqslant k \leqslant n-1;$$

i.e., only one power is out of line.

By defining the vectors

$$c_{\mathrm{I}} = (c_1, c_2, \dots, c_{k+1})^T, \qquad c_{\mathrm{II}} = (c_{k+2}, c_{k+3}, \dots, c_{n+1})^T,$$

$$f_{\mathrm{I}} = (f_1, f_2, \dots, f_{k+1})^T, \qquad f_{\mathrm{II}} = (f_{k+2}, f_{k+3}, \dots, f_{n+1})^T,$$

the system (1.4) can be written in the form

$$\begin{bmatrix} V(x_1,\dots,x_k) & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} c_{\mathrm{I}} \\ c_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} f_{\mathrm{I}} \\ f_{\mathrm{II}} \end{bmatrix}, \tag{5.2}$$

where

$$M_{12}=\begin{bmatrix} x_1^{k+p+1} & \cdots & x_1^{n+p} \\ \cdot & \cdots & \cdot \\ x_{k+1}^{k+p+1} & \cdots & x_{k+1}^{n+p} \end{bmatrix}, \qquad M_{21}=\begin{bmatrix} 1 & x_{k+2} & \cdots & x_{k+2}^{k} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_{n+1} & \cdots & x_{n+1}^{k} \end{bmatrix},$$

$$M_{22}=\begin{bmatrix} x_{k+2}^{k+p+1} & \cdots & x_{k+2}^{n+p} \\ \cdot & \cdots & \cdot \\ x_{n+1}^{k+p+1} & \cdots & x_{n+1}^{n+p} \end{bmatrix}.$$

Applying block Gaussian elimination to (5.2), we obtain

$$\begin{bmatrix} V(x_1,\dots,x_k) & M_{12} \\ 0 & M_{22}^{(2)} \end{bmatrix}\begin{bmatrix} C_{\rm I} \\ C_{\rm II} \end{bmatrix}=\begin{bmatrix} f_{\rm I} \\ f_{\rm II}^{(2)} \end{bmatrix},$$

where

$$M_{22}^{(2)}=M_{22}-M_{21}V^{-1}(x_1,\dots,x_k)M_{12},$$

$$f_{\rm II}^{(2)}=f_{\rm II}-M_{21}V^{-1}(x_1,\dots,x_k)f_{\rm I}.$$

After having solved the system

$$M_{22}^{(2)}\cdot c_{\rm II}=f_{\rm II}^{(2)} \tag{5.3}$$

by ordinary Gaussian elimination, we can obtain $c_{\rm I}$ from

$$c_{\rm I}=V^{-1}(x_1,\dots,x_k)(f_{\rm I}-M_{12}c_{\rm II}).$$

The computation of $M_{22}^{(2)}$ and $f_{\rm II}^{(2)}$, as well as $c_{\rm II}$, from the system (5.3) may be susceptable to instability. But $V^{-1}(x_1,\dots,x_k)$ can be obtained with high precision by means of the Björck-Pereyra method.

When $k=n-1$, this case reduces to that of the example in Sec. 3. When $p=1$, the formula (4.5.b) is applicable; the corresponding method of D-determinants is more accurate, but slower than the method of block Gaussian elimination as described here.

III

A natural generalization of cases I and II is that the powers $j_1, j_2, \ldots, j_{n+1}$ are "piecewise in line", i.e. take on values

$$p_1, p_1+1, \ldots, p_1+n_1; p_2, p_2+1, \ldots, p_2+n_2; \cdots; p_m, p_m+1, \ldots, p_m+n_m,$$

where $p_k > p_{k-1}+n_{k-1}+1$, $k=2,3,\ldots,m$, and $\sum_{i=1}^{m} n_i + m = n+1$. Again, in seeking a mechanizable solution method, we can partition the generalized Vandermonde matrix into an $m \times m$ block matrix, and think of a block Gaussian elimination. But practically, if two or more powers are out of line (i.e., $m > 2$), this method has no advantage at all over ordinary Gaussian elimination.

6. GENERALIZED CONFLUENT VANDERMONDE MATRICES AND SYSTEMS

We modify the interpolation problem (1.2), (1,3) as follows: determine the coefficients c_i of $\Phi(x)$ such that

$$\Phi^{(s)}(x_k) = f^{(s)}(x_k), \qquad s = 0, 1, \ldots, m_k, \quad k = 1, 2, \ldots, N, \tag{6.1}$$

with all x_k positive, and $\sum_{i=1}^{N} m_i + N = n+1$. This problem has a unique solution. Indeed, if the matrix M of the linear system corresponding to (6.1) were singular, then its columns would be linearly dependent, which means that the function $\Phi(x)$ could have positive zeros x_k of multiplicity m_k+1, $k=1,2,\ldots,N$, i.e., $n+1$ positive zeros; it is easy to see that this is impossible.

The matrix M can be reduced to a generalized type of confluent Vandermonde matrix. To illustrate the numerical treatment of the interpolation problem (6.1), we reconstruct concisely the foregoing theory of D-determinants for the special case $N=n$, all $m_k=0$ except $m_p=1$, $1 \leqslant p \leqslant n$. Then, taking $\Phi(x_p) = f(x_p)$ and $\Phi'(x_p) = f'(x_p)$ as the first and the second equation, respectively, we have

$$M \equiv \begin{bmatrix} x_p^{j_1} & x_p^{j_2} & \cdots & x_p^{j_{n+1}} \\ j_1 x_p^{j_1-1} & j_2 x_p^{j_2-1} & \cdots & j_{n+1} x_p^{j_{n+1}-1} \\ x_1^{j_1} & x_1^{j_2} & \cdots & x_1^{j_{n+1}} \\ \cdot & \cdot & \cdots & \cdot \\ x_{p-1}^{j_1} & x_{p-1}^{j_2} & \cdots & x_{p-1}^{j_{n+1}} \\ x_{p+1}^{j_1} & x_{p+1}^{j_2} & \cdots & x_{p+1}^{j_{n+1}} \\ \cdot & \cdot & \cdots & \cdot \\ x_n^{j_1} & x_n^{j_2} & \cdots & x_n^{j_{n+1}} \end{bmatrix}.$$

The linear system can be reduced to

$$V_{i_1,\ldots,i_n}(x_p;x_1,x_2,\ldots,x_{p-1},x_{p+1},\ldots,x_n)\cdot c=f, \tag{6.2}$$

where the matrix

$$V_{i_1,\ldots,i_n}(x_p;x_1,\ldots,x_{p-1},x_{p+1},\ldots,x_n)\equiv\begin{bmatrix} 1 & x_p^{i_1} & \cdots & x_p^{i_n} \\ 0 & i_1x_p^{i_1} & \cdots & i_nx_p^{i_n} \\ 1 & x_1^{i_1} & \cdots & x_1^{i_n} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_{p-1}^{i_1} & \cdots & x_{p-1}^{i_n} \\ 1 & x_{p+1}^{i_1} & \cdots & x_{p+1}^{i_n} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_n^{i_1} & \cdots & x_n^{i_n} \end{bmatrix} \tag{6.3}$$

can be considered as a generalized confluent Vandermonde matrix (although usually the powers in the second row are $i_1-1,\ldots,i_n-1$; but this is only a matter of how the second equation is written down).

The equivalents of the formulas (2.2), (2.3) are

$$\det\left[V_{i_1,\ldots,i_n}(x_p;x_1,\ldots,x_{p-1},x_{p+1},\ldots,x_n)\right]$$
$$=(-1)^{p-1}\bar{D}_{i_1,\ldots,i_n}(x_p;x_1,\ldots,x_{p-1},x_{p+1},\ldots,x_n)\cdot\det\left[V(x_1,\ldots,x_n)\right], \tag{6.4}$$

with

$$\bar{D}_{i_1,\ldots,i_n}(x_p;x_1,\ldots,x_{p-1},x_{p+1},\ldots,x_n)$$
$$=\left|i_kP_{i_k}(x_p)\,P_{i_k-1}(x_p,x_1)\cdots P_{i_k-p+1}(x_p,x_1,\ldots,x_{p-1})\,P_{i_k-p}(x_1,\ldots,x_{p+1})\cdots\right.$$
$$\left.P_{i_k-n+1}(x_1,\ldots,x_n)\right|, \tag{6.5a}$$

$$\bar{D}_j(x_p)=jx_p^j. \tag{6.5.b}$$

By using Eq. (A4) properly, it is easy to show that a similar identity holds for $\bar{D}$-determinants. It is therefore possible to express successive steps of Gaussian elimination applied to the system (6.2) (and where the first two equations remain unaltered during the first step) in terms of these quantities.

Putting

$$s=\begin{cases} q-2 & \text{if} \quad p \geqslant q-1, \\ q-1 & \text{if} \quad p<q-1, \end{cases}$$

and further using the notation of Sec. 3, the results are

$$\left.\begin{aligned} &a_{qr}^{(2)}=D_{i_{r-1}}(x_p,x_s), \quad r=2,3,\ldots,n+1, \quad q=3,4,\ldots,n+1; \\ &f_q^{(2)}=\frac{f_q-f_1}{x_s-x_p}, \quad q=3,4,\ldots,n+1; \end{aligned}\right\} \tag{6.6.a}$$

$$\left.\begin{aligned} &a_{qr}^{(3)}=\bar{D}_{i_1,i_{r-1}}(x_p;x_s), \quad q,r=3,4,\ldots,n+1; \\ &f_q^{(3)}=f_q^{(2)}\bar{D}_{i_1}(x_p)-f_2^{(2)}D_{i_1}(x_p,x_1), \quad q=3,4,\ldots,n+1; \end{aligned}\right\} \tag{6.6.b}$$

$$\left.\begin{aligned} &a_{qr}^{(k)}=\bar{D}_{i_1,\ldots,i_{k-2},i_{r-1}}(x_p;x_1,\ldots,x_{k-3},x_s), \quad q,r=k,k+1,\ldots,n+1; \\ &f_q^{(k)}=\frac{f_q^{(k-1)}\bar{D}_{i_1,\ldots,i_{k-2}}(x_p;x_1,\ldots,x_{k-3})-f_{k-1}^{(k-1)}\bar{D}_{i_1,\ldots,i_{k-2}}(x_p;x_1,\ldots,x_{k-4},x_s)}{(x_s-x_{k-3})\bar{D}_{i_1,\ldots,i_{k-3}}(x_p;x_1,\ldots,x_{k-4})}, \\ &q=k,k+1,\ldots,n+1, \quad k=4,5,\ldots,n+1. \end{aligned}\right\} \tag{6.6.c}$$

The following remark is in order here. The $\bar{D}$-determinants of successively higher degree appearing in Eqs. (6.6) naturally lead us to consider such determinants for which $p>n$. In that case we agree that in the notation $\bar{D}_{i_1,\ldots,i_m}(x_p;x_1,\ldots,x_m)$ the argument x_p does not occur in the sequence $x_1,x_2,\ldots,x_m$. Equations (6.6) must be read in accordance with this convention.

For the evaluation of $\bar{D}$-determinants we can again follow the procedure of Theorem 2. If

$$i_1-1 \geqslant i_{n-1}-n+1,$$

i.e., if (5.1.a) holds (meaning that at most two powers, i_1 and i_n, are out of line), we find the following equivalent of the formula (4.5.b):

$$\begin{aligned}
&\bar{D}_{i_1,\ldots,i_m}(x_p;x_1,\ldots,x_{m-s+1})\\
&\quad = x_{m-s+1}^{i_{m-1}-m+2}P_{i_m-i_{m-1}-1}(x_p,x_1,\ldots,x_{m-s-1})\bar{D}_{i_1,\ldots,i_{m-1}}(x_p;x_1,\ldots,x_{m-s})\\
&\quad\quad - x_p^{i_1}(x_1,\ldots,x_{m-s})^{i_1-i_{m-1}+m-2}\\
&\quad\quad \times P_{i_{m-1}-m+1}\left(\prod_{k\neq p}^{1,m-s+1}x_k,\ \prod_{k\neq m-s+1}^{1,m-s+1}x_k,\ldots,\ \prod_{k\neq 1}^{1,m-s+1}x_k\right)\\
&\quad\quad \times \bar{D}_{i_2-i_1,\ldots,i_m-i_1}(x_p;x_1,\ldots,x_{m-s})
\end{aligned}\tag{6.7}$$

with $s=1$ if $p<m$ and $s=2$ if $p\geqslant m$. Equation (6.7) again allows for a recursive, and hence easily automatizable, computation of all $\bar{D}$-determinants in Eqs. (6.6). But observe that now a minus sign appears in the recurrence relation. As a consequence, Theorem 1 no longer holds true, and in solving the system (6.2) by this method, stability is less well guaranteed than in the nonconfluent case.

However, still favorable for stability is the fact that a $\bar{D}$-determinant is never equal to zero, and in this respect it is useful to determine its signature. Since all $\bar{D}$-determinants of the same degree m have equal sign regardless of the values of the powers $i_1,\ldots,i_n$, we can solve this problem by considering the simple case that $i_k=k$, $k=1,2,\ldots,n$. Then (6.7) reduces to

$$\bar{D}_{1,2,\ldots,m}(x_p;x_1,\ldots,x_{m-s+1})=(x_{m-s+1}-x_p)\bar{D}_{1,2,\ldots,m-1}(x_p;x_1,\ldots,x_{m-s}).$$

From this we conclude that

$$\begin{aligned}
&\bar{D}_{1,2,\ldots,m}(x_p;x_1,\ldots,x_{m-s+1})\\
&\quad=\begin{cases}(-1)^{m-1}x_p(x_p-x_1)\cdots(x_p-x_{m-1}), & p\geqslant m,\\ (-1)^{p-1}x_p(x_p-x_1)\cdots(x_p-x_{p-1})\cdot(x_m-x_p)\cdots(x_{p+1}-x_p), & p<m,\end{cases}
\end{aligned}$$

and further that

$$\operatorname{sgn}\bar{D}_{i_1,\ldots,i_m}(x_p;x_1,\ldots,x_{m-s+1})=\begin{cases}(-1)^{m-1}, & p\geqslant m,\\ (-1)^{p-1}, & p<m,\end{cases}$$

and

$$\det\left[V_{i_1,\dots,i_m}(x_p;x_1,\dots,x_{m-s+1})\right]>0.$$

APPENDIX—PROOF OF FORMULAS (A1)–(A4)

The formula (A1) can be established by repeated application of (2.1.a). By taking $r=2$ in (A1), we see that

$$P_p(x_1,x_2)=\frac{x_2^{p+1}-x_1^{p+1}}{x_2-x_1}. \tag{A1$'$}$$

To prove (A2), we use (A1) and (A1′), to give successively

$$\begin{aligned}
&P_p(x_1,\dots,x_q,x_i)-P_p(x_1,\dots,x_q,x_{q+1})\\
&=\left(x_i^p-x_{q+1}^p\right)+\left(x_i^{p-1}-x_{q+1}^{p-1}\right)P_1(x_1,\dots,x_q)+\cdots+(x_i-x_{q+1})P_{p-1}(x_1,\dots,x_q)\\
&=(x_i-x_{q+1})\left[P_{p-1}(x_{q+1},x_i)+P_{p-2}(x_{q+1},x_i)P_1(x_1,\dots,x_q)+\cdots\right.\\
&\qquad\qquad\left.+P_{p-1}(x_1,\dots,x_q)\right]\\
&=(x_i-x_{q+1})\left\{x_i^{p-1}+x_i^{p-2}\left[x_{q+1}+P_1(x_1,\dots,x_q)\right]+\cdots\right.\\
&\qquad\left.+\left[x_{q+1}^{p-1}+x_{q+1}^{p-2}P_1(x_1,\dots,x_q)+\cdots+P_{p-1}(x_1,\dots,x_q)\right]\right\}\\
&=(x_i-x_{q+1})\left[x_i^{p-1}+x_i^{p-2}P_1(x_1,\dots,x_{q+1})+\cdots+P_{p-1}(x_1,\dots,x_{q+1})\right]\\
&=(x_i-x_{q+1})P_{p-1}(x_1,\dots,x_{q+1},x_i).
\end{aligned}$$

The proof of (A3) is by induction. For $r=2$ we have

$$\begin{aligned}
P_p(x_1,x_2)-x_1^{p-q}P_q(x_1,x_2)&=x_2^p+x_2^{p-1}P_1(x_1)+\cdots+x_2^{q+1}P_{p-q-1}(x_1)\\
&=x_2^{q+1}P_{p-q-1}(x_1,x_2).
\end{aligned}$$

Hence (A3) is true for $r=2$. Assume now that it is true for $r=t-1$. On account of this hypothesis we then have

$$
\begin{aligned}
&P_p(x_1,\dots,x_t) - x_1^{p-q}P_q(x_1,\dots,x_t)\\
&\qquad = x_t^p + x_t^{p-1}P_1(x_1,\dots,x_{t-1}) + \cdots + x_t^{q+1}P_{p-q-1}(x_1,\dots,x_{t-1})\\
&\qquad\quad + x_t^q\left[P_{p-q}(x_1,\dots,x_{t-1}) - x_1^{p-q}\right]\\
&\qquad\quad + x_t^{q-1}\left[P_{p-q+1}(x_1,\dots,x_{t-1}) - x_1^{p-q}P_1(x_1,\dots,x_{t-1})\right]\\
&\qquad\quad + \cdots\\
&\qquad\quad + P_p(x_1,\dots,x_{t-1}) - x_1^{p-q}P_q(x_1,\dots,x_{t-1})\\
&\qquad = x_t^{q+1}P_{p-q-1}(x_1,\dots,x_t)\\
&\qquad\quad + x_t^p\sum_{i=0}^{t-3} x_{t-1-i}P_0(x_{t-1-i},\dots,x_{t-1})P_{p-q-1}(x_1,\dots,x_{t-1-i})\\
&\qquad\quad + \cdots\\
&\qquad\quad + \sum_{i=0}^{t-3} x_{t-1-i}P_q(x_{t-1-i},\dots,x_{t-1})P_{p-q-1}(x_1,\dots,x_{t-1-i})\\
&\qquad = x_t^{q+1}P_{p-q-1}(x_1,\dots,x_t) + \sum_{i=1}^{t-2} x_{t-i}P_{p-q-1}(x_1,\dots,x_{t-i})\\
&\qquad\quad \times\left[\sum_{j=0}^{q} x_t^jP_{q-j}(x_{t-i},\dots,x_{t-1})\right]\\
&\qquad = \sum_{i=0}^{t-2} x_{t-i}P_q(x_{t-i},\dots,x_t)P_{p-q-1}(x_1,\dots,x_{t-i}).
\end{aligned}
$$

The proof of (A4) involves a very elaborate, but elementary, calculation, which we sketch briefly here. Let D^* be the determinant in the left member. When developing the (1,1)- and (2,2)-determinants in it according to the elements of their last columns, we find

$$D^* = T_1 + T_2, \tag{A4$'$}$$

with

$$T_1 = D_{i_1,\ldots,i_{p-1}}(x_1,\ldots,x_p)\Big\{P_{i_q-p}(x_1,\ldots,x_p,x_{q+1})\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p}(x_1,\ldots,x_{p+1})\big|$$
$$-P_{i_q-p}(x_1,\ldots,x_{p+1})\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p}(x_1,\ldots,x_p,x_{q+1})\big|\Big\},$$

$$T_2 = \sum_{k=1}^{p-1}(-1)^{k+p}D_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_{p-1},i_q}(x_1,\ldots,x_p)$$
$$\times\Big\{P_{i_k-p}(x_1,\ldots,x_p,x_{q+1})\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p}(x_1,\ldots,x_{p+1})\big|$$
$$-P_{i_k-p}(x_1,\ldots,x_{p+1})\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p}(x_1,\ldots,x_p,x_{q+1})\big|\Big\}.$$

On account of (A2) we can rewrite the second factor in the expression for T_1 as

$$P_{i_q-p}(x_1,\ldots,x_p)\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p}(x_1,\ldots,x_{p+1})-P_{i_j-p}(x_1,\ldots,x_p,x_{q+1})\big|$$
$$+\big|P_{i_j-1}(x_1,x_2)\cdots x_{q+1}P_{i_q-p-1}(x_1,\ldots,x_p,x_{q+1})P_{i_j-p}(x_1,\ldots,x_{p+1})$$
$$-x_{p+1}P_{i_j-p-1}(x_1,\ldots,x_{p+1})P_{i_j-p}(x_1,\ldots,x_p,x_{q+1})\big|,$$

or

$$(x_{q+1}-x_{p+1})\Big\{-P_{i_q-p}(x_1,\ldots,x_p)$$
$$\times\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p+1}(x_1,\ldots,x_p)P_{i_j-p-1}(x_1,\ldots,x_{p+1},x_{q+1})\big|$$
$$+P_{i_q-p-1}(x_1,\ldots,x_{p+1},x_{q+1})\big|P_{i_j-1}(x_1,x_2)\cdots P_{i_j-p}(x_1,\ldots,x_{p+1})\big|$$
$$-x_{p+1}P_{i_q-p-1}(x_1,\ldots,x_{p+1})\big|P_{i_j-1}(x_1,x_2)\cdots$$
$$P_{i_j-p+1}(x_1,\ldots,x_p)\,P_{i_j-p-1}(x_1,\ldots,x_{p+1},x_{q+1})\big|\Big\}.$$

By virtue of (2.1.a) we then obtain

$$T_1 = (x_{q+1} - x_{p+1}) D_{i_1,\dots,i_{p-1}}(x_1,\dots,x_p)$$

$$\times \Big\{ P_{i_q-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \big| P_{i_l-1}(x_1,x_2) \cdots P_{i_l-p}(x_1,\dots,x_{p+1}) \big|$$

$$- P_{i_q-p}(x_1,\dots,x_{p+1}) \big| P_{i_{l-1}}(x_1,x_2) \cdots$$

$$P_{i_l-p+1}(x_1,\dots,x_p)\, P_{i_l-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \big| \Big\}.$$

The kth term in the expression for T_2 can be transformed into

$$(-1)^{k+p} D_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_{p-1},i_q}(x_1,\dots,x_p)$$

$$\times \sum_{l\neq k}^{1,p} (-1)^{l+p} D_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_p}(x_1,\dots,x_p)$$

$$\times \begin{vmatrix} P_{i_k-p}(x_1,\dots,x_p,x_{q+1}) & P_{i_l-p}(x_1,\dots,x_p,x_{q+1}) \\ P_{i_k-p}(x_1,\dots,x_{p+1}) & P_{i_l-p}(x_1,\dots,x_{p+1}) \end{vmatrix},$$

or, on account of (A2),

$$(x_{q+1} - x_{p+1})(-1)^{k+p} D_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_{p-1},i_q}(x_1,\dots,x_p)$$

$$\times \sum_{l\neq k}^{1,p} (-1)^{l+p+1} D_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_p}(x_1,\dots,x_p)$$

$$\times \begin{vmatrix} P_{i_k-p}(x_1,\dots,x_{p+1}) & P_{i_k-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \\ P_{i_l-p}(x_1,\dots,x_{p+1}) & P_{i_l-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \end{vmatrix}.$$

When developing the first D-determinant in this expression according to the elements of its last row, we obtain for T_2 the result

$$T_2 = (x_{q+1} - x_{p+1}) \sum_{m=1}^{p-1} (-1)^{p+m-1} P_{i_q - m}(x_1, \dots, x_{m+1})$$

$$\times \Bigg\{ D_{i_1,\dots,i_{p-1}}(x_1,\dots,x_p) \sum_{k=1}^{p-1} \Bigg[(-1)^{k+p+1} \big| P_{i_j-1}(x_1,x_2) \cdots P_{i_j-m+1}(x_1,\dots,x_m)\, P_{i_j-m-1}(x_1,\dots,x_{m+2}) \cdots P_{i_j-p+1}(x_1,\dots,x_p) \big|_{j=1,2,\dots,p-1;\, j \neq k}$$

$$\times \begin{vmatrix} P_{i_k-p}(x_1,\dots,x_{p+1}) & P_{i_k-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \\ P_{i_p-p}(x_1,\dots,x_{p+1}) & P_{i_p-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \end{vmatrix} \Bigg]$$

$$+ \sum_{k=1}^{p-1} \Bigg[(-1)^{k+p} \big| P_{i_j-1}(x_1,x_2) \cdots P_{i_j-m+1}(x_1,\dots,x_m)\, P_{i_j-m-1}(x_1,\dots,x_{m+1}) \cdots P_{i_j-p-1}(x_1,\dots,x_p) \big|_{j=1,2,\dots,p-1;\, j \neq k}$$

$$\times \sum_{l \neq k}^{1,p-1} (-1)^{l+p+1} D_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_p}(x_1,\dots,x_p)$$

$$\times \begin{vmatrix} P_{i_k-p}(x_1,\dots,x_{p+1}) & P_{i_k-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \\ P_{i_l-p}(x_1,\dots,x_{p+1}) & P_{i_l-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \end{vmatrix} \Bigg] \Bigg\}.$$

The last sum in this expression becomes, after lengthy rearrangements of terms and simplifications,

$$D_{i_1,\dots,i_{p-1}}(x_1,\dots,x_p) \sum_{k<l}^{1,p-1} (-1)^{k+l+1}$$

$$\big| P_{i_j-1}(x_1,x_2)\, P_{i_j-m+1}(x_1,\dots,x_m)\, P_{i_j-m-1}(x_1,\dots,x_{m+2})\, P_{i_j-p+1}(x_1,\dots,x_p) \big|_{j=1,2,\dots,p;\, j\neq k,\, \neq l}$$

$$\times \begin{vmatrix} P_{i_k-p}(x_1,\dots,x_{p+1}) & P_{i_k-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \\ P_{i_l-p}(x_1,\dots,x_{p+1}) & P_{i_l-p-1}(x_1,\dots,x_{p+1},x_{q+1}) \end{vmatrix}.$$

After having substituted this result in the expression for T_2, we observe that the summation over k is precisely the Laplacian development of the determinant

$$|P_{i_l-1}(x_1,x_2)\cdots P_{i_l-m+1}(x_1,\ldots,x_m)\;P_{i_l-m-1}(x_1,\ldots,x_{m+2})\cdots$$
$$P_{i_l-p-1}(x_1,\ldots,x_{p+1},x_{q+1})|$$

according to its minors out of the last two columns (see Muir [3, Sec. 93]). Substitution of the results for T_1 and T_2 in (A4′) yields

$$D^*=(x_{q+1}-x_{p+1})D_{i_1,\ldots,i_{p-1}}(x_1,\ldots,x_p)\sum_{m=1}^{p+1}(-1)^{p+m+1}P_{i_q-m}(x_1,\ldots,x_{m+1})$$
$$\times|P_{i_l-1}(x_1,x_2)\cdots P_{i_l-m+1}(x_1,\ldots,x_m)\;P_{i_l-m-1}(x_1,\ldots,x_{m+2})\cdots$$
$$P_{i_l-p-1}(x_1,\ldots,x_{p+1},x_{q+1})|\qquad\text{where}\quad x_{p+2}\quad\text{means}\quad x_{q+1}$$
$$=(x_{q+1}-x_{p+1})D_{i_1,\ldots,i_{p-1}}(x_1,\ldots,x_p)D_{i_1,\ldots,i_p,i_q}(x_1,\ldots,x_{p+1},x_{q+1}).$$

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