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LINEARLY ORDERED EBERLEIN COMPACT SPACES

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In this note we prove that every Eberlein compact linearly ordered space is metrizable. (By an Eberlein compact space we mean a topological space which can be embedded as a compact subset of a Banach space with the weak topology.)

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1. Introduction

A topological space X is *Eberlein compact* if X embeds topologically [1, 7] as a compact subset of a Banach space with the weak topology. Equivalently X is Eberlein compact if X is compact, Hausdorff, and has a collection $\mathcal{F} = \bigcup\{\mathcal{F}(n): n \geq 1\}$ such that:

(a) each $\mathcal{F}(n)$ is a point-finite collection of open F_σ-subsets of X; and

(b) if p and q are distinct points of X, then either some $F \in \mathcal{F}$ has $p \in F \subset X - \{q\}$ or else some $F \in \mathcal{F}$ has $q \in F \subset X - \{p\}$. The collection $\mathcal{F}$ is called an *EC-structure* for X.

A collection of subsets of X which satisfies (b), above, is said to "weakly separate" the points of X. A related property, called "strong separation" of points of X, is

(b)* if p and q are distinct points of X, then some $F \in \mathcal{F}$ has $p \in F \subset X - \{q\}$. It is known that there are non-metrizable Eberlein compact spaces (cf. (3.11)), a fact which shows that weak and strong separation of points are distinct notions because one can prove:

1.1. Theorem *A compact Hausdorff space X is metrizable if and only if there is a σ-point finite collection of open F_σ-subsets of X which strongly separates points of X.*

* This research was completed while the second author was a visiting professor at Vrije Universiteit in Amsterdam.

Theorem 1.1 can be viewed as a corollary of the following well-known metrization theorem:

1.2. Theorem *A compact Hausdorff space with a σ-point finite base is metrizable.*

In this note we prove that a linearly ordered topological space (LOTS), equipped with the usual open interval topology, is metrizable whenever it is Eberlein compact, and we obtain related metrization theorems for certain generalized ordered spaces which admit certain types of point-separating open covers.

2. Notations and elementary lemmas

2.1. Lemma. *If $\mathscr{G}$ and $\mathscr{H}$ are point-finite collections of subsets of a set X, then so are the collections*

(a) $\{G \cap H: G \in \mathscr{G}$ *and* $H \in \mathscr{H}\}$,

(b) $\mathscr{G} \cup \mathscr{H}$;

(b) $\mathscr{G}' = \{S: S$ is a finite intersection of members of $\mathscr{G}\}$.

2.2 Lemma. *Let $\mathscr{G}$ be a point-finite collection of subsets of a set X and for each $G \in \mathscr{G}$ let $\mathscr{C}(G)$ be a countable collection of subsets of G. Then the collection $\bigcup\{\mathscr{C}(G): G \in \mathscr{G}\}$ is σ-point-finite.*

2.3. Lemma. *If X is Eberlein compact, then X has an EC-structure $\mathscr{F} = \bigcup\{\mathscr{F}(n): n \geq 1\}$ which satisfies:*

(a) $X \in \mathscr{F}(1)$;

(b) $\mathscr{F}(n) \subset \mathscr{F}(n+1)$ *for each* $n \geq 1$;

(c) *each* $\mathscr{F}(n)$ *is closed under finite intersections.*

2.4. Notation. Let $\mathscr{F}$ be an EC-structure for X as described in (2.3).

(a) For $p \in X$ and $n \geq 1$, let $C(p, n) = \bigcap\{F \in \mathscr{F}(n): p \in F\}$.

(b) For $p, q \in X$ define $p \sim_n q$ to mean $C(p, n) = C(q, n)$.

(c) For $p \in X$ and $n \geq 1$ let $E(p, n) = \{q \in X: q \sim_n p\}$.

Obviously each relation $\sim_n$ is an equivalence relation on X.

(d) Let $X(n)$ be a subset of X containing exactly one point from each equivalence class $E(p, n)$.

2.5. Lemma. *For $p \in X$ and $n \geq 1$, $E(p, n) = C(p, n) - \bigcup\{C(q, n): C(q, n) \subsetneqq C(p, n)\}$ and hence each set $E(p, n)$ is an F_σ-subset of X.*

2.6. Notation. Use the notations defined in (2.4).

(e) If $p_i \in X(i)$ for $1 \leq i \leq n$, let $C(p_1, p_2, \ldots, p_n) = \bigcap\{C(p_i, i): 1 \leq i \leq n\}$.

(f) For each $p \in X(n)$, use the conclusion of (2.5) to write $E(p, n) = \bigcup\{E(p, n, k): k \geq 1\}$ where each set $E(p, n, k)$ is a closed (and hence compact) subset of X.

(g) If $p_i \in X(i)$ and if $k_i \geq 1$ for $1 \leq i \leq n$, let $E(p_1, \ldots, p_n; k_1, \ldots, k_n) = \bigcap\{E(p_i, i, k_i): 1 \leq i \leq n\}$.

2.7. Lemma. *With notations as in* (2.6),

(a) *the collection* $\mathscr{C}(n) = \{C(p_1, \ldots, p_n): p_i \in X(i)$ for $1 \leq i \leq n\}$ *is a subcollection of* $\mathscr{F}(n)$ *and so is point-finite;*

(b) *if* $p_i \in X(i)$ *and* $k_i \geq 1$ *for* $1 \leq i \leq n$, *then* $E(p_1, \ldots, p_n; k_1, \ldots, k_n)$ *is a subset of* $C(p_1, \ldots, p_n)$.

2.8. Lemma. *For any* $p \in X$, $\bigcap\{E(p, n): n \geq 1\} = \{p\}$.

Proof. Clearly $p \in \bigcap\{E(p, n): n \geq 1\}$ and if $q \neq p$ has $q \in \bigcap\{E(p, n): n \geq 1\}$, then $C(q, n) = C(p, n)$ for every $n \geq 1$. Since $\mathscr{F}$ is an EC-structure, we may choose some $F \in \mathscr{F}$, say $F \in \mathscr{F}(n_1)$, such that either $p \in F \subset X - \{q\}$ or else $q \in F \subset X - \{p\}$. In the first case we see that $C(p, n_1) \subset F$ so that $q \notin C(p, n_1)$ and in the second $C(q, n_1) \subset F$ and $p \notin C(q, n_1)$; each case is impossible because $C(p, n_1) = C(q, n_1)$. □

2.9. Lemma. *Let* $K_1 \supset K_2 \supset \cdots$ *be nonempty compact subsets of a space* X, *and suppose that* W *is an open set in* X *having* $\bigcap\{K_n: n \geq 1\} \subset W$. *Then for some* $n \geq 1$, $K_n \subset W$.

3. Eberlein compact ordered spaces

3.1. Lemma. *Any Eberlein compact LOTS is first countable.*

Proof. If X were a non-first countable Eberlein compact LOTS, then we could embed the usual ordinal space $\Omega = [0, \omega_1]$ as a closed subset of X, and it would follow that Ω is Eberlein compact.

Let $\mathscr{F}$ be an EC-structure for Ω as described in (2.3). Let $\mathscr{F}' = \{F \in \mathscr{F}: \omega_1 \in F\}$. Then $\mathscr{F}'$ is countable so that there must be an ordinal $\alpha < \omega_1$ such that $(\alpha, \omega_1) \subset \bigcap\mathscr{F}'$. But then if $\beta \neq \gamma$ belong to (α, ω_1), any member $F \in \mathscr{F}$ which separates β and γ must fail to contain ω_1. Such an F, being σ-compact, must be bounded away from ω_1 so that the collection $\mathscr{F} - \mathscr{F}'$ is seen to be a point-countable cover of (α, ω_1) by bounded open sets, and an easy application of the Pressing Down Lemma shows that such a cover cannot exist. □

3.2. Lemma. *Any Eberlein compact LOTS* X *has an EC-structure* $\mathscr{F}$ *satisfying* (a), (b) *and* (c) *of* (2.3), *plus*

(d) *each member of* $\mathscr{F}$ *is an open interval in* X.

Proof. Let $\mathcal{F}'$ be any EC-structure for X and let $\mathcal{F}''$ be the family of all convex-components (=maximal convex subsets) of members of $\mathcal{F}'$. Since X is first countable and compact, each member of $\mathcal{F}''$ is an open F_σ-subset of X. Now obtain $\mathcal{F}$ by reproving Lemma (2.3) starting with the collection $\mathcal{F}''$. □

3.3. Notation. Let $\mathcal{F}$ be an EC-structure for the LOTS X as described in (3.2), and let the sets $E(p_1, \ldots, p_n; k_1, \ldots, k_n)$ be defined from $\mathcal{F}$ as in (2.6). Each is compact and therefore has an infimum and supremum which will belong to the set $C(p_1, \ldots, p_n)$ if the set $E(p_1, \ldots, p_n; k_1, \ldots, k_n) \neq \emptyset$. Let:

(h) $S = \{q : q$ is the infimum or supremum of some nonvoid set $E(p_1, \ldots, p_n; k_1, \ldots, k_n)\}$

(i) $H(p_1, \ldots, p_n; k_1, \ldots, k_n) =$
$= (\text{convex hull of } E(p_1, \ldots, p_n; k_1, \ldots, k_n)) -$
$- \{\text{infimum and supremum of } E(p_1, \ldots, p_n; k_1, \ldots, k_n)\}$.

Then $H(p_1, \ldots, p_n; k_1, \ldots, k_n)$ is a convex open subset of X contained in $C(p_1, \ldots, p_n)$. We remind the reader that the sets $H(p_1, \ldots, p_n; k_1, \ldots, k_n)$ are defined only in case $p_i \in X(i)$ and $k_i \geq 1$ for $1 \leq i \leq n$.

(j) For fixed n and $k_1, \ldots, k_n \geq 1$, let

$$\mathcal{H}(k_1, \ldots, k_n) = \{H(p_1, \ldots, p_n; k_1, \ldots, k_n) : p_i \in X(i) \text{ for } 1 \leq i \leq n \text{ and } E(p_1, \ldots, p_n; k_1, \ldots, k_n) \neq \emptyset\};$$

(k) and let $\mathcal{H} = \bigcup\{\mathcal{H}(k_1, \ldots, k_n) : n \geq 1 \text{ and } k_1, \ldots, k_n \geq 1\}$.

3.4. Lemma. *The collection $\mathcal{H}$ is a σ-point finite collection of open* F_σ *sets.*

Proof. Fix $n \geq 1$ and $k_i \geq 1$ for $1 \leq i \leq n$. According to (2.7), the collection $\mathcal{C}(n) = \{C(p_1, \ldots, p_n) : p_i \in X(i) \text{ for } 1 \leq i \leq n\}$ is point finite and $E(p_1, \ldots, p_n; k_1, \ldots, k_n) \subset C(p_1, \ldots, p_n)$. Being a member of $\mathcal{F}(n)$, $C(p_1, \ldots, p_n)$ is an interval so that $H(p_1, \ldots, p_n; k_1, \ldots, k_n) \subset C(p_1, \ldots, p_n)$. According to (2.2) the collection $\mathcal{H}(k_1, \ldots, k_n)$ must be point-finite and so $\mathcal{H}$ is σ-point finite. Since each member of $\mathcal{H}$ is an interval in a first countable LOTS, each member of $\mathcal{H}$ is an F_σ-set in X. □

3.5. Lemma. *Let S be as defined in* (3.3). *Then there is a σ-point finite collection $\mathcal{G}$ of open* F_σ*-sets in X which contains a neighborhood base at each point of S.*

Proof. Fix $k_1, \ldots, k_n \geq 1$. If $p_i \in X(i)$ for $1 \leq i \leq n$ and if $E(p_1, \ldots, p_n; k_1, \ldots, k_n)$ is not empty, then the infimum and supremum of that set belong to $C(p_1, \ldots, p_n)$ so that, X being first countable, there is a countable collection $\mathcal{B}(p_1, \ldots, p_n; k_1, \ldots, k_n)$ of open intervals of X, each contained in $C(p_1, \ldots, p_n)$, and containing a neighborhood base at each of the two extreme points of $E(p_1, \ldots, p_n; k_1, \ldots, k_n)$. Since the collection $\mathcal{C}(n) = \{C(p_1, \ldots, p_n) : p_i \in X(i) \text{ for } 1 \leq i \leq n\}$ is σ-point finite, so is the collection

$\mathcal{B}(k_1,\ldots,k_n)=\bigcup\{\mathcal{B}(p_1,\ldots,p_n;k_1,\ldots,k_n): p_i\in X(i)$ for $1\leq i\leq n\}$. Therefore the collection $\mathcal{G}=\bigcup\{\mathcal{B}(k_1,\ldots,k_n): n\geq 1$ and $k_i\geq 1$ for $1\leq i\leq n\}$ is σ-point finite and contains the required neighborhood bases. □

3.6: Theorem. *Any Eberlein compact LOTS is metrizable.*

Proof. Let $\mathcal{H}$ and $\mathcal{G}$ be as constructed in (3.3), (3.4) and (3.5). Let $\mathcal{I}=\mathcal{G}\cup\mathcal{H}$. Then $\mathcal{I}$ is a σ-point finite collection of open F_σ-subsets of X. Fix distinct points p and q of X. We show that some $I\in\mathcal{I}$ has $p\in I$ and $q\notin I$. In the light of (1.1), that will complete the proof.

Choose an open convex set W with $p\in W\subset X-\{q\}$. If $p\in S$ (cf. (3.3)), then $\mathcal{I}$ contains a neighborhood base at p so that some $I\in\mathcal{I}$ has $p\in I\subset W$, as required. So assume $p\notin S$. For each $n\geq 1$, choose the unique point p_n of $X(n)$ having $p\sim_n p_n$. Then $p\in E(p,n)=E(p_n,n)=\cup\{E(p_n,n,k): k\geq 1\}$ so that we may choose the first index k_n having $p\in E(p_n,n,k_n)$. Then for each $n\geq 1$, $p\in E(p_1,\ldots,p_n;k_1,\ldots,k_n)\subset E(p_n,n)=E(p,n)$ so we conclude from (2.8) that $\bigcap\{E(p_1,\ldots,p_n;k_1,\ldots,k_n): n\geq 1\}=\{p\}\subset W$. Because the compact sets $E(p_1,\ldots,p_n;k_1,\ldots,k_n)$ are nested, Lemma (2.9) applies and yields an index n having $E(p_1,\ldots,p_n;k_1,\ldots,k_n)\subset W$. Since W is convex and $p\notin S$ we must have $p\in H(p_1,\ldots,p_n;k_1,\ldots,k_n)\subset W\subset X-\{q\}$ showing that $H(p_1,\ldots,p_n;k_1,\ldots,k_n)$ is the required member of $\mathcal{I}$. □

Certain generalizations of Theorem 3.6 are available due to the special metrization theory already known for generalized ordered spaces. By a *generalized ordered space* we mean a triple $(X,\mathcal{T},<)$ where $<$ is a linear ordering of X and $\mathcal{T}$ is a topology on X such that $\mathcal{T}$ contains the usual open-interval topology of $<$ and such that $\mathcal{T}$ has a base of open, order-convex sets. It is known that the class of generalized ordered spaces coincides with the class of topological spaces which can be embedded in some LOTS [L].

3.7. Theorem. *Let X be a σ-compact generalized ordered space which has a σ-point finite family $\mathcal{F}$ of open F_σ-sets such that if $p\neq q$ belong to X then some $F\in\mathcal{F}$ either has $p\in F\subset X-\{q\}$ or else $q\in F\subset X-\{p\}$. Then X is metrizable.*

Proof. Write $X=\bigcup\{Y(n): n\geq 1\}$ where each $Y(n)$ is a compact subset of X. Then each $Y(n)$ is a compact generalized ordered space, and hence a LOTS. By restricting the structure $\mathcal{F}$ to the subspace $Y(n)$, we see that $Y(n)$ is an Eberlein compact LOTS and therefore is metrizable. But then X, being a countable union of closed metrizable subspaces, is at least semi-stratifiable in the sense of [2]. But a semistratifiable generalized ordered space is known to be metrizable ([6] or [3]). □

To obtain a version of (3.6) for the class of locally compact generalized ordered spaces, one uses Smirnov's metrization theorem (A space is metrizable if it is paracompact and locally metrizable [4, p. 415]), once the next lemma is established.

The elementary results about stationary sets used in the proof of the lemma may be found in [5] or [3]. (We say that a subset of S of $[0, \kappa)$, where κ is a regular uncountable cardinal, is *stationary* in κ if S meets every cofinal closed subset of the usual ordinal space $[0, \kappa)$.)

3.8. Lemma. *Let X be a generalized ordered space which has a σ-point finite, weakly point separating collection of open sets. Then X is hereditarily paracompact.*

Proof. According to [5], if such an X is not hereditarily paracompact, there is an uncountable regular cardinal κ and a stationary subset S of κ such that S embeds topologically in X. By restricting the σ-point finite open collection in X to the subspace S, we obtain a collection $\mathcal{F} = \bigcup\{\mathcal{F}(n): n \geq 1\}$ of open subsets of S such that:

(a) each $\mathcal{F}(n)$ is point finite;

(b) $\mathcal{F}$ weakly separates points of S.

As usual, we may assume:

(c) each member of $\mathcal{F}$ is a convex subset of S;

(d) $\mathcal{F}(n) \subset \mathcal{F}(n+1)$ for each $n \geq 1$.

Let $\mathcal{F}' = \{F \in \mathcal{F}: F$ is a bounded subset of $S\}$ and let $\mathcal{F}'' = \mathcal{F} - \mathcal{F}'$. The following assertion is an easy consequence of the Pressing Down Lemma:

(*) If T is a stationary subset of κ, then there cannot be a point countable covering of T by bounded open sets.

Therefore, the set $\bigcup\mathcal{F}'$ cannot be stationary in κ so that the set $\bigcup\mathcal{F}''$ must be stationary in κ. Let $\mathcal{F}''(n) = \mathcal{F}'' \cap \mathcal{F}(n)$ for each $n \geq 1$. Since the set $\bigcup\{(\bigcup\mathcal{F}''(n)): n \geq 1\}$ is stationary, one of the sets $\bigcup\mathcal{F}''(n)$ must be stationary. Because of (d), above, we may assume that

(e) for each $n \geq 1$, $\bigcup\mathcal{F}''(n)$ is stationary.

Now let $T(n)$ be the set of non-isolated points of the set $\bigcup\mathcal{F}''(n)$. Then $T(n)$ is stationary and for each $p \in T(n)$ there is a point $f_n(p) < p$ of S such that $[f_n(p), p] \cap S \subset C(p, n) = \bigcap\{F \in \mathcal{F}''(n): p|F\}$. According to the Pressing Down Lemma, there is a stationary set $R(n) \subset T(n)$ and a point q_n such that $f_n(r) = q_n$ for each $r \in R(n)$. Now let $F \in \mathcal{F}''(n)$. Since F is a cofinal convex subset of S and since $R(n)$ is cofinal in S, we may choose $r \in R(n) \cap F$. Then $q_n \in [q_n, r] \cap S \subset C(r, n) \subset F$. Hence each member of $\mathcal{F}''(n)$ contains the point q_n. Since $\mathcal{F}''(n)$ is point finite, $\mathcal{F}''(n)$ is actually finite. Hence $\mathcal{F}''$ is countable. Because κ is regular and uncountable, it follows that there must be a point $p \in S$ having $S \cap [p, \rightarrow) \subset \bigcap\mathcal{F}''$. If there were two distinct points q and r of $S \cap [p, \rightarrow)$ such that neither belongs to $\bigcup\mathcal{F}'$, then no member of $\mathcal{F}$ could be used to separate q and r, contrary to hypothesis. Hence there is a point $q \geq p$ such that the set $T = S \cap [q, \rightarrow) \subset \bigcup\mathcal{F}'$, and that contradicts observation (*), above

Therefore, X is hereditarily paracompact. □

3.9. Theorem. *A locally compact generalized ordered space X which has a σ-point finite collection of open F_σ-sets which weakly separates points of X must be metrizable.*

Proof. According to (3.8), such a space is paracompact. According to (3.6), X is locally metrizable. According to Smirnov's metrization theorem, quoted above, X is metrizable. □

A slight extension of (3.9) is available.

3.10. Corollary. *Suppose* $Y = \prod\{X(n); n \geq 1\}$ *where each* $X(n)$ *is a locally compact* GO-*space. If* Y *has a weakly point separating,* σ*-point finite collection of open* F_σ*-sets, then* Y *is metrizable.*

Proof. Each $X(n)$ embeds in Y and inherits a weakly point-separating σ-point finite collection of relatively open, relative F_σ-sets, so that each $X(n)$ is metrizable. □

J. van Mill has asked whether Theorem 3.6 could be proved for *dendrons*, i.e., compact, connected Hausdorff spaces with the property that any two points can be separated by a third point. (Such spaces are intimately related to linearly ordered spaces since it follows from a theorem of Ward [9] that any dendron is the continuous image of a compact, connected LOTS.) An example suggested by van Mill provides a negative answer to that question.

3.11. Example. There is a non-metrizable, Eberlein compact dendron.

Proof. The underlying set of the space X is the unit square $[0, 1]\times[0, 1]$. For each $x \in (0, 1]$, the subspace $\{x\}\times(0, 1]$ is topologized as a copy of the usual space $(0, 1]$. Basic neighborhoods of a point $(x, 0)$ of X have the form

$$N(x, \varepsilon, t) = (((x-\varepsilon, x+\varepsilon)\times[0, 1]) - (\{x\}\times[t, 1]))\cap X$$

where $\varepsilon > 0$ and $0 < t \leq 1$. The resulting space is a dendron and is non-metrizable since $\{(x, \frac{1}{2}): 0 \leq x \leq 1\}$ is an uncountable relatively discrete subspace of X. To see that X is Eberlein compact, let $\{I(n): n \geq 1\}$ be any countable base of intervals for $(0, 1]$ and let $\mathscr{B}(n) = \{\{x\}\times I(n): x \in [0, 1]\}$. Let $\{J(n): n \geq 1\}$ be and countable base of intervals for $[0, 1]$ and let $\mathscr{C}(n) = \{J(n)\times[0, 1]\}$. Then $\bigcup\{\mathscr{B}(n)\cup\mathscr{C}(n): n \geq 1\}$ is the required σ-point finite collection of open F_σ-sets which weakly separates points of X. □

3.12. Remark. After discussing this paper with the authors, Professor M.E. Rudin was able to prove a related result, viz., that a first countable compact LOTS which has a point-countable weakly separating collection of open sets must be metrizable. First countability is a necessary hypothesis in Rudin's result, as can easily be seen by considering the usual space $[0, \omega_1]$. Rudin's methods are quite different from the ones used in this paper.

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