Input-to-state stability for Lur’e stochastic distributed parameter control systems

Zhixin Tai
Department of Automation, College of Engineering, Bohai University, 121013 Jinzhou, PR China

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ABSTRACT
In this work, the stochastic input-to-state stability (SISS) of Lur’e distributed parameter control systems has been addressed. Using a comparison principle, delay-dependent sufficient conditions for the stochastic input-to-state stability in Hilbert spaces are established in terms of linear operator inequalities (LOIs). Finally, the stochastic wave equation illustrates our result.

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1. Introduction

The input-to-state stability problem, which was initiated in [1], has been extensively investigated by many authors (see [2–4] and the references therein). For the case where a more general type of feedback is allowed, input-to-state stabilizability was verified in [2] to hold even for systems that are not linear in control; this was further extended by Jiang [3] to the discrete-time case. Moreover, Hu and Liu [4] have established the notion of input-to-state stability (ISS) of Runge–Kutta methods for nonlinear control systems. So far, however, input-to-state stability results are only available for systems governed by ordinary differential equations (ODEs) rather than by partial differential equations (PDEs), not to mention by stochastic partial differential equations (SPDEs). Motivated by the fact that distributed parameter systems described by SPDEs are more general, there is a real need to discuss the input-to-state stability problem of such systems. Very recently, the linear operator inequalities (LOIs) approach [5] has provided new insights into the control theory of distributed parameter systems. In this work, the concept of stochastic input-to-state stability (SISS) will be extended to the infinite-dimensional case where, using a comparison principle [6], delay-dependent sufficient conditions for input-to-state stability of Lur’e stochastic distributed parameter control systems are established in the form of linear operator inequalities (LOIs).

2. Preliminaries

Consider the input-to-state stability of Lur’e stochastic control systems in a Hilbert space $H$ described by

\[
\Sigma_0: \begin{cases}
    dx(t) = [Ax(t) + Bx(t - h) + Ew(t) + Fu(t)]dt + [Cx(t) + Dx(t - h)]d\omega(t) \\
    z(t) = Mx(t) + Nx(t - h) + Ru(t) \\
    w(t) = -\varphi(t, z(t)) \\
    x(t) = \phi(t), \quad \text{for } \forall t \in [-h, 0]
\end{cases}
\]  

E-mail address: taizhixin01@163.com.
where \( x(t), z(t) \in H \) are the states, \( w(t), u(t) \in U \) are the control inputs, \( h \) denotes positive constant delay, \( \phi \in C([-h, 0], H) \) is the given initial state, \( \varphi(t, z(t)) : R \times H \rightarrow H \) is an abstract nonlinear function satisfying the following sector condition:

\[
\langle \varphi(t, z(t)) - K_1 z(t), \varphi(t, z(t)) - K_2 z(t) \rangle \leq 0, \tag{2}
\]

and \( \omega(t) \) is a zero-mean real scalar Wiener process on probability space \((\Omega, \mathcal{F}, \mathcal{P})\) with the assumption that \( E[\omega(t)] = 0, E[\omega^2(t)] = dt \).

Without loss of generality, it is assumed that:

(i) Operator \( A \) generates a \( C_0 \)-semigroup \( T(t), t \geq 0 \).
(ii) Operators \( B, C, D, E, F, M, N, K_1, K_2 \) are all linear and bounded.
(iii) Operators \( K_1 \) and \( K_2 \) are linear and may be unbounded.

In what follows, we shall introduce some notation and definitions.

The set of such controls that are measurable and locally essentially bounded in a Hilbert space \( U \) with the supremum norm \( \|u\|_{\text{sup}} := \text{sup}\{\|u(t)\| : t \geq -h\} < \infty \) is denoted by \( \mathcal{L}_\infty \).

For each \( \phi \in C([-h, 0], H) \) and \( u \in \mathcal{L}_\infty \), we denote by \( x(t, \phi, u) \) the solution trajectory of system \( \Sigma_0 \) with initial state \( \phi \) and control input \( u \).

**Definition 2.1.** A function \( \gamma : R_+ \rightarrow R_+ \) is said to be a class \( K \)-function if it is continuous, zero at zero and strictly increasing. A function \( \beta : R_+ \times R_+ \rightarrow R_+ \) is said to be a class \( KL \)-function if for each fixed \( t \geq 0 \), the function \( \beta(\cdot, t) \) is a class \( K \)-function and for each fixed \( s \geq 0 \), the function \( \beta(s, \cdot) \) is decreasing and \( \beta(s, t) \rightarrow 0 \) as \( t \rightarrow \infty \).

Consequently, we are now in a position to define the concept of stochastic input-to-state stability (SISS) in Hilbert spaces.

**Definition 2.2.** System \( \Sigma_0 \) is called stochastic input-to-state stable (SISS) if there exist a class \( KL \)-function \( \gamma : R_+ \rightarrow R_+ \) and a class \( K \)-function \( \beta : R_+ \rightarrow R_+ \) such that for any initial state \( \phi \in C([-h, 0], H) \) and any bounded control input \( u \in \mathcal{L}_\infty \), it holds that

\[
E\{\|x(t, \phi, u)\|\} \leq \beta(\|\phi\|_h, t) + \gamma(\|u\|_{\text{sup}})
\]

where \( E \) denotes the mathematical expectation, and \( \|\phi\|_h := \text{sup}\{\|\phi(\theta)\| : -h \leq \theta \leq 0\} \).

As a key tool for developing our result in this work, some lemmas will be introduced as follows.

**Lemma 2.1** (Comparison Principle [6]). If the function \( g(x, y) \) is continuous and satisfies a Lipschitz condition, then the implication

\[
\begin{align*}
D_+ m(x) & \leq g(x, m(x)) \\
D_+ u(x) & \geq g(x, u(x)) \\
m(x_0) & \leq u(x_0)
\end{align*}
\]

is true for continuous functions \( m(x) \) and \( u(x) \).

**Lemma 2.2** (Wirtinger's Inequality [5]). Let \( z \in W^{1,2}([a, b], R) \) be a scalar function with \( z(a) = z(b) = 0 \). Then

\[
\int_a^b z^2(\xi) d\xi \leq \frac{(b-a)^2}{\pi^2} \int_a^b \left( \frac{dz(\xi)}{d\xi} \right)^2 d\xi.
\]

### 3. Stochastic input-to-state stability in a Hilbert space

In this section, a delay-dependent absolute stochastic input-to-state stability condition for system \( \Sigma_0 \) is presented in terms of linear operator inequalities (LOIs).

In the case where the nonlinear function \( \varphi(t, z(t)) \) belongs to the sector \([0, K]\), i.e.,

\[
\langle \varphi(t, z(t)), \varphi(t, z(t)) - K z(t) \rangle \leq 0,
\]

the following result is given.

**Theorem 3.1.** Given a positive constant delay \( h > 0 \), let there exist positive constants \( \beta > 0, \varepsilon > 0 \), a linear positive definite operator \( Q_1 : D(A) \rightarrow H \) and nonnegative definite operators \( Q_2 \in \mathcal{L}(H) \) and \( Q_3 \in \mathcal{L}(U) \) with the following inequalities:

\[
\alpha(x, x) \leq (x, Q_1 x) \leq \gamma_{q_1} \left[ (x, x) + (Ax, Ax) \right], \tag{5}
\]

\[
(x, Q_2 x) \leq \gamma_{q_2} (x, x), \tag{6}
\]

\[
(u(t), Q_3 u(t)) \leq \gamma_{q_3} (u(t), u(t)) \tag{7}
\]

\[
\int_a^b z^2(\xi) d\xi \leq \frac{(b-a)^2}{\pi^2} \int_a^b \left( \frac{dz(\xi)}{d\xi} \right)^2 d\xi.
\]
for some positive constants $\alpha, \gamma_1, \gamma_2, \gamma_3$ such that the LOI

$$\mathcal{Z} := \begin{bmatrix}
Q_1(A + \beta I) + (A + \beta I)^*Q_1 + C^*Q_1C + Q_2 & Q_1B + C^*Q_1D & Q_1E - \varepsilon M^*K^* & Q_1F \\
\ast & \ast & 0 & 0 \\
\ast & \ast & -\varepsilon N^*K^* & 0 \\
\ast & \ast & -2\varepsilon I & -2Q_3
\end{bmatrix} < 0 \quad (8)$$

holds in the Hilbert space $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$. Then system $\Sigma_0$ is absolutely stochastically input-to-state stable (SISS) in the sector $[0, K]$.

**Proof.** Choose the following positive semi-definite Lyapunov–Krasovskii functional in Hilbert spaces:

$$V(x_t) = \langle x(t), Q_1x(t) \rangle + \int_0^te^{2\beta\theta} \langle x(t + \theta), Q_2x(t + \theta) \rangle \, d\theta. \quad (9)$$

It follows from (5)–(6) and (9) that the estimate for functional $V(x_t)$ can be obtained as follows:

$$\alpha\|x(t)\|^2 \leq V(t) \leq \gamma_1 \left( \|x(t)\|^2 + \|Ax(t)\|^2 \right) + h\gamma_2 \|x_t\|^2. \quad (10)$$

The stochastic differential $dV(t, x_t)$ can be computed, by using the Itô formula, as

$$dV(t, x_t) = \mathcal{L}V(t, x_t)dt + 2\langle x(t), Q_1(Cx(t) + Dx(t - h)) \rangle \, d\omega(t) \quad (11)$$

where

$$\mathcal{L}V(t, x_t) = 2\langle x(t), Q_1(Ax(t) + Bx(t - h) + Eu(t) + Fu(t)) \rangle + \langle x(t), Q_2x(t) \rangle$$

$$- e^{-2\beta t} \langle x(t - h), Q_2x(t - h) \rangle - 2\beta \int_{t-h}^t e^{2\beta(s-t)} \langle x(s), Q_2x(s) \rangle \, ds$$

$$+ \langle (Cx(t) + Dx(t - h)), Q_1(Cx(t) + Dx(t - h)) \rangle. \quad (12)$$

It follows from (9) and (12) that

$$\mathcal{L}V(t, x_t) + 2\beta V(x_t) - 2\langle u(t), Q_3u(t) \rangle = \langle \eta(t), \Theta \eta(t) \rangle \quad (13)$$

where

$$\eta(t) := \begin{bmatrix}
x(t - h) \\
w(t)
\end{bmatrix}, \quad \Theta := \begin{bmatrix}
Q_1(A + \beta I) + (A + \beta I)^*Q_1 + C^*Q_1C + Q_2 & Q_1B + C^*Q_1D & Q_1E - \varepsilon M^*K^* & Q_1F \\
\ast & \ast & 0 & 0 \\
\ast & \ast & -\varepsilon N^*K^* & 0 \\
\ast & \ast & -2\varepsilon I & -2Q_3
\end{bmatrix}.$$

In view of LOI (8), the inequality $\langle \eta(t), \mathcal{Z} \eta(t) \rangle < 0$, i.e.,

$$\langle \eta(t), \Theta \eta(t) \rangle - 2\varepsilon \langle w(t), w(t) \rangle - 2\varepsilon \langle u(t), K(Mx(t) + Nx(t - h) + Ru(t)) \rangle < 0,$$

holds for $\forall \eta(t) \neq 0$, which implies that the inequality $\langle \eta(t), \Theta \eta(t) \rangle < 0$ holds for $\forall 0 \neq \eta(t)$ satisfying (4), and hence from equality (13), along the solution trajectories of $\Sigma_0$, we have that

$$\mathcal{L}V(t, x_t) \leq -2\beta V(t, x_t) + 2\langle u(t), Q_3u(t) \rangle. \quad (14)$$

In view of Dynkin's formula [7], Lemma 2.1 and inequalities (5)–(7), it is easy to obtain that

$$\alpha \mathbb{E} \left\{ \|x(t, \phi, u)\|^2 \right\} \leq \mathbb{E} \{V(t, x_t)\} \leq e^{-2\beta t} V(0) + \frac{1}{\beta} \sup_{\tau \geq 0} \langle u(\tau), Q_3u(\tau) \rangle$$

$$\leq (2\gamma_1 + h\gamma_2)e^{-2\beta t} \cdot \max \left\{ \|\phi\|_h, \|A\phi(0)\| \right\} + \frac{\gamma_2}{\beta} \|u\|_{sup}^2$$

$$\leq \left( \sqrt{2\gamma_1 + h\gamma_2} e^{-\beta t} \cdot \max \left\{ \|\phi\|_h, \|A\phi(0)\| \right\} + \sqrt{\frac{\gamma_2}{\beta}} \|u\|_{sup} \right)^2 \quad (15)$$

which implies that

$$\mathbb{E} \left\{ \|x(t, \phi, u)\| \right\} \leq \frac{1}{\sqrt{\alpha}} \left( \sqrt{2\gamma_1 + h\gamma_2} e^{-\beta t} \cdot \max \left\{ \|\phi\|_h, \|A\phi(0)\| \right\} + \sqrt{\frac{\gamma_2}{\beta}} \|u\|_{sup} \right). \quad (16)$$

And hence from Definition 2.2, the proof is completed. \(\square\)

Under the more general circumstances where the nonlinear function $\phi(t, z(t))$ satisfies the sector condition (2), using the loop transformation technique [8], one comes to the conclusion that the absolute stochastic input-to-state stability of
Consider the 3D stochastic wave equations

$$dz(t) = \left( a\nabla^2 z(\xi, \eta, \zeta, t) - \mu_0 z_t(\xi, \eta, \zeta, t) - \mu_1 z_{tt}(\xi, \eta, \zeta, t - h) - a_0 z(\xi, \eta, \zeta, t - h) - a_1 z(\xi, \eta, \zeta, t - h) \right) dt + \left( b_z(\xi, \eta, \zeta, t - h) - b_0 z(\xi, \eta, \zeta, t - h) - b_1 z(\xi, \eta, \zeta, t - h) \right) dw(t)$$

with Neumann boundary conditions

$$z^{(i)}(\xi, \eta, \zeta, 0, t) = z^{(i)}(\xi, \eta, \zeta, t, 0) = 0 \quad (i = 0, 1)$$

$$z^{(i)}_{\xi}(\xi, \eta, \zeta, 0, t) = z^{(i)}_{\xi}(\xi, \eta, \zeta, t, 0) = 0 \quad (i = 0, 1)$$

$$z^{(i)}_{\eta}(\xi, \eta, \zeta, 0, t) = z^{(i)}_{\eta}(\xi, \eta, \zeta, t, 0) = 0 \quad (i = 0, 1)$$

where $\nabla^2$ denotes the Laplace operator, i.e., $\nabla^2 := \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}$, with constant parameters $a > 0, \mu_0 > 0,$ and $\xi \in [0, \pi], t \geq 0.$

The boundary-value problem (23)–(26) can be rewritten as Eq. (1) in a Hilbert space:

$$\mathcal{H} = \left\{ z \in W^{2,2}(0, \pi) \times (0, \pi) \times (0, \pi), \mathbf{R} \text{ s.t. boundary conditions (24)–(26)} \right\}$$

with operators

$$A = \begin{bmatrix} 0 & 0 \\ a\nabla^2 & -a_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -a_1 & -a_1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ b & -b_0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ -b_1 & -b_1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ r_1 & r_2 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 \\ k_{21} & k_{22} \end{bmatrix}$$

and with the state $x(t) := \begin{bmatrix} z(\xi, \eta, \zeta, t) \\ z_t(\xi, \eta, \zeta, t) \end{bmatrix}$ and control $u(t) := \begin{bmatrix} u_1(\xi, \eta, \zeta, t) \\ u_2(\xi, \eta, \zeta, t) \end{bmatrix}.$
Theorem 4.1. Given the scalars $\beta > 0$, $a > 0$, $\mu_0 > 0$, $\mu_1$, $a_0$, $a_1$, $b$, $b_0$, $b_1$, $c_1$, $c_2$, $m_1$, $m_2$, $n_1$, $n_2$, $k_1$, $k_2$, $k_3$, $p_{12}$, $k_{21}$, $k_{22}$, $k_{23}$ and $p_{22}$, let there exist a positive constant $\varepsilon > 0$, a symmetric positive definite matrix $Q_m = \begin{bmatrix} q_{01} & q_{02} & q_{03} \\ q_{02} & q_{02} & q_{03} \\ q_{03} & q_{03} & q_{03} \end{bmatrix} > 0$ (where $q_{03} > 0$) and nonnegative definite matrices $Q_2 \geq 0$ and $Q_3 \geq 0$ such that the following LMIs hold:

$$
q_{02} - \beta q_{03} > 0 \quad \text{(27)}
$$

$$
\begin{bmatrix}
\Pi Q_m(B - EK_{1m}N) + C^T Q_mD & Q_mE - \varepsilon M^T(K_{2m} - K_{1m})^T & Q_m(F - EK_{1m}R) \\
* & -\varepsilon N^T(K_{2m} - K_{1m})^T & -\varepsilon (K_{2m} - K_{1m})R \\
* & * & -2Q_3
\end{bmatrix} < 0 \quad \text{(28)}
$$

where

$$
\Pi := Q_m \begin{bmatrix} -a_0 - 3a - h_1 - 3h_3 & \beta - \mu_0 - h_2 \\
1 & -a_0 + 3a - h_1 - 3h_3 \end{bmatrix} + \begin{bmatrix} \beta & -a_0 - 3a - h_1 - 3h_3 \\
1 & \beta - \mu_0 - h_2 \end{bmatrix} Q_m + C^T Q_m C + Q_2
$$

$$
h_1 := c_1 k_{11} m_1 + c_2 p_{12} m_1, \quad h_2 := c_1 k_{12} m_2 + c_2 k_{13} m_2, \quad h_3 := c_2 p_{11} m_1 > 0
$$

$$
K_{1m} := \begin{bmatrix} k_{11} & k_{12} \\ p_{12} & k_{13} \end{bmatrix}, \quad K_{2m} := \begin{bmatrix} k_{21} & k_{22} \\ p_{22} & k_{23} \end{bmatrix}.
$$

Then boundary-value problem (23)–(26) is absolutely stochastically input-to-state stable in the sector $[K_1, K_2]$.

Proof. In the case of the stochastic wave equation (23), consider the Lyapunov–Krasovskii functional $V$ taken in (9) where

$$
Q_1 = \begin{bmatrix} q_{01} - aq_{03} \nabla^2 + h_3 q_{03} & \frac{\partial^4}{\partial \eta^4} + \frac{\partial^4}{\partial \zeta^4} \\
q_{02} & q_{02} \\
q_{03} & q_{03} \end{bmatrix}, \quad Q_2 \geq 0. \quad \text{(29)}
$$

The proof is given in the following steps.

Step 1. Integrating by parts and utilizing Wirtinger’s inequality given in Lemma 2.1, direct computation can obtain that

$$
\langle x, Q_1 x \rangle = -a q_{03} \iint_\Omega z \cdot \nabla^2 z \, dv + h_3 q_{03} \iint_\Omega z \cdot \left( \frac{\partial^4}{\partial \eta^4} + \frac{\partial^4}{\partial \zeta^4} \right) z \, dv + \langle x, \begin{bmatrix} q_{01} & q_{02} & q_{03} \end{bmatrix} \rangle \begin{bmatrix} x \\
q_{02} \\
q_{03} \end{bmatrix} \rangle
$$

$$
\geq 3(a + h_3) q_{03} \iint_\Omega z^2 \, dv + \langle x, \begin{bmatrix} q_{01} & q_{02} & q_{03} \end{bmatrix} \rangle \begin{bmatrix} x \\
q_{02} \\
q_{03} \end{bmatrix} \rangle = \langle x, Q_{m} x \rangle > 0 \quad \text{for} \quad x \neq 0 \quad \text{(30)}
$$

where region $\Omega := \{ (\xi, \eta, \zeta) : 0 \leq \xi \leq \pi, 0 \leq \eta \leq \pi, 0 \leq \zeta \leq \pi \}$, which, together with the self-adjointness of operator $Q_1$, implies that operator $Q_1$ is positive definite.

Step 2. In view of Wirtinger’s inequality given in Lemma 2.1 and the inequality (27), we have that

$$
\langle x, (E - K_{1m} M + \beta I)^* Q_1 + Q_1 (E - K_{1m} M + \beta I) x \rangle
$$

$$
= \langle x, \left( a \nabla^2 - h_3 \left( \frac{\partial^4}{\partial \eta^4} + \frac{\partial^4}{\partial \zeta^4} \right) \right) - a_0 - h_1 \beta - \mu_0 - h_2 \right)^* \begin{bmatrix} 1 \\
q_{01} - a q_{03} \nabla^2 + h_3 q_{03} & \frac{\partial^4}{\partial \eta^4} + \frac{\partial^4}{\partial \zeta^4} \\
q_{02} & q_{02} \\
q_{03} & q_{03} \end{bmatrix} \rangle \begin{bmatrix} x \\
q_{02} \\
q_{03} \end{bmatrix} \rangle
$$

$$
\leq \langle x, Q_m \left( \begin{bmatrix} -a_0 - 3a - h_1 - 3h_3 & \beta - \mu_0 - h_2 \\
1 & \beta - \mu_0 - h_2 \end{bmatrix} q_{03} > 0 \right) \rangle
$$

From the above analysis it follows that if LMIs (27) and (28) hold, then LOI (22) is satisfied, and hence by Theorem 3.2, the proof is completed. □
**Remark 4.1.** Utilizing Theorem 4.1 for the stochastic wave equation (23) with coefficients $a_0 = 30$, $a_1 = -0.2$, $a_2 = 0.14$, $a_3 = -0.12$, $b_0 = 1.2$, $b_1 = 1.5$, $b_2 = 0.2$, $b_3 = -0.1$, $c_1 = 1.3$, $c_2 = 0.2$, $d_1 = 0.5$, $d_2 = -0.3$, $M = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 0 \\ 0 & 0.3 \end{bmatrix}$, $R = \begin{bmatrix} 0 & 0.7 & 0.4 \end{bmatrix}$, $K_{1m} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$, $K_{2m} = \begin{bmatrix} 2 & 0 \\ 0 & 0.9 \end{bmatrix}$, $p_{11} = 1$, $p_{21} = 1.3$, $p_{31} = 0.08$, and $h_3 = 0.2$ yields that system (23)–(26) is absolutely stochastically input-to-state stable in the sector $[K_1, K_2]$ with decay rate $\beta = 1.5$ and maximum delay $h_{\text{max}} = 2.2801$ where operators $K_1 = \begin{bmatrix} p_{11} & 0 \\ \frac{\partial}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} & 0 \end{bmatrix} + K_{1m}$ and $K_2 = \begin{bmatrix} \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} & 0 \end{bmatrix} + K_{2m}$.

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