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Rota's Alternating Procedure with Non-positive Operators

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Let (T_n) be a sequence of linear contractions on all L^p spaces, $1 \leqslant p \leqslant \infty$. We show that $\lim_n T_1^* T_2^* \cdots T_n^* T_n \cdots T_2 T_1 f$ exists a.e. for each function $f \in L \operatorname{Log} L$. This extends to the non-positive case (G. C. Rota, *Bull. Amer. Math. Soc.* **68**, 95–102; N. Starr, *Trans. Amer. Math. Soc.* **121**, 90–115). We obtain also the a.e. convergence of products $J_\nu T_1^* \cdots T_n^* J_\mu T_n \cdots T_1 f$ in L^p for some non-positive contractions on L^p, $1 < p < \infty$.

INTRODUCTION

Let (Ω, a, μ) be a σ-finite measure space and L^p the usual Banach spaces constructed on (Ω, a, μ). A linear operator (all operators in this paper are linear) $T: L^p \to L^p$ is a contraction if $\|T\|_p \leqslant 1$, $1 \leqslant p \leqslant \infty$. If T is a contraction of all L^p spaces $1 \leqslant p \leqslant \infty$, simultaneously then we will say that T is an $L^1 - L^\infty$ contraction. The operator T is positive if $Tf \geqslant 0$ a.e. if $f \geqslant 0$ a.e.

In [5] G. C. Rota introduced the following procedure for $L^1 - L^\infty$ positive contractions. He considered the products $T_1^* T_2^* \cdots T_n^* T_n \cdots T_1$ and proved the pointwise convergence in L^p, $1 < p < \infty$. He assumed that $T_i \mathbf{1} = \mathbf{1}$ for all i. This assumption was removed by N. Starr, who obtained the following result:

THEOREM 1. [6]. *Let T_n be a sequence of positive $L^1 - L^\infty$ contractions. Then $\int \sup_{n \geqslant 1} |T_1^* T_2^* \cdots T_n^* T_n \cdots T_1 f| \, d\mu < \infty$ for $f \in L \operatorname{Log} L$ and $T_1^* T_2^* \cdots T_n^* T_n \cdots T_1 f$ converges a.e.*

This result also generalized earlier work of D. L. Burkholder and Y. S. Chow [3], J. L. Doob [4], and E. M. Stein [7].

We want to show in this paper that in Theorem 1 we can get the pointwise convergence without the positivity assumption, giving a proof of a result announced already in [2].

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If we use the notation $|T|$ for the linear modulus of an L^1-L^∞ contraction T we see easily that

$$\sup_n |T_1^*T_2^* \cdots T_n^*T_n \cdots T_1 f| \in L^1 \qquad \text{for} \quad f \in L \operatorname{Log} L$$

because $|T_1^* \cdots T_n^*T_n \cdots T_1 f| \leqslant |T_1|^* \cdots |T_n|^*|T_n| \cdots |T_1|\, |f|$.

If we wish to apply Banach's principle we need a dense subset on which the poinwise convergence holds. But the existence of such a set does not seem obvious.

In Theorem 4 we will obtain the pointwise convergence of the products $J_\nu^*T_1^* \cdots T_n^*J_\mu T_n \cdots T_1 f$ in L^p, $1<p<\infty$, when each contraction T_n is of the form $T_n f = h_n \cdot \phi_n(f)$ (support separating contractions) and

$$\varlimsup_{\substack{x\to 0\\ x>0}} \frac{(\mu(x))^{q-1}}{x} < \infty \qquad \text{and} \qquad \varlimsup_{\substack{x\to 0\\ x>0}} \frac{(\nu(x))^{p-1}}{x} < \infty \left(q = \frac{p}{p-1}\right).$$

As in [2], ν and μ belong to the set C of real strictly increasing continuous functions defined on $[0, +\infty[$ verifying $\mu(0)=0$. The spaces L^r, $1<r<\infty$, being smooth there exist a unique map J_μ for each $\mu \in C$, $J_\mu : L^p \to L^q$ satisfying

(i) for any f in L^p, $(J_\mu f, f) = \|J_\mu f\|\,\|f\|$,

(ii) for any f in L^p, $\|J_\mu f\|_q = \mu(\|f\|_p)$.

THEOREM 2. *Let T_n be a sequence of L^1-L^∞ contractions. For any $f \in L \operatorname{Log} L$, $T_1^*T_2^* \cdots T_n^*T_n \cdots T_1 f$ converges a.e.*

Proof. We can assume that (Ω, a, μ) is a probability measure space. For each T_n, L^1-L^∞ contraction we associate the following L^1-L^∞ positive contraction T_n^+, T_n^-, where

$$T_n^+ = \frac{|T_n| + T_n}{2} \qquad \text{and} \qquad T_n^- = \frac{|T_n| - T_n}{2}.$$

Then we consider the spaces $\tilde{L}^p(\Omega_1 \cup \Omega_2, a \cup a, \tilde{\mu})$ (disjoint copies of (Ω, a, μ)) and on these spaces the $\tilde{L}^1 - \tilde{L}^\infty$ positive contractions $\tilde{T}_n$ defined by

$$\begin{aligned}\tilde{T}_n(\mathbf{1}_{\Omega_1} f_1 + \mathbf{1}_{\Omega_2} f_2) = {} & (T_n^+(\mathbf{1}_{\Omega_1} f_1) + T_n^-(\mathbf{1}_{\Omega_2} f_2))\,\mathbf{1}_{\Omega_1} \\ & + (T_n^-(\mathbf{1}_{\Omega_1} f_1) + T_n^+(\mathbf{1}_{\Omega_2} f_2))\,\mathbf{1}_{\Omega_2}.\end{aligned}$$

(A simple calculation shows that $\tilde{T}_n$ are $\tilde{L}^1 - \tilde{L}^\infty$ contractions.) Furthermore the adjoint operator is

$$\tilde{T}_n^*(\mathbf{1}_{\Omega_1} g_1 + \mathbf{1}_{\Omega_2} g_2) = (T_n^{+*}(\mathbf{1}_{\Omega_1} g_1) + T_n^{-*}(\mathbf{1}_{\Omega_2} g_2)) \cdot \mathbf{1}_{\Omega_1}$$
$$+ (T_n^{-*}(\mathbf{1}_{\Omega_1} g_1) + T_n^{+*}(\mathbf{1}_{\Omega_2} g_2)) \mathbf{1}_{\Omega_2}.$$

We observe that if we start with the function $\tilde{f} = \mathbf{1}_{\Omega_1} f - \mathbf{1}_{\Omega_2} f$ ($f \in L^1$) we have

$$\mathbf{1}_{\Omega_1} \tilde{T}_1^* \cdots \tilde{T}_n^* \tilde{T}_n \cdots \tilde{T}_1 \tilde{f} = \mathbf{1}_{\Omega_1} T_1^* \cdots T_n^* T_n \cdots T_1 f.$$

So if $f \in L \operatorname{Log} L$ then the pointwise convergence follows from Theorem 1.

Remark 3. In [1] M. A. Akcoglu obtained recently the pointwise convergence in complex L^p spaces for f in any L^p, $1 < p < \infty$. It is easy to see by using Banach's principle that the same conclusion holds in $L \log L$. The proof given here is simpler (no use of a reduction to the finite dimensional case and dilation's methods).

For the next theorem we will use some notations given in the Introduction. For instance, $\mathscr{C}$ is the set of real continuous strictly increasing functions defined on $[0, +\infty[$ verifying $\mu(0) = 0$.

THEOREM 4. *Let (Ω, α, μ) be a σ-finite measure space, T_n a sequence of support separating contractions on $L^p(\mu)$, $1 < p < +\infty$ (not necessarily positive), and ν and μ two functions in $\mathscr{C}$.*

(i) *If ν and μ satisfy the conditions $\overline{\lim}_{x \to 0}((\mu(x))^{q-1}/x) < +\infty$ and $\overline{\lim}_{x \to 0}((\nu(x))^{p-1}/x) < +\infty$, then for any f in L^p, $J_\nu^* T_1^* T_2^* \cdots T_n^* J_\mu T_n \cdots T_1 f$ converges a.e. in L^p.*

(ii) *If we have $\lim_n \|T_n \cdots T_1 f\|_p > 0$ for all functions f, $f \neq 0$, then for any f in L^p the sequence*

$$J_\nu^* T_1^* T_2^* \cdots T_n^* J_\mu T_n \cdots T_1 f \textit{ converges a.e. in } L^p.$$

Proof. We remark first that from the equality $(J_\mu f, f) = \|J_\mu f\|_q \cdot \|f\|_p$ and the fact that $f^* = |f|^{p-1} \operatorname{sgn} f$ is the unique function in L^q such that $(f^*, f) = \|f\|_p^p$ we have

$$J_\mu f = f^* \|J_\mu f\|_q \, \|f\|_p^{1-p}. \qquad (*)$$

When T is a support separating contraction on L^p we know that $Tf = h \cdot \phi f$ (ϕ being multiplicative). We have $\int |h|^p \phi f \, d\mu = \int f \cdot D(h) \, d\mu$, where $D(h)$ is the Radon–Nykodim derivative of the measure $m(E) = \int_{\phi(E)} |h|^p \, d\mu$ with respect to μ. So $\|D(h)\|_\infty = \|T\|^p$.

Now we compute, for any support separating contraction T and any ν, μ in C, $J_\nu^* T^* J_\mu Tf$. In view of (*) it is enough to know $T^* J_\mu Tf$. We have for any g in $L^q(\mu)$

$$\begin{aligned}(g, T^* J_\mu Tf) &= (Tg, J_\mu Tf)\\ &= \int h \cdot \phi g \cdot (Tf)^* \, \|J_\mu(Tf)\|_q \, \|Tf\|_p^{1-p} \, d\mu\\ &= \int h \cdot \phi g \cdot |h|^{p-1} \operatorname{sgn} h \cdot \phi(|f|^{p-1}) \operatorname{sgn} \phi f\\ &\quad \cdot \|J_\mu(Tf)\|_q \, \|Tf\|_p^{1-p} \cdot d\mu\\ &= \|J_\mu(Tf)\|_q \cdot \|Tf\|_p^{1-p} \cdot \int |h|^p \cdot \phi(g \cdot |f|^{p-1} \operatorname{sgn} f) \cdot d\mu\\ &= \|J_\mu(Tf)\|_q \cdot \|Tf\|_p^{1-p} \cdot \int g \cdot |f|^{p-1} \cdot \operatorname{sgn} f \cdot D(h) \, d\mu.\end{aligned}$$

So $T^* J_\mu Tf = D(h) \cdot (|f|^{p-1} \cdot \operatorname{sgn} f) \cdot \|J_\mu(Tf)\|_q \cdot \|Tf\|_p^{1-p}$ and $J_\nu^* T^* J_\mu Tf = (D(h))^{q-1} \cdot f \cdot \|J_\mu(Tf)U_q^{q-1} \cdot \|Tf\|_p^{-1} \cdot \|J_\nu^* T^* J_\mu Tf\|_p \cdot \|T^* J_\mu Tf\|_q^{1-q}$.

We can use this last equality for the support separating contraction $\tilde{T}_n = T_n \cdots T_1$. If we denote $\tilde{T}_n f = \tilde{h}_n \tilde{\phi}_n(f)$ then

$$\begin{aligned}J_\nu^* T_n^* J_\mu T_n f = (D(\tilde{h}_n))^{q-1} \cdot f \cdot \|J_\mu(\tilde{T}_n f)\|_q^{q-1} \cdot \|\tilde{T}_n f\|_p^{-1}\\ \cdot \|J_\nu^* \tilde{T}_n^* J_\mu \tilde{T}_n f\|_p \cdot \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_q^{1-q}.\end{aligned}$$

As $\|D(\tilde{h}_n)\|_\infty = \|\tilde{T}_n\|^p \leqslant 1$ we have $0 \leqslant D(\tilde{h}_n) \leqslant 1$ a.e. The sequence $D(\tilde{h}_n)$ is a decreasing one. In fact if we denote $\tilde{T}_n f = \tilde{h}_n \cdot \tilde{\phi}_n f$, $T_{n+1} f = h_{n+1} \cdot \phi_{n+1} f$ then we have $\tilde{T}_{n+1} f = \tilde{h}_{n+1} \cdot \tilde{\phi}_{n+1}(f) = h_{n+1} \cdot \phi_{n+1}(\tilde{T}_n f)$,

$$\begin{aligned}\int |h_{n+1}|^p \cdot \tilde{\phi}_{n+1}(|f|^p) \, d\mu = \int |f|^p \cdot D(\tilde{h}_{n+1}) \cdot d\mu\\ = \int |h_{n+1}|^p \cdot \phi_{n+1}(|\tilde{T}_n f|^p) \, d\mu\\ = \int D(h_{n+1}) \cdot |\tilde{T}_n f|^p \cdot d\mu\\ \leqslant \int |h_n|^p \cdot \tilde{\phi}_n(|f|^p) \, d\mu\\ = \int |f|^p \cdot D(\tilde{h}_n) \, d\mu.\end{aligned}$$

So $D(\tilde{h}_n)$ converges a.e. to the function $\tilde{h}$. Let us denote $A = \{\tilde{h}(\omega) > 0 \text{ a.e.}\}$.

(i) In this case we write

$$J_\nu^* \tilde{T}_n J_\mu \tilde{T}_n f = (D(\tilde{h}_n))^{q-1} \cdot f \cdot \frac{(\mu(\|\tilde{T}_n f\|_p))^{q-1}}{\|\tilde{T}_n f\|_p} \cdot \frac{\nu(\|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p)}{\|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p^{q-1}};$$

(a) if $\lim_n \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p > 0$ (which exists by Theorem 1) then we must have also $\lim_n \|\tilde{T}_n f\|_p > 0$ and $T_\nu^* \tilde{T}_n T_\mu \tilde{T}_n f$ converges a.e. to

$$(\tilde{h})^{q-1} \cdot f \cdot \lim_n \frac{(\mu(\|\tilde{T}_n f\|_p))^{q-1}}{\|\tilde{T}_n f\|_p} \cdot \lim_n \frac{\nu(\|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p)}{\|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p^{q-1}};$$

(b) if $\lim_n \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p = 0$ and $\lim_n \|\tilde{T}_n f\|_p = 0$ then $\int \tilde{h} \cdot |f|^p \, d\mu = 0$ and $f = 0$ on A. By using the assumptions $\overline{\lim}_{x \to 0}((\mu(x))^{q-1}/x) < +\infty$ and $\overline{\lim}_{x \to 0}((\nu(x))^{p-1}/x) < +\infty$ we conclude that $J_\nu^* T_n J_\mu T_n f \to 0$ a.e.;

(c) If $\lim_n \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_p = 0$ and $\lim_n \|\tilde{T}_n f\|_p > 0$ we have then $\int \tilde{h} \cdot |f|^p \, d\mu > 0$ and $f \neq 0$ on A. By using (**) as in (i) we conclude that

$$\lim_n \|J_\mu(\tilde{T}_n f)\|_q^{q-1} \cdot \|\tilde{T}_n f\|_p^{-1} \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_q^{1-q} = \frac{1}{\|\tilde{h}^{q-1} \cdot f\|_p} < +\infty$$

and

$$\lim_n J_\nu^* \tilde{T}_n^* J_\mu \tilde{T}_n f = \tilde{h}^{q-1} \cdot f \cdot \lim_n \|J_\nu^* \tilde{T}_n^* J_\nu \tilde{T}_n^* J_\mu \tilde{T}_n f\|_p \cdot \frac{1}{\|\tilde{h}^{q-1} \cdot f\|_p}.$$

(ii) If $\lim_n \|\tilde{T}_n f\|_p > 0$ for all functions f in L^p, $f \neq 0$, then as

$$\lim_n \int |\tilde{T}_n f|^p \, d\mu = \int |h_n|^p \cdot \tilde{\phi}_n(|f|^p) \, d\mu$$
$$= \int D(\tilde{h}_n) \cdot |f|^p \, d\mu \xrightarrow[n]{} \int \tilde{h} \cdot |f|^p \, d\mu$$

we must have $\tilde{h}(\omega) > 0$ a.e. and $A = \Omega$. We can get the pointwise convergence as follows:

for $f \in L^p$, $f \neq 0$ we have

$$\|D(\tilde{h}_n)^{q-1} \cdot f\|_p \, \|J_\mu(\tilde{T}_n f)\|_q^{q-1} \, \|\tilde{T}_n f\|_p^{-1} \, \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_q^{1-q} = 1 \qquad (**)$$

so

$$\lim_n \|J_\mu(\tilde{T}_n f)\|_q^{q-1} \, \|\tilde{T}_n f\|_p^{-1} \, \|\tilde{T}_n^* J_\mu \tilde{T}_n f\|_q^{1-q} = \frac{1}{\lim_n \|D(\tilde{h}_n)^{q-1} f\|_p}$$

$$= \frac{1}{\|\tilde{h}^{q-1} \cdot f\|_p} < +\infty.$$

We now use Corollary 2 in [2] to conclude that

$$\lim_n J_\nu^* \tilde{T}_n^* J_\mu \tilde{T}_n f = \tilde{h}^{q-1} \cdot f \cdot \lim_n \|J_\nu^* \tilde{T}_n^* J_\mu \tilde{T}_n f\|_p \cdot \frac{1}{\|\tilde{h}^{q-1} \cdot f\|_p}.$$

Remark 5. The assumptions

$$\overline{\lim_{x \to 0}}\, ((\mu(x))^{q-1}/x) < +\infty, \qquad \overline{\lim_{x \to 0}}\, ((\nu(x))^{p-1}/x) < +\infty$$

are satisfied by the maps $\mu(x) = x^{p-1}$, $\nu(x) = x^{q-1}$ which defined the classical duality maps $f \to |f|^{p-1} \operatorname{sgn} f$ and $g \to |g|^{q-1} \operatorname{sgn} g$.

REFERENCES

1. M. A. AKCOGLU, Alternating sequences with non positive operators, preprint.
2. I. ASSANI, Alternating procedures in uniformly smooth Banach spaces, *Proc. Amer. Math. Soc.*, in press.
3. D. L. BURKHOLDER AND Y. S. CHOW, Iterates of conditional expectations operators, *Proc. Amer. Math. Soc.* **12**, 490–495.
4. J. L. DOOB, A ratio operator limit theorem, *Wahrsch.* **1**, 288–294.
5. G. C. ROTA, An Alterniernde Verfahren for general positive operators, *Bull. Amer. Math. Soc.* **68**, 95–102.
6. N. STARR, Operator limit theorem, *Trans. Amer. Math. Soc.* **121**, 90–115.
7. E. M. STEIN, On the maximal ergodic theorem, *Proc. Nat. Acad. Sci. U.S.A.* **47**, 1894–1897.