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Higher-order symmetric duality in nondifferentiable multiobjective programming problems ☆

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Abstract

In this paper, a pair of nondifferentiable multiobjective programming problems is first formulated, where each of the objective functions contains a support function of a compact convex set in R^n. For a differentiable function $h : R^n \times R^n \to R$, we introduce the definitions of the higher-order F-convexity (F-pseudo-convexity, F-quasi-convexity) of function $f : R^n \to R$ with respect to h. When F and h are taken certain appropriate transformations, all known other generalized invexity, such as η-invexity, type I invexity and higher-order type I invexity, can be put into the category of the higher-order F-invex functions. Under these the higher-order F-convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems related to a properly efficient solution.

1. Introduction

Symmetric duality in nonlinear programming problem was first introduced by Dorn [6], who defined a mathematical programming problem and its dual to be symmetric if the dual of the dual is the primal problem, that is, when the dual is recast in the form of primal, its dual is primal. Later, Dantzig et al. [2] and Mond [11] formulated a pair of symmetric dual programs for scalar function $f(x, y)$ that is convex in the first variable and that is concave

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in the second variable, respectively. Under the weaker convexity assumptions imposed on f, Mond and Weir [13] gave another different pair of symmetric dual problem.

Mond [10] first formulated second-order symmetric dual models, introduced the concept of second-order convex function, and proved second-order symmetric duality theorems. Bector and Chandra [1] established the second-order symmetric and self duality results under second-order pseudo-convexity and pseudo-concavity assumptions. Devi [5] formulated a pair of second-order symmetric dual programs and established duality results involving second-order invex functions. Pandey [15] introduced second-order η-invex function for multiobjective fractional programming problem and established weak and strong duality theorems.

Mond and Schechter [12] constructed two new symmetric dual pairs in which the objective functions contain a support function of a compact convex set in R^n, and are therefore nondifferentiable. Under the second-order F-pseudo-convexity assumptions, Hou and Yang [7] gave the second-order symmetric duality.

Higher-order duality in nonlinear programs have been studied by some researchers. Mangasarian [8] formulated a class of higher-order dual problems for the nonlinear programming problem "$\min\{f(x) \mid g(x) \geqslant 0\}$" by introducing twice differentiable function $h: R^n \times R^n \to R$ and $k: R^n \times R^n \to R^m$. Mond and Zhang [14] obtained duality results for various higher-order dual programming problems under higher-order invexity assumptions. Recently, under invexity-type conditions, such as higher-order type I, higher-order pseudo-type I, and higher-order quasi-type I conditions, Mishra and Rueda [9] gave various duality results, which included Mangasarian higher-order duality and Mond–Weir higher-order duality. Chen [3] also discussed the duality theorems under the higher-order F-convexity (F-pseudo-convexity, F-quasi-convexity) for a pair of nondifferentiable programs.

Up to now, there is no literature, as known by author, in which the higher-order symmetric duality for multiobjective programming problems is discussed. In this paper, we first formulate a pair of symmetric higher-order multiobjective programming problems by introducing a differentiable function, where each of objective functions contains a support function of a compact convex set in R^n. For a differentiable function $h: R^n \times R^n \to R$, we also introduce the definitions of the higher-order F-convexity (F-pseudo-convexity, F-quasi-convexity) with respect to h. All known other generalized invexity, such as η-invexity, type I invexity and higher-order type I invexity, can be put into the category of the higher-order F-invex functions by taking certain appropriate transformations of F and h. Under these the higher-order F-convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems related to a properly efficient solution.

2. Preliminaries and lemmas

Throughout this paper, denote by R^n the n-dimension Euclidean space, and R^n_+ the nonnegative orthant of R^n, respectively.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) := \max\{x^T y \mid y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential [16], that is, there exits $z \in R^n$ such that

$$s(y|C) \geqslant s(x|C) + z^T(y - x) \quad \text{for all } y \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C \mid z^T x = s(x|C)\}.$$

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) := \{y \in R^n \mid y^T(z - x) \leqslant 0 \text{ for all } z \in D\}.$$

It is obvious that for a compact convex set C, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Consider the following multiobjective programming problem:

$$\text{(P)} \quad \text{minimize } f(x) \quad \text{subject to } g(x) \leqslant 0, \quad x \in X,$$

where $f: R^n \to R^k$, $g: R^n \to R^l$ and $X \subset R^n$. Denote by Y the set of feasible solutions of (P).

Definition 1. A point $\bar{x} \in Y$ is said to be an efficient solution of (P) if there exists no other $x \in Y$ such that $f(\bar{x}) - f(x) \in R^k_+ \backslash \{0\}$, that is, $f_i(x) \leqslant f_i(\bar{x})$ for all $i \in \{1, 2, \ldots, k\}$, and at least one $j \in \{1, 2, \ldots, k\}$, $f_j(x) < f_j(\bar{x})$; $\bar{x} \in Y$ is said to be a weak efficient solution of (P) if there exists no other $x \in Y$ such that for all $i \in \{1, 2, \ldots, k\}$, $f_i(\bar{x}) > f_i(x)$.

Definition 2. $\bar{x} \in Y$ is said to be a Geoffrion properly efficient solution of (P), if $\bar{x}$ is an efficient solution, and there exists a real number $M > 0$ such that for all $i \in \{1, 2, \ldots, p\}$, $x \in Y$, and $f_i(x) < f_i(\bar{x})$,

$$f_i(\bar{x}) - f_i(x) \leqslant M[f_j(x) - f_j(\bar{x})]$$

for some $j \in \{1, 2, \ldots, k\}$ such that $f_j(\bar{x}) < f_j(x)$.

Lemma 1 [4]. *If $\bar{x} \in Y$ is a properly efficient solution of* (P)*, there exist $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T \in R^k$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l)^T \in R^l$ such that*

$$\sum_{i=1}^{k} \alpha_i \nabla_x f_i(\bar{x}) + \sum_{j=1}^{l} \beta_j \nabla_x g_j(\bar{x}) = 0, \quad \alpha \geqslant 0,\ \beta \geqslant 0,\ (\alpha^T, \beta^T) \neq 0.$$

For a real-valued twice differentiable function $g(x, y)$ defined on an open set in $R^n \times R^m$, denote by $\nabla_x g(\bar{x}, \bar{y})$ the gradient vector of g with respect to x at $(\bar{x}, \bar{y})$, $\nabla_{xx} g(\bar{x}, \bar{y})$ the Hessian matrix with respect to x at $(\bar{x}, \bar{y})$. Similarly, $\nabla_y g(\bar{x}, \bar{y})$, $\nabla_{xy} g(\bar{x}, \bar{y})$ and $\nabla_{yy} g(\bar{x}, \bar{y})$ are also defined.

Definition 3. A function $F: X \times X \times R^n \to R$ (where $X \subseteq R^n$) is sublinear with respect to the third variable if for all $(x, u) \in X \times X$,

(i) $F(x, u; a_1 + a_2) \leqslant F(x, u; a_1) + F(x, u; a_2)$ for all $a_1, a_2 \in R^n$,

(ii) $F(x, u; \alpha a) = \alpha F(x, u; a)$, $\alpha \geqslant 0$, for all $a \in R^n$.

Now, we give the definitions of a class of higher-order F-invexity.

Definition 4. Suppose that $h : X \times R^n \to R$ is a differentiable function, F is sublinear with respect to the third variable. f is said to be higher-order F-convex at $u \in X$ with respect to h, if for all $(x, p) \in X \times R^n$,

$$f(x) - f(u) \geqslant F\big(x, u; \nabla_x f(u) + \nabla_p h(u, p)\big) + h(u, p) - p^T\big[\nabla_p h(u, p)\big].$$

If for all $(x, p) \in X \times R^n$,

$$\begin{aligned} &F\big(x, u; \nabla_x f(u) + \nabla_p h(u, p)\big) \geqslant 0 \\ &\quad \Rightarrow \quad f(x) \geqslant f(u) + h(u, p) - p^T\big[\nabla_p h(u, p)\big], \end{aligned}$$

then f is said to be higher-order F-pseudo-convex at $u \in X$ with respect to h.

If for all $(x, p) \in X \times R^n$,

$$\begin{aligned} &f(x) \leqslant f(u) + h(u, p) - p^T\big[\nabla_p h(u, p)\big] \\ &\quad \Rightarrow \quad F\big(x, u; \nabla_x f(u) + \nabla_p h(u, p)\big) \leqslant 0, \end{aligned}$$

then f is said to be higher-order F-quasi-convex at $u \in X$ with respect to h.

If f is higher-order F-convex (F-pseudo-convex, F-quasi-convex) at each point $u \in X$ with respect to same function h, then f is said to be higher-order F-convex (F-pseudo-convex, F-quasi-convex) on X with respect to h.

If $-f$ is higher-order F-convex (F-pseudo-convex, F-quasi-convex) at $u \in X$ with respect to h, then f is said to be higher-order F-concave (F-pseudo-concave, F-quasi-concave) at $u \in X$ with respect to h.

Remark 1. (i) When $h(u, p) = (1/2)p^T \nabla_{xx} f(u)p$ and $F(x, u; a) = \eta(x, u)^T a$, where η is a function from $X \times X$ to R^n, the higher-order F-convexity (F-pseudo-convexity, F-quasi-convexity) reduces to η-bonvexity (η-pseudo-bonvexity, η-quasi-bonvexity) in [5,15].

(ii) When $h(u, p) = (1/2)p^T \nabla_{xx} f(u)p$, the higher-order F-convexity (higher-order F-pseudo-convexity, higher-order F-quasi-convexity) reduces to the second-order F (pseudo-, quasi-) invexity in [7].

(iii) When $h(u, p) = -p^T \nabla_x f(u) + k(u, p)$ and $F(x, u; a) = \alpha(x, u)a^T \eta(x, u)$, where $\alpha : X \times X \to R_+ \backslash \{0\}$, $\eta : X \times X \to R^n$ are positive functions, and $k : X \times R^n \to R$ is a differentiable function, the higher-order F-convex (higher-order F-pseudo-convex, higher-order F-quasi-convex) function becomes the higher-order type I (higher-order pseudo-type I, and higher-order quasi-type I) function in [9,14].

From now on, suppose that the sublinear function F satisfies the following condition:

$$F(x, y; a) + a^T y \geqslant 0 \quad \text{for all } a \in R^n_+. \tag{1}$$

3. Higher-order symmetric duality

In this section, we consider the following multiobjective symmetric dual problems:

$$
\begin{aligned}
\text{(MP)}\quad & \text{minimize } \big(f_1(x,y)+s(x|C_1)-y^Tz_1 \\
&\qquad + h_1(x,y,p_1)-p_1^T\big[\nabla_{p_1}h_1(x,y,p_1)\big],\ldots, \\
&\quad f_k(x,y)+s(x|C_k)-y^Tz_k \\
&\qquad + h_k(x,y,p_k)-p_k^T\big[\nabla_{p_k}h_k(x,y,p_k)\big]\big) \\
& \text{subject to } \sum_{i=1}^k \lambda_i\big[\nabla_y f_i(x,y)-z_i+\nabla_{p_i}h_i(x,y,p_i)\big]\leqslant 0, \qquad (2)\\
&\quad y^T\sum_{i=1}^k \lambda_i\big[\nabla_y f_i(x,y)-z_i+\nabla_{p_i}h_i(x,y,p_i)\big]\geqslant 0, \qquad (3)\\
&\quad z_i\in D_i,\ i=1,2,\ldots,k,\ \lambda>0,\ \lambda^Te=1, \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(MD)}\quad & \text{maximize } \big(f_1(u,v)-s(v|D_1)+u^Tw_1 \\
&\qquad + g_1(u,v,r_1)-r_1^T\big[\nabla_{r_1}g_1(u,v,r_1)\big],\ldots, \\
&\quad f_k(u,v)-s(v|D_k)+u^Tw_k \\
&\qquad + g_k(u,v,r_k)-r_k^T\big[\nabla_{r_k}g_k(u,v,r_k)\big]\big) \\
& \text{subject to } \sum_{i=1}^k \lambda_i\big[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big]\geqslant 0, \qquad (5)\\
&\quad u^T\sum_{i=1}^k \lambda_i\big[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big]\leqslant 0, \qquad (6)\\
&\quad w_i\in C_i,\ i=1,2,\ldots,k,\ \lambda>0,\ \lambda^Te=1, \qquad (7)
\end{aligned}
$$

where C_i and D_i is a compact convex sets in R^n and R^m, respectively; $f_i : R^n \times R^m \to R$, $h_i : R^n \times R^m \times R^m \to R$ and $g_i : R^n \times R^m \times R^n \to R$ are twice differentiable functions, $i = 1, 2, \ldots, k$. Since the objective functions contain the function $s(x|C_i)$ and $s(v|D_i)$, $i = 1, 2, \ldots, k$, they are nondifferentiable multiobjective programming problems.

Remark 2. (1) If $h_i(x, y, p_i) = (1/2)p_i^T\nabla_{yy} f_i(x, y)p_i$, $p_i = p$; $g_i(u, v, r_i) = (1/2)r_i^T \times \nabla_{xx} f_i(u, v)r_i$, $r_i = r$; and $k = 1$, then (MP) and (MD) become the problems considered by Hou and Yang [7].

(2) If $k = 1$, then (MP) and (MD) become the nondifferentiable programming problems (SP) and (SD) considered by Chen [3].

We first give the weak duality theorem under the higher-order F-convexity assumptions.

Theorem 1 (Weak duality). *For each feasible solution* $(x, y, \lambda, z_1, z_2, \ldots, z_k, p_1, p_2, \ldots, p_k)$ *of* (MP) *and each feasible solution* $(u, v, \lambda, w_1, , w_2, \ldots, w_k, r_1, r_2, \ldots, r_k)$ *of* (MD), *if* $f_i(\cdot, v) + (\cdot)^T w_i$ *is higher-order* F*-convex at* u *with respect to* $g_i(u, v, r_i)$;

$-[f_i(x,\cdot)-(\cdot)^T z_i]$ *is higher-order* G*-convex at* y *with respect to* $-h_i(x,y,p_i)$, $i=1,2,\ldots,k$, *where sublinear functions* $F: R^n\times R^n\times R^n\to R$ *and* $G: R^m\times R^m\times R^m\to R$ *satisfy the condition* (1), *then the following inequalities cannot hold simultaneously*:

(I) *for all* $i\in\{1,2,\ldots,k\}$,

$$f_i(x,y)+s(x|C_i)-y^T z_i+h_i(x,y,p_i)-p_i^T\big[\nabla_{p_i}h_i(x,y,p_i)\big] \leqslant f_i(u,v)-s(v|D_i)+u^T w_i+g_i(u,v,r_i)-r_i^T\big[\nabla_{r_i}g_i(u,v,r_i)\big], \tag{8}$$

(II) *for at least one* $j\in\{1,2,\ldots,k\}$,

$$f_j(x,y)+s(x|C_j)-y^T z_j+h_j(x,y,p_j)-p_j^T\big[\nabla_{p_i}h_j(x,y,p_j)\big] < f_j(u,v)-s(v|D_j)+u^T w_j+g_j(u,v,r_j)-r_j^T\big[\nabla_{r_j}g_j(u,v,r_j)\big]. \tag{9}$$

Proof. For each feasible solution $(x,y,\lambda,z_1,z_2,\ldots,z_k,p_1,p_2,\ldots,p_k)$ of (MP) and each feasible solution $(u,v,\lambda,w_1,w_2,\ldots,w_k,r_1,r_2,\ldots,r_k)$ of (MD), by (1) and (5), we have

$$F\left(x,u;\sum_{i=1}^k\lambda_i\big[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big]\right) + \sum_{i=1}^k\lambda_i\big[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big]^T u\geqslant 0, \tag{10}$$

and from (6), (10) yields

$$F\left(x,u;\sum_{i=1}^k\lambda_i\big[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big]\right)\geqslant 0. \tag{11}$$

Using the higher-order F-convexity of $f_i(\cdot,v)+(\cdot)^T w_i$ at u with respect to $g_i(u,v,r_i)$, we have

$$\begin{aligned}&\big[f_i(x,v)+x^T w_i\big]-\big[f_i(u,v)+u^T w_i\big]\\ &\quad\geqslant F\big(x,u;\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big)\\ &\qquad+g_i(u,v,r_i)-r_i^T\big[\nabla_{r_i}g_i(u,v,r_i)\big].\end{aligned} \tag{12}$$

Since F is a sublinear function about the third variable, and $\lambda>0$, $\lambda^T e=1$, from (5), (11) and (12), it holds

$$\begin{aligned}&\sum_{i=1}^k\lambda_i\big[f_i(x,v)+x^T w_i\big]-\sum_{i=1}^k\lambda_i\big[f_i(u,v)+u^T w_i\big]\\ &\quad\geqslant F\left(x,u;\sum_{i=1}^k\lambda_i\big[\nabla_u f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)\big]\right)\\ &\qquad+\sum_{i=1}^k\lambda_i\big\{g_i(u,v,r_i)-r_i^T\big[\nabla_{r_i}g_i(u,v,r_i)\big]\big\}\end{aligned}$$

$$\geqslant \sum_{i=1}^{k} \lambda_i \big\{ g_i(u, v, r_i) - r_i^T \big[\nabla_{r_i} g_i(u, v, r_i) \big] \big\},$$

that is,

$$\sum_{i=1}^{k} \lambda_i f_i(x, v) \geqslant \sum_{i=1}^{k} \lambda_i \big\{ f_i(u, v) - x^T w_i + u^T w_i + g_i(u, v, r_i) - r_i^T \big[\nabla_{r_i} g_i(u, v, r_i) \big] \big\}. \tag{13}$$

On the other hand, from (2) and (1), we get

$$G\left(v, y; -\sum_{i=1}^{k} \lambda_i \big[\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i) \big] \right) + y^T \left\{ -\sum_{i=1}^{k} \lambda_i \big[\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i) \big] \right\} \geqslant 0. \tag{14}$$

From (3), (14) implies

$$G\left(v, y; -\sum_{i=1}^{k} \lambda_i \big[\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i) \big] \right) \geqslant 0. \tag{15}$$

By the higher-order G-convexity of $-[f_i(x, \cdot) - (\cdot)^T z_i]$ at y with respect to $-h_i(x, y, p_i)$, and from (1), we have

$$\begin{aligned} &-\big[f_i(x, v) - v^T z_i \big] + \big[f_i(x, y) - y^T z_i \big] \\ &\quad \geqslant G\big(v, y; -\nabla_y f_i(x, y) + z_i - \nabla_{p_i} h_i(x, y, p_i) \big) \\ &\qquad + \big[-h_i(x, y, p_i) \big] - p_i^T \big\{ \nabla_{p_i} \big[-h_i(x, y, p_i) \big] \big\} \\ &\quad = G\big(v, y; -\big[\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i) \big] \big) \\ &\qquad - h_i(x, y, p_i) + p_i^T \big[\nabla_{p_i} h_i(x, y, p_i) \big]. \end{aligned} \tag{16}$$

Similarly, from the sublinearity of G, $\lambda > 0$, (2), (15) and (16), we have

$$\begin{aligned} &-\sum_{i=1}^{k} \lambda_i \big[f_i(x, v) - v^T z_i \big] + \sum_{i=1}^{k} \lambda_i \big[f_i(x, y) - y^T z_i \big] \\ &\quad \geqslant -\sum_{i=1}^{k} \lambda_i h_i(x, y, p_i) + \sum_{i=1}^{k} \lambda_i p_i^T \big[\nabla_{p_i} h_i(x, y, p_i) \big], \end{aligned}$$

that is,

$$\sum_{i=1}^{k} \lambda_i f_i(x, v) \leqslant \sum_{i=1}^{k} \lambda_i \big\{ f_i(x, y) + v^T z_i - y^T z_i + h_i(x, y, p_i) - p_i^T \big[\nabla_{p_i} h_i(x, y, p_i) \big] \big\}. \tag{17}$$

From (13) and (17), we obtain

$$\sum_{i=1}^{k}\lambda_i\big\{f_i(u,v)-v^Tz_i+u^Tw_i+g_i(u,v,r_i)-r_i^T\big[\nabla_{r_i}g_i(u,v,r_i)\big]\big\}$$
$$\leqslant \sum_{i=1}^{k}\lambda_i\big\{f_i(x,y)+x^Tw_i-y^Tz_i+h_i(x,y,p_i)-p_i^T\big[\nabla_{p_i}h_i(x,y,p_i)\big]\big\}. \tag{18}$$

Noting that $x^Tw_i \leqslant s(x|C_i)$ and $v^Tz_i \leqslant s(v|D_i)$, (18) yields

$$\sum_{i=1}^{k}\lambda_i\big\{f_i(u,v)-s(v|D_i)+u^Tw_i+g_i(u,v,r_i)-r_i^T\big[\nabla_{r_i}g_i(u,v,r_i)\big]\big\}$$
$$\leqslant \sum_{i=1}^{k}\lambda_i\big\{f_i(x,y)+s(x|C_i)-y^Tz_i+h_i(x,y,p_i)-p_i^T\big[\nabla_{p_i}h_i(x,y,p_i)\big]\big\},$$

this implies that the conclusion holds. □

Remark 3. From the process of the proof in Theorem 1, we can also obtain that (8) and (9) cannot hold simultaneously if the sublinear functions F and G satisfy the condition (1), and for each feasible solution $(x, y, \lambda, z_1, z_2, \dots, z_k, p_1, p_2, \dots, p_k)$ of (MP) and each feasible solution $(u, v, \lambda, w_1, w_2, \dots, w_k, r_1, r_2, \dots, r_k)$ of (MD), one of the following conditions holds:

(1) $f_i(\cdot, v) + (\cdot)^T w_i$ is higher-order F-pseudo-convex at u with respect to $g_i(u, v, r_i)$, $-[f_i(x,\cdot) - (\cdot)^T z_i]$ is higher-order G-pseudo-convex at y with respect to $-h_i(x, y, p_i)$;

(2) $f_i(\cdot, v) + (\cdot)^T w_i$ is higher-order F-quasi-convex at u with respect to $g_i(u, v, r_i)$, $-[f_i(x,\cdot) - (\cdot)^T z_i]$ is higher-order G-quasi-convex at y with respect to $-h_i(x, y, p_i)$.

The following result indicates that under some conditions, a properly efficient solution of (MP) is also the ones of (MD) and the two objective values are correspondingly equal.

Theorem 2 (Strong duality). *Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_k, \bar{p}_1, \bar{p}_2, \dots, \bar{p}_k)$ be a properly efficient solution of* (MP), *$f_i : R^n \times R^n \to R$ is twice differentiable at $(\bar{x}, \bar{y})$, $h_i : R^n \times R^n \times R^n \to R$ is twice differentiable at $(\bar{x}, \bar{y}, \bar{p}_i)$, $g_i : R^n \times R^n \times R^n \to R$ is differentiable at $(\bar{x}, \bar{y}, \bar{p}_i)$, $i = 1, 2, \dots, k$. If the following conditions hold*:

- (I) $h_i(\bar{x}, \bar{y}, 0) = 0$, $g_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_{p_i} h_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_y h_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_x h_i(\bar{x}, \bar{y}, 0) = \nabla_{r_i} g_i(\bar{x}, \bar{y}, 0)$, $i = 1, 2, \dots, k$;
- (II) *for all $i \in \{1, 2, \dots, k\}$, the Hessian matrix $\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p})$ is positive definite or negative definite*;
- (III) *the set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\}_{i=1}^k$ is linearly independent*;
- (IV) *for some $\alpha \in R^k$ $(\alpha > 0)$ and $p_i \in R^n$, $p_i \neq 0$ $(i = 1, 2, \dots, k)$ implies that* $\sum_{i=1}^k \alpha_i p_i^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] \neq 0$.

Then

(i) $\bar{p}_i = 0$, $i = 1, 2, \ldots, k$;
(ii) *there exists* $\bar{w}_i \in C_i$ *such that* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{r}_1 = \bar{r}_2 = \cdots = \bar{r}_k = 0)$ *is a feasible solution of* (MD).

Furthermore, if the hypotheses in Theorem 1 *are satisfied, then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{p}_1 = \bar{p}_2 = \cdots = \bar{p}_k = 0)$ *is a properly efficient solution of* (MD)*, and the two objective values are equal.*

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k, \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_k)$ is properly efficient for (MP), from Lemma 1, there exists $\alpha \in R^k$, $\beta \in R^n$, $\mu \in R$, and $\nu_i \in C_i$ such that

$$\sum_{i=1}^k \alpha_i \big[\nabla_x f_i(\bar{x}, \bar{y}) + \nu_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] + \sum_{i=1}^k \bar{\lambda}_i \big(\nabla_{yx} f_i(\bar{x}, \bar{y})\big)^T (\beta - \mu \bar{y}) + \sum_{i=1}^k \big(\nabla_{p_i x} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big)^T (\lambda_i \beta - \alpha_i \bar{p}_i - \bar{\lambda}_i \mu \bar{y}) = 0, \tag{19}$$

$$\sum_{i=1}^k \alpha_i \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] + \sum_{i=1}^k \bar{\lambda}_i \big(\nabla_{yy} f_i(\bar{x}, \bar{y})\big)^T (\beta - \mu \bar{y}) + \sum_{i=1}^k \big(\nabla_{p_i y} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big)^T (\bar{\lambda}_i \beta - \alpha_i \bar{p}_i - \bar{\lambda}_i \mu \bar{y}) - \mu \sum_{i=1}^k \bar{\lambda}_i \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0, \tag{20}$$

$$(\beta - \mu \bar{y})^T \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0, \quad i = 1, 2, \ldots, k, \tag{21}$$

$$\sum_{i=1}^k \big(\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big)^T (\bar{\lambda}_i \beta - \alpha_i \bar{p}_i - \bar{\lambda}_i \mu \bar{y}) = 0, \tag{22}$$

$$\beta^T \sum_{i=1}^k \bar{\lambda}_i \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0, \tag{23}$$

$$\mu \bar{y}^T \sum_{i=1}^k \bar{\lambda}_i \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0, \tag{24}$$

$$\alpha_i \bar{y} - \bar{\lambda}_i \beta + \bar{\lambda}_i \mu \bar{y} \in N_{D_i}(\bar{z}_i), \quad i = 1, 2, \ldots, k, \tag{25}$$

$$\nu_i^T \bar{x} = s(\bar{x} | C_i), \quad i = 1, 2, \ldots, k, \tag{26}$$

$$\nu_i \in C_i, \quad i = 1, 2, \ldots, k, \qquad (\alpha, \beta, \mu) \geqslant 0, \quad (\alpha, \beta, \mu) \neq 0. \tag{27}$$

From the condition (II), (22) yields

$$\bar{\lambda}_i \beta - \alpha_i \bar{p}_i - \bar{\lambda}_i \mu \bar{y} = 0. \tag{28}$$

We claim that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T \neq 0$. Otherwise, if $\alpha = 0$, then (28) becomes

$$\beta = \mu \bar{y},$$

and (20) yields

$$\mu \sum_{i=1}^{k} \bar{\lambda}_i \big(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big) = 0.$$

Since $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\}_{i=1}^{k}$ is linearly independent, and $\bar{\lambda} > 0$, we have $\mu = 0$, and so $\beta = 0$. These contradict (27).

Subtracting (24) from (23) yields

$$(\beta - \mu \bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0.$$

Using (28), we get

$$\sum_{i=1}^{k} \alpha_i \bar{p}_i^T \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0.$$

By the condition (IV), we have $\bar{p}_i = 0$, $i = 1, 2, \ldots, k$. And from (28) and $\beta \geqslant 0$, $\bar{y} \geqslant 0$.

By (28), $\bar{p}_i = 0$, $i = 1, 2, \ldots, k$, the condition (I) and $\bar{\lambda} > 0$, (19) and (20) become

$$\sum_{i=1}^{k} \alpha_i \big[\nabla_x f_i(\bar{x}, \bar{y}) + \nu_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0 \tag{29}$$

and

$$\sum_{i=1}^{k} (\alpha_i - \mu \bar{\lambda}_i) \big[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0, \tag{30}$$

respectively. Similarly, from the condition (III), we have $\alpha = \mu \bar{\lambda}$. Thus, from (29) and $\mu > 0$, it holds

$$\sum_{i=1}^{k} \bar{\lambda}_i \big[\nabla_x f_i(\bar{x}, \bar{y}) + \nu_i + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0,$$

and from the condition (I), we have

$$\sum_{i=1}^{k} \bar{\lambda}_i \big[\nabla_x f_i(\bar{x}, \bar{y}) + \nu_i + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{p}_i)\big] = 0.$$

Taking $\bar{w}_i = \nu_i$, $i = 1, 2, \ldots, k$, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{r}_1 = \bar{r}_2 = \cdots = \bar{r}_k = 0)$ satisfies (5)–(7), that is, it is a feasible solution of (MD).

Under Theorem 1 assumptions, if $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{r}_1 = \bar{r}_2 = \cdots = \bar{r}_k = 0)$ is not an efficient solution of (MD), then there exists other feasible solution $(u, v, \lambda, w_1, w_2, \ldots, w_k, r_1, r_2, \ldots, r_k)$ of (MD) such that for all $i \in \{1, 2, \ldots, k\}$,

$$f_i(\bar{x},\bar{y}) - s(\bar{y}|D_i) + \bar{x}^T\bar{w}_i + g_i(\bar{x},\bar{y},\bar{r}_i) - \bar{r}_i^T\big[\nabla_{r_i} g_i(\bar{x},\bar{y},\bar{r}_i)\big]$$
$$\leqslant f_i(u,v) - s(v|D_i) + u^T w_i + g_i(u,v,r_i) - r_i^T\big[\nabla_{r_i} g_i(u,v,r_i)\big], \tag{31}$$

and for at least one $j \in \{1,2,\ldots,k\}$,

$$f_j(\bar{x},\bar{y}) - s(\bar{y}|D_j) + \bar{x}^T\bar{w}_j + g_j(\bar{x},\bar{y},\bar{r}_j) - \bar{r}_j^T\big[\nabla_{r_j} g_j(\bar{x},\bar{y},\bar{r}_j)\big]$$
$$< f_j(u,v) - s(v|D_j) + u^T w_j + g_j(u,v,r_j) - r_j^T\big[\nabla_{r_j} g_j(u,v,r_j)\big]. \tag{32}$$

Since $\alpha > 0$ and $\beta = \mu\bar{y}$, (25) yields $\bar{y} \in N_{D_i}(\bar{z}_i)$, that is, $s(\bar{y}|D_i) = \bar{y}^T\bar{z}_i$, $i = 1,2,\ldots,k$. Therefore, using (26), $\bar{p}_i = 0$, $i = 1,2,\ldots,k$, and the condition (I), we obtain that for all $i \in \{1,2,\ldots,k\}$,

$$\begin{aligned}
& f_i(\bar{x},\bar{y}) + s(\bar{x}|C_i) - \bar{y}^T\bar{z}_i + h_i(\bar{x},\bar{y},\bar{p}_i) - \bar{p}_i^T\big[\nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i)\big] \\
&= f_i(\bar{x},\bar{y}) - s(\bar{y}|D_i) + \bar{x}^T\bar{w}_i + g_i(\bar{x},\bar{y},\bar{r}_i) - \bar{r}_i^T\big[\nabla_{r_i} g_i(\bar{x},\bar{y},\bar{r}_i)\big] \\
&\leqslant f_i(u,v) - s(v|D_i) + u^T w_i + g_i(u,v,r_i) - r_i^T\big[\nabla_{r_i} g_i(u,v,r_i)\big],
\end{aligned}$$

and for at least one $j \in \{1,2,\ldots,k\}$,

$$\begin{aligned}
& f_j(\bar{x},\bar{y}) + s(\bar{x}|C_j) - \bar{y}^T\bar{z}_j + h_j(\bar{x},\bar{y},\bar{p}_j) - \bar{r}_j^T\big[\nabla_{r_j} g_j(\bar{x},\bar{y},\bar{r}_j)\big] \\
&= f_j(\bar{x},\bar{y}) - s(\bar{y}|D_j) + \bar{x}^T\bar{w}_j + g_j(\bar{x},\bar{y},\bar{r}_j) - \bar{r}_j^T\big[\nabla_{r_j} g_j(\bar{x},\bar{y},\bar{r}_j)\big] \\
&< f_j(u,v) - s(v|D_j) + u^T w_j + g_i(u,v,r_j) - r_j^T\big[\nabla_{r_j} g_j(u,v,r_j)\big],
\end{aligned}$$

which contradict Theorem 1.

If $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_k,\bar{r}_1=\bar{r}_2=\cdots=\bar{r}_k=0)$ is not a properly efficient solution of (MD), then there exists a feasible solution $(u,v,\lambda,w_1,w_2,\ldots,w_k,r_1,r_2,\ldots,r_k)$ of (MD) such that for some $i \in \{1,2,\ldots,k\}$ and any real $M > 0$,

$$\begin{aligned}
&\big\{f_i(u,v) - s(v|D_i) + u^T w_i + g_i(u,v,r_i) - r_i^T\big[\nabla_{r_i} g_i(u,v,r_i)\big]\big\} \\
&\quad - \big\{f_i(\bar{x},\bar{y}) - s(\bar{y}|D_i) + \bar{x}^T\bar{w}_i + g_i(\bar{x},\bar{y},\bar{r}_i) - \bar{r}_i^T\big[\nabla_{r_i} g_i(\bar{x},\bar{y},\bar{r}_i)\big]\big\} > M.
\end{aligned}$$

It is similar to the above discussion that we have

$$\begin{aligned}
&\big\{f_i(u,v) - s(v|D_i) + u^T w_i + g_i(u,v,r_i) - r_i^T\big[\nabla_{r_i} g_i(u,v,r_i)\big]\big\} \\
&\quad - \big\{f_i(\bar{x},\bar{y}) + s(\bar{x}|C_i) - \bar{y}^T\bar{z}_i + h_i(\bar{x},\bar{y},\bar{p}_i) - \bar{p}_i^T\big[\nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i)\big]\big\} > M,
\end{aligned}$$

which contradicts Theorem 1 again.

Furthermore, we also obtain

$$\begin{aligned}
& f_i(\bar{x},\bar{y}) + s(\bar{x}|C_i) - \bar{y}^T\bar{z}_i + h_i(\bar{x},\bar{y},\bar{p}_i) - \bar{p}_i^T\big[\nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i)\big] \\
&= f_i(\bar{x},\bar{y}) + \bar{x}^T\nu_i - s(\bar{y}|D_i) \\
&= f_i(\bar{x},\bar{y}) - s(\bar{y}|D_i) + \bar{x}^T\bar{w}_i + g_i(\bar{x},\bar{y},\bar{r}_i) - \bar{r}_i^T\big[\nabla_{r_i} g_i(\bar{x},\bar{y},\bar{r}_i)\big]
\end{aligned}$$

for all $i \in \{1,2,\ldots,k\}$, which indicates that the objective values of (MP) at $(\bar{x},\bar{y},\bar{\lambda},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{p}_1=\bar{p}_2=\cdots=\bar{p}_k=0)$ and the objective values of (MD) at $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_k,\bar{r}_1=\bar{r}_2=\cdots=\bar{r}_k=0)$ are correspondingly equal. $\square$

Similarly, we have the following converse duality.

Theorem 3 (Converse duality). *Let* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \dots, \bar{w}_k, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_k)$ *be a properly efficient solution of* (MD), $f_i : R^n \times R^n \to R$ *is differentiable at* $(\bar{u}, \bar{v})$, $g_i : R^n \times R^n \times R^n \to R$ *is twice differentiable at* $(\bar{u}, \bar{v}, \bar{r}_i)$, $h_i : R^n \times R^n \times R^n \to R$ *is differentiable at* $(\bar{u}, \bar{v}, \bar{r}_i)$. *If the following conditions hold*:

(I) $h_i(\bar{u}, \bar{v}, 0) = 0$, $g_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_{r_i} g_i(\bar{u}, \bar{u}, 0) = 0$, $\nabla_x g_i(\bar{u}, \bar{u}, 0) = 0$, $\nabla_y g_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} h_i(\bar{u}, \bar{v}, 0)$, $i = 1, 2, \dots, k$;

(II) *for all* $i \in \{1, 2, \dots, k\}$, *the Hessian matrix* $\nabla_{r_i r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)$ *is positive definite or negative definite*;

(III) *the set of vectors* $\{\nabla_x f_i(\bar{u}, \bar{v}) - \bar{w}_i + \nabla_{r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)\}_{i=1}^k$ *is linearly independent*;

(IV) *for some* $\alpha \in R^k$ $(\alpha > 0)$ *and* $r_i \in R^n$, $r_i \neq 0$ $(i = 1, 2, \dots, k)$ *implies that* $\sum_{i=1}^k \alpha_i r_i^T [\nabla_x f_i(\bar{u}, \bar{v}) - \bar{w}_i + \nabla_{r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)] \neq 0$.

Then

(i) $\bar{r}_i = 0$, $i = 1, 2, \dots, k$;

(ii) *there exists* $\bar{z}_i \in D_i$ *such that* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_k, \bar{p}_1 = \bar{p}_2 = \cdots = \bar{p}_k = 0)$ *is a feasible solution of* (MP).

Furthermore, if the hypotheses in Theorem 1 *are satisfied, then* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_k, \bar{p}_1 = \bar{p}_2 = \cdots = \bar{p}_k = 0)$ *is a properly efficient solution of* (MP), *and the two objective values are correspondingly equal.*

In the above, we formulate a pair of the higher-order symmetric multiobjective programming problem in which the objective functions contain a support function of a compact convex set in R^n or R^m. Under the higher-order F-convexity (higher-order F-pseudo-convexity, higher-order F-quasi-convexity) assumption, we give the higher-order weak, higher-order strong and higher-order converse duality. In our models, if $h_i(x, y, p) = (1/2) p^T \nabla_{yy} f_i(x, y) p$, $g_i(u, v, r) = (1/2) r^T \nabla_{uu} f_i(u, v) r$, and $k = 1$, then (MP) and (MD) reduce to the second-order symmetric models in [7]. So, our results include some of the known results in [1,7–10,14,15].

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