JOURNAL OF ALGEBRA 111, 1–13 (1987)

On $PSU(3, q)$ as Collineation Groups

CHAT YIN HO

Department of Mathematics, University of Florida, Gainesville, Florida 32611

AND

ADILSON GONÇALVES

Department of Mathematics, University Federal Pernambuco, Recife, Pe, Brazil

Communicated by Walter Feit

Received August 27, 1984

1. INTRODUCTION

In the study of groups of Lie type as collineation groups of projective planes, the linear groups of dimension 2 or 3 and the unitary groups of dimension 3 are the important families related to the Desarguesian planes. The unitary family, $PSU(3, q)$, has been studied, among others, by Hering, Hoffer, Kantor, Walker, and Seib. In this paper we study $PSU(3, q)$ as a collineation group of a finite projective plane such that its involutions are perspectivities. As the situation involving elations has been handled by Hering and Walker [4], we assume here that the involutions are homologies. Using geometric method, the group structure of $PSU(3, q)$, and a character theoretical-like method based on Proposition 2.1, we obtain an invariant subplane of order q^2 under various conditions.

THEOREM A. *Let π be a finite projective plane. Then the following are equivalent.*

(a) *π is a Desarguesian plane of order q^2, q odd.*

(b) *π admits a collineation group $G \cong PSU(3, q)$, q odd such that G does not leave any point or line invariant, an involution in G is a homology, and the order of the stabilizer or any point of π is even.*

THEOREM B. *Let $G \cong PSU(3, q)$, $q \in \{3, 5, 7\}$, be a collineation group of a finite projective plane of order n such that its involutions are homologies*

0021-8693/87 $3.00

and G does not fix any point or line. Assume $n \neq 1 \pmod 3$ *when* $q = 3$. *If the stabilizer of any point of* π *in G is not trivial, then* π *contains a G-invariant Desarguesian subplane of order* q^2 *where G acts naturally on.*

More general results are presented in Theorems 4.1 and 5.1.

2. Notation and Preliminaries

Let $\pi = (\mathscr{P}, \mathscr{L})$ be a projective plane of order n and G a collineation group of π. For $H \leqslant G$, set $\mathscr{P}(H) = \{P \in \mathscr{P} \mid P^\sigma = P \text{ for all } \sigma \in H\}$ and $\mathscr{L}(H) = \{l \in \mathscr{L} \mid l^\sigma = l \text{ for all } \sigma \in H\}$, and $\text{Fix}(H) = (\mathscr{P}(H), \mathscr{L}(H))$. For $A \in \mathscr{P}$, $l \in \mathscr{L}$, $\mathscr{X} \leqslant \mathscr{P}$, and $\mathscr{B} \leqslant \mathscr{L}$, let $[A] = \{b \in \mathscr{L} \mid A \in b\}$, $(l) = \{P \in \mathscr{P} \mid P \in l\}$.

We call a collineation $1 \neq \sigma$ a *perspectivity* if $\mathscr{P}(\sigma) = (a) \cup \{A\}$ and $\mathscr{L}(\sigma) = [A] \cup \{a\}$ for some $A \in P$ and $a \in \mathscr{L}$, and say that $A = C(\sigma)$ is the *center* and $a = a(\sigma)$ is the *axis* of σ. A perspectivity is called a *homology* if its center does not lie on its axis. We call a collineation σ *planar* (resp. *triangular*) if $\text{Fix}(\sigma)$ is a subplane (resp. triangle). If $\mathscr{P}(\sigma) = \{A\}$ and $\mathscr{L}(\sigma) = \{a\}$ such that $A \notin a$, then σ is caled an *anti-flag* collineation. The same terminology applies to collineation groups also. A collineation group, which does not leave any point, line, triangle, or subplane invariant, is *strongly irreducible*. A *G-incidence matrix* of π is an integral matrix whose rows (resp. columns) are indexed by the line (resp. point) orbits of G such that for a line orbit q and a point orbit Q the (q, Q)-entry is the number of lines in q passing through a point in Q, which is denoted by qQ. Any permutation among the columns or rows yields an equivalent G-incidence matrix. In order to include the information about the orbit sizes of G, we write on top (resp. to the left) of the column (resp. row) indexed by Q (resp. q) the number $|Q|$ (resp. $|q|$). This is called the *embroidered G-incidence matrix of* π

For a finite set of integers $\{k_i \mid i \in I\}$, let $\sum_{i \in I}^{+} k_i$ be the sum of all nonnegative integers in this set.

For any prime p, $\text{Syl}\, p(G)$ denotes the set of Sylow p-subgroups of G. For any subset S of G, $\langle S \rangle$ denotes the subgroup of G generated by S and $C_G(S)$ denotes the centralizer of S in G. Let K be a subgroup of the symmetric group of a set Ω and let $a \in \Omega$. The stabilizer of a in K is denoted by K_a. Other notation and terminology concerning groups (resp. projective planes) can be found in [2] (resp. [1]). For the convenience of the reader, we record some known results used in this paper.

2.1. Proposition. *Let* $Q_1, Q_2, \ldots, Q_v$ *be the point orbits of* G *and let* $q_1, \ldots, q_v$ *be the line orbits of* G. *For* $1 \leqslant t$, $s \leqslant v$, *define* $[q_t \mid q_s]_G = \sum_{j=1}^{v} |P_j|\, (q_t P_j)(q_s P_j)$. *Then* $[q_t \mid q_s]_G = |q_t|\, |q_s| + n\, |q_t \cap q_s|$.

Proof. The conclusion for $[q_t \mid q_s]_G$ is a special case of 2.1 of [6].

2.2. LEMMA [3, Lemma 5.3]. *Perspectivity leaving invariant a subplane has its center and axis belonging to this subplane.*

2.3. THEOREM [3, Theorem 3.18]. *Let G be a strongly irreducible group of collineations of π and let M be a minimal normal subgroup of G. Then one of the following statements holds.*

(a) *Each element of M is regular or planar.*

(b) *M is elementary abelian of order 9 and $C_G(M) = M$.*

Either each subgroup of M is triangular, or M contains two triangular and two planar subgroups of order 3.

(c) *M is not solvable, and each subgroup of any component of M is regular or planar.*

(d) *M is non-abelian simple and $C_G(M)$ is a $\{2, 3\}$-group. If $C_G(M) \neq M$, then each subgroup of M is regular, planar, or triangular.*

3. EMBEDDING OF A UNITAL IN π

In this section let $\pi = (\mathbb{P}, \mathbb{L})$ be a finite projective plane of order n admitting a collineation group $G \cong PSU(3, q)$, where q is a power of an odd prime p, such that $\mathrm{Fix}(G) = (\phi, \phi)$ and an involution in G is a homology. Hence n is odd. Since G has only one conjugacy class of involutions, all involutions in G are homologies. Let $P \in \mathrm{Syl}\, p(G)$ and $H = N_G(P)$.

3.1. LEMMA. *Different involutions in G have different centers and axes.*

Proof. Let $\alpha \neq \beta$ be two involutions of G. If α and β have a common center, then $G = \langle C_G(\alpha), C_G(\beta) \rangle$ fixes this common center. This contradicts $\mathrm{Fix}(G) = (\phi, \phi)$. Similarly, α and β cannot have a common axis.

3.2. PROPOSITION. *The q^2 distinct centers (resp. axes) of the involutions of H are collinear (resp. concurrent) except possibly for $q = 3$, $n \equiv 1 \pmod 3$, $Z(P)$ is planar, and H is strongly irreducible on the subplane of order congruent to 1 modulo 3, generated by the centers and axes of involutions in H.*

Proof. The following structure of H is used in this proof: $H = PC$, where C is a cyclic subgroup; $Z(P)$ is the unique minimal normal subgroup of H; $H/Z(P)$ is a Frobenius group with kernel $P/Z(P)$; and $C \cap P = 1$.

First we claim that H fixes a point or a line, unless we are in the exceptional situation mentioned in the Proposition. Assume not. Suppose H leaves invariant a triangle Δ. Let $H(\Delta)$ be the kernel of the action of H on Δ, and $\bar{H}=H/H(\Delta)$. Since $\bar{H}$ is isomorphic to a subgroup of the symmetric group on three letters and $|H|=q^3(q^2-1)/(q+1, 3)$, we get $H(\Delta)\geqslant P$ by the structure of H and q^2 does not divide 6. Hence $\bar{H}$ is cyclic of order 1, 2, or 3. If $|\bar{H}|\leqslant 2$, then H fixes a point, a contradiction. Therefore $|\bar{H}|=3$, and so all involutions of H belong to $H(\Delta)$. Thus the vertex (resp. sides) of Δ are centers (resp. axes) of three involutions in $H(\Delta)$. Hence these three involutions commute with each other. However, this contradicts the fact that C is cyclic. Therefore we may assume that H does not leave invariant any triangle. Let π_1 be the substructure of π generated by the centers and axes of the involution of H. Hence π_1 is an H-invariant subplane. By Lemma 2.2, H is strongly irreducible on π_1. Let H_1 be the kernel of the action of H on π_1. Suppose $H_1\neq 1$. Then $H_1\geqslant Z(P)$. By the structure of H and Theorem 2.3 we get that $H_1=Z(P)$, $P/Z(P)\cong C_3\times C_3$, $q=3$, and non-trivial elements of $P/Z(P)$ are triangular, on π_1, which implies they are also triangular on $\mathrm{Fix}(Z(P))$. Since an element in $P/Z(P)$ is a product of two involutions, it is not planar. This implies $n\equiv 1 \pmod 3$. Therefore we may assume $H_1=1$. By Lemma 2.2, $Z(P)\cong C_3\times C_3$ and $C_H(Z(P))=Z(P)$. However, this contradicts the fact that $C_H(Z(P))$ contains a subgroup of order $q+1=10$. Our claim is established.

Next we prove that if H fixes a point, then H fixes a line containing a center of an involution of H. Suppose not. Let the fixed point be P^*. By Lemma 3.1, there is an involution $\alpha\in H$ such that its center $\mathscr{C}(\alpha)\neq P^*$. Since any two involutions of H do not commute, $[P^*]$ contains all axes of involutions of H. Let $x(\alpha)=P^*\mathscr{C}(\alpha)$. Then $C_H(\alpha)=Z(P)\,C\leqslant H_{x(\alpha)}$. Since $C_H(\alpha)$ is a maximal subgroup of H, we may assume $C_H(\alpha)=H_{x(\alpha)}$. Hence $\mathscr{C}(\alpha)$ is the unique center of an involution of H incident with $x(\alpha)$. For $\alpha\in P\setminus Z(P)$, let $I(\sigma)=\{i\in H \mid i^2=1 \text{ and } \sigma^i=\sigma^{-1}\}$. Thus the centers of involutions of $I(\sigma)$ are collinear. Denote this line by $l(\sigma)$. For any line $l\notin[P^*]$, if $l^\sigma=l$, then $l=l(\sigma)$. Now $P=AB$, where A and B are elementary abelian normal subgroups of order q^2 containing $Z(P)$. Let $a\in A\setminus Z(P)$. For $a1\in A\setminus Z(P)$ we have $I(a)^{a1}=(a)$. Hence $l(a)^{a1}=(a)$. Since $l(a)\notin[P^*]$, $l(a)=l(a1)$. Let $b\in B\setminus Z(P)$. As $A\trianglelefteq P$, $a^b\in A\setminus Z(P)$. Hence $l(a^b)=l(a)$. But $l(a^b)=(l(a))^b$. Therefore $l(a)^b=l(a)$. Since $l(a)\notin[P^*]$, $l(a)=l(b)$. As $A=\langle A\setminus Z(P)\rangle$ and $B=\langle B\setminus Z(P)\rangle$, we get $l(a)^A=l(a)=l(a)^B$. Hence $l(a)^P=l(a)$. This implies $l(\sigma)=l(a)$ for any $\sigma\in P\setminus Z(P)$. Thus $l(a)^H=l(a)$. Therefore H always leaves invariant a line containing a center of an involution of H.

By duality we get that if H fixes a line then H fixes a point incident with an axis of an involution of H. Since we prove that H fixes a point or a line, H fixes a point P^* and a line l^* such that P^* (resp. l^*) is incident with an axis (resp. a center) of an involution of H. The action of P on $[P^*]$ (resp.

(l^*)) shows that all q^2 axes (resp. centers) of involutions of H are incident with P^* (resp. l^*).

3.3. COROLLARY. (a) *If* $q \neq 3$ *or* $n \neq 1$ (mod 3), *then* π *contains a* G*-invariant configuration isomorphic naturally to a unital of the Desarguesian plane of order* q^2.

(b) (Kantor [8]) *If a fixed point of H is incident with a fixed line of H, then* π *contains a G-invariant subplane naturally isomorphic to the Dearguesian plane of order* q^2.

Proof. Consider the natural action of G on $PG(2, q^2) = \mathscr{D}$.

(a) By Proposition 3.2, H fixes a point P^* in π. Applying the argument in the proof of Proposition 5.3 of [4] yields that $(P^*)^G$ is a configuration isomorphic naturally to a unital of $PG(2, q^2)$.

(b) Let $P^* \in \mathbb{P}(H)$ and $l^* \in \mathbb{L}(H)$ such that $P^* \in l^*$. By Lemma 3.1 we get that all q^2 centers of involutions of H are incident with l^*. Since H is a maximal subgroup of G and $\mathrm{Fix}(G) = (\phi, \phi)$, $|(l^*) \cap \{\text{centers of involutions of } G\}| = q^2$. An easy counting argument on the axes of involutions of G shows that any line joining two points in P^{*G} is an axis of an involution of G. Hence $|(l^*) \cap P^{*G}| = 1$. Let $\mathbb{P}^* = P^{*G} \cup \{\text{centers of involutions of } G\}$ and $\mathbb{L}^* = l^{*G} \cup \{\text{axes of involutions of } G\}$. Then $(\mathbb{P}^*, \mathbb{L}^*)$ is a G-invariant subplane of order q^2. Therefore $(\mathbb{P}^*, \mathbb{L}^*)$ is a Desarguesian plane by Hoffer [6].

4. THEOREM A

In this section we retain the conditions and notations of Section 3. The exceptional situation mentioned at the end of Proposition 3.2 will be called

Case E. $q = 3$, $n \equiv 1$ (mod 3), $Z(P)$ *is planar, and* H *is strongly irreducible on the subplane of order congruent to* 1 *modulo* 3, *generated by the center and axes of involutions in H.*

The following result improves Theorem A except in Case E.

4.1. THEOREM. *In addition to the conditions, we assume that* $|G_V|$ *is either even or a power of q. Then* π *contains a Desarguesian subplane of order* q^2 *where G acts naturally on, except possibly in Case* E.

Proof. Suppose this theorem is false. Let the fixed line and point of H in Proposition 3.2 be l^* and P^*. By Corollary 3.3(b) we may assume that

$P^* \notin l^*$. Let α be an involution of H and let $B = a(\alpha) \cap l^*$. So $B^{Z(P)} = B$, and $|\mathscr{C}(\alpha)^G \cap l^*| = q^2 = |B^G \cap l^*|$ by counting axes of involutions.

If $|G_V|$ is always even, then $l^* = (\mathscr{C}(\alpha)^G \cap l^*) \cup (B^G \cap l^*)$. Since $Z(P)$ fixes each point in these two sets, $Z(P)$ consists of homologies with common axis (resp. center) l^* (resp. P^*). Hence $2q^2 - 1 = n \equiv 1 \pmod q$, a contradiction. Therefore there exists a point U of l^* such that $|G_U|$ is a power of q. Let $\mathscr{C}(\alpha)^G$, B^G, 01, ..., 0s be the G-orbits of points whose intersections with l^* are not empty. For $i = 1, ..., s$ let $Ui \in 0i \cap l^*$. Thus $|G_{Ui}|$ is a power of q. Let $W1^G, ..., Wr^G$ (resp. $V1^G, ..., Vt^G$) be the G-orbits of points whose intersections with $a(\alpha)$ (resp. $P^*\mathscr{C}(\alpha)$) are non-empty.

If all elements in $P \backslash Z(P)$ are anti-flag, then $\mathbb{P}(Z(P)) \geqslant (l^*)$, which implies $n + 1 \equiv 0 \pmod p$ and $n - 1 \equiv 0 \pmod p$. This contradiction shows that there exists a non-antiflag element $\sigma \in P \backslash Z(P)$. Thus σ fixes a point on l^*. Since σ is a product of two involutions in H, $\mathbb{P}(\sigma) \leqslant \{P^*\} \cup (l^*)$. Thus $n \equiv 1 \pmod p$. The action of P on l^* now implies the existence of $Z \in l^*$ such that $G_Z = P$.

We now use the method introduced in [5] based on Proposition 2.1 to construct the following part of the embroidered G-incidence matrix involving the line orbits $a(\alpha)$, $P^*\mathscr{C}(\alpha)$, and l^*. For the convenience of the reader, the number of points of a G-orbit on a line of a line orbit is put in the lower corner of that entry for some cases. Also we indicate the representative of G-orbits involved.

Some remarks concerning Table I are in order. Note that the points in the line orbits represented by $a(\alpha)$, $P^*\mathscr{C}(\alpha)$, l^* are contained in the point orbits shown in Table I, where $V = (q+1, 3)$ and $U = (q^2 - 1)/V$.

Clearly $|\mathscr{C}(\alpha)^G| = |a(\alpha)^G| = q^2(q^2 - q + 1)$. Since $H = G_{l^*} = G_{P^*}$, $|l^{*G}| = q^3 + 1 = |P^{*G}|$. As $|C_G(\alpha): C_G(\alpha) \cap H| = q + 1$, $|[\mathscr{C}(\alpha)] \cap l^{*G}| \geqslant q + 1$. From $|l^* \cap \mathscr{C}(\alpha)^G| = q^2$, we get $|[B] \cap l^{*G}| = 1$. Hence $G_B \leqslant G_{l^*} = H$ and so $G_B = H_B = Z(P)\, C$. Therefore $|B^G| = q^2(q^3 + 1)$ and $|[\mathscr{C}(\alpha)] \cap l^{*G}| = q + 1$. Similarly $|P^*\mathscr{C}(\alpha)^G| = q^2(q^3 + 1)$ and $|P^*\mathscr{C}(\alpha) \cap \mathscr{C}(\alpha)^G| = q + 1$.

The columns indexed by $\mathscr{C}(\alpha)$, B, P^* are easily obtained by counting axes and centers of involutions.

Counting incidence in $\{x \cap y \mid x \neq y \in a(\alpha)^G\}$ yields the remaining upper corner entries of the first row. Computing [first row | third row] yields the entries of the third row indexed by the W's, which are zeros. Counting incidence in $\{x \cap y \mid x \neq y \in l^{*G}\}$ yields the remaining upper corner entries of the third row.

Let $Uj^g \in P^*\mathscr{C}(\alpha)$. Suppose $G_{Uj^g} \cap Z(P) \neq 1$. Then $G_{Uj^g} \cap Z(P) \leqslant P^g$ implies $P^g = P$ by the structure of $PSU(3, q)$. Hence $g \in H$ and so $P^{*g} = P^*$. Thus $P^*\mathscr{C}(\alpha) = P^*Uj^g = (P^*U_j)^g = l^{*g}$, a contradiction. Therefore $Z(P) \cap G_{Uj^g} = 1$, and $P^g \neq P$. As $P_{P^*\mathscr{C}(\alpha)} = Z(P)$, $|(P^*\mathscr{C}(\alpha))^{G_{Uj^g}}| = |G_{Uj^g}| = |G_{Uj}|$. This shows that $|[Uj^g] \cap (P^*\mathscr{C}(\alpha))^G| = |G_{Uj}|\, y_j$, where $y_j \geqslant 1$. Thus $|[Uj] \cap (P^*\mathscr{C}(\alpha))^G| = |G_{Uj}|\, yj$. Computing $[P^*\mathscr{C}(\alpha)^G \mid l^{*G}]$ yields

TABLE I

		$q^2(q^2-q+1)$	$q^2(q^3+1)$	q^3+1	$(q^2-q+1)\,k_1$	$\cdots$	$(q^2-q+1)\,k_r$						
		$\mathscr{C}(\alpha)$	B	P^*	$W1$	$\cdots$	Wr	$U1$	$\cdots$	Us	$V1$	$\cdots$	Vt
$q^2(q^2-q+1)$	$a(\alpha)$	q^2-q / q^2-q	1 / $1+q$	q^2 / $q+1$	1 / k_1	$\cdots$	1 / kr	0	$\cdots$	0	0	$\cdots$	0
$q^2(q^3+1)$	$P^*\mathscr{C}(\alpha)$	$q+1$ / 1	0	q^2 / 1	a_1 / $q(q-1)\,b_1$	$\cdots$	a_r / $q(q-1)\,b_r$	$\|G_{UL}\|\ y_1$ / qUy_1	$\cdots$	$\|G_{Us}\|\ y_s$ / qUy_s	C_1 / Ud_1	$\cdots$	C_t / Ud_t
q^3+1	l^*	$q+1$ / q^2	1 / q^2	0	0	$\cdots$	0	1 /	$\cdots$	1 /	0	$\cdots$	0

$q^2(q^3+1)^2 = q^2(q^2-q+1)(q+1)(q+1) + \sum_{j=1}^{s} |Uj^G|(|G_{Uj}|\, y_j)$. Since $|G| = q^3(q^3+1)(q^2-1)/V$, the above equation gives

$$V = \sum_{j=1}^{s} y_j. \tag{1}$$

Since the cyclic subgroup of order $q-1$ of C is inverted by an involution, it is not planar.

There is exactly one axis $a(\beta)$ of an involution β of G passing through W for any $W \in \bigcup_{i=1}^{r} W_i^G$. Therefore $G_W \leqslant G_{a(\beta)} = C_G(\beta)$. The axis of any involution commuting with $Z(P)$ is in $[P^*]$. Hence $|Z(P) \cap G_W| = 1$ for $W \in Wi^G \cap P^*\mathscr{C}(\alpha)$, $i = 1, ..., r$. This and the last paragraph now imply that $|Wi^G \cap P^*\mathscr{C}(\alpha)| = q(q-1)\, b_i$ for some non-negative integer b_i. Since $P^*\mathscr{C}(\alpha) \cap \mathscr{C}(\alpha)^G = \{\mathscr{C}(\alpha)\}$, $(P^*\mathscr{C}(\alpha))^\beta \neq P^*\mathscr{C}(\alpha)$. Hence $a_i \geqslant 2$.

Let X be a point in the G-orbits of the U's or V's. Then $|G_X|$ is a power of q. Hence $|G_X \cap C| = 1$. Therefore C acts semi-regularly on $X^G \cap P^*\mathscr{C}(\alpha)$. This shows that $|V_i^G \cap P^*\mathscr{C}(\alpha)| = ud_i$ for some non-negative integer d_i. Assume now X is a point in the G-orbits of the U's. Then $|G_X \cap Z(P)| = 1$. Therefore $|Uj^G \cap P^*\mathscr{C}(\alpha)| = qUx_j$ for some non-negative integer x_j. Since $|Uj^G| \cdot |G_{Uj}|\, y_j = q^2(q^3+1)(qUx_j)$ by the incidence structure $(P^*\mathscr{C}(\alpha)^G, Uj^G)$, $x_j = y_j$.

$|\bigcup_{i=1}^{r} W_i^G \cap P^*\mathscr{C}(\alpha)| = (q^2-q)(q^2-1)$ by counting the intersection of axes of involutions with $P^*\mathscr{C}(\alpha)$. On the other hand, this number is $\sum_{i=1}^{r} q(q-1)\, b_i$. Hence

$$\sum_{i=1}^{r} b_i = q^2 - 1. \tag{2}$$

Counting the points on $P^*\mathscr{C}(\alpha)$ we get

$$n \equiv 1 \quad \left(\operatorname{mod} \frac{q^2-1}{V}\right). \tag{3}$$

Since $Z^P = Z$, $(P^*Z)^P = P^*Z$. As $\mathbb{P}(\sigma) \leqslant l^* \cup \{P^*\}$ for $\sigma \in P \setminus Z(P)$, the action of P on the points of P^*Z yields $n \equiv 1 \pmod{q^2}$. By (3) we get

$$n = 1 + \lambda \frac{q^2(q^2-1)}{V} \qquad \text{for some integer } \lambda. \tag{4}$$

Counting the points on $P^*\mathscr{C}(\alpha)$, we now get

$$1 + \lambda \frac{q^2(q^2-1)}{V} = n = 1 + q(q-1) \sum_{i=1}^{r} b_i + \frac{q(q^2-1)}{V} \sum_{i=1}^{s} y_i$$
$$+ \frac{q^2-1}{V} \sum_{i=1}^{t} d_i.$$

By (1) and (2), the last equation, after we first cancel 1 from both sides and then divide them by $(q^2-1)/V$, becomes $\lambda q^2 = q(q-1)V + qV + \sum_{i=1}^{t} d_i = q^2V + \sum_{i=1}^{t} d_i$. Therefore,

$$\sum_{i=1}^{t} d_i = (\lambda - V)\, q^2. \tag{5}$$

Counting incidences in $\{x \cap y \mid x \neq y \in (P^*\mathscr{C}(\alpha))^G\}$, we get $(q^2(q^3+1))(q^2(q^3+1)-1) = q^2(q^2-q+1)(q+1)\cdot q + (q^3+1)\,q^2(q^2-1) + \sum_{i=1}^{r+} q^2(q^3+1)\,q(q-1)\,b_i(a_i-1) + \sum_{i=1}^{s+} |U_i^G|\,|G_{Ui}|\,y_i(|G_{Ui}|\,y_i - 1) + \sum_{i=1}^{t+} ((q^2-1)/V)\,d_i \cdot q^2(q^3+1)(C_i-1)$. After dividing by $q^2(q^3+1)$, this equation simplifies to

$$q(q^4-1) = q(q-1)\sum_{i=1}^{r+} b_i(a_i-1) + \frac{q(q^2-1)}{V}\sum_{i=1}^{s+} y_i(|G_{Ui}|\,y_i - 1) + \frac{q^2-1}{V}\sum d_i(C_i-1).$$

By (1), (2), and the fact that $a_i \geqslant 2$ and $|G_{Ui}| \geqslant q$, we get from the last equation that

$$q(q^4-1) \geqslant q(q-1)(q^2-1) + \frac{q(q^2-1)}{V}\cdot V(q-1) + \frac{q^2-1}{V}\sum_{i=1}^{t} d_i(C_i-1),$$

which, after dividing by q^2-1, yields

$$q(q^2+1) \geqslant q(q-1) + q(q-1) + \frac{1}{V}\sum_{i=1}^{t} d_i(C_i-1). \tag{6}$$

If $\lambda = V$, then by (5) we have $d_i = 0$ for $i = 1, ..., t$. This means that the V's do not exist. We recall that $G_{Uj}g \cap Z(P) = 1 = G_W \cap Z(P)$. Therefore $Z(P)$ fixes exactly two points on $P^*\mathscr{C}(\alpha)$, and so $Z(P)$ is not planar. Thus $\mathbb{P}(Z(P)) \leqslant l^* \cup \{P^*\}$. Hence $\mathbb{P}(P) \leqslant l^* \cup \{P^*\}$. Now the action of P on P^*Z yields $n \equiv 1 \pmod{q^3}$, which gives $n \equiv 1 \pmod{q^3((q^2-1)/V)}$ by (3). However, $\lambda = 1$ implies $n = 1 + q^2(q^2-1)/V$ by (4), which contradicts the last congruence. Therefore we may assume $\lambda > V$ and $t \geqslant 1$.

Let $i \in \{1, ..., t\}$. Since $\mathbb{P}(P \backslash Z(P)) \leqslant l^* \cup \{P^*\}$ and $V_i^G \cap l^* = \phi$, we obtain $G_{Vi} \cap (P \backslash Z(P))^G = \varnothing$. This implies that $G_{Vi} = Z(P^g)$ for some $g \in G$, as $|G_{Vi}|$ is a power of q. Thus there are two kinds of $Z(P)\,C$ orbits of $Vi^G \cap P^*C$. The first kind has size $(q^2-1)/V$ and the stabilizer of a point is $Z(P)$. The second kind has size $q((q^2-1)/V)$ and the stabilizer in $Z(P)\,C$ of a point is 1. Let r_i = number of $Z(P)\,C$ orbits of the first kind and f_i = number of $Z(P)\,C$ orbits of the second kind. By the incidence structure

$((P^*\mathscr{C}(\alpha))^G, Vi^G)$ and $|G_{Vi}| = q$, we get that $d_i = C_i$. Counting $|Vi^G \cap P^*\mathscr{C}(\alpha)|$ yields

$$\frac{q^2-1}{V} d_i = \frac{q^2-1}{V} r_i + q \frac{(q^2-1)}{V} f_i.$$

Hence

$$d_i = qf_i + r_i \qquad \text{for} \quad i = 1, ..., t. \tag{7}$$

Let $m = 1 + ((q^2-1)/V) \sum_{i=1}^t r_i$. Since the fixed points of $Z(P)$ on $P^*\mathscr{C}(\alpha)$ different from P^* and $\mathscr{C}(\alpha)$ are in $\sum_{i=1}^t (Vi^G \cap P^*\mathscr{C}(\alpha))$, $m = |\mathbb{P}(Z(P)) \cap P^*\mathscr{C}(\alpha)| - 1$.

Let $\Omega_1 = \{1 \leqslant j \leqslant t \mid f_j \neq 0\}$. Suppose $\Omega_1 = \varnothing$. Then for $1 \leqslant j \leqslant t$ we have $f_j = 0$ and so $Vj^G \cap P^*\mathscr{C}(\alpha) \leqslant \mathbb{P}(Z(P))$. Since $t \geqslant 1$, $\mathrm{Fix}(Z(P))$ is a subplane of order m. Thus $m = 1 + ((q^2-1)/V) \sum_{i=1}^t d_i = 1 + (\lambda - V) q^2((q^2-1)/V)$ by (5). Since $\lambda > V$ and $q \geqslant 3$, $m > q$. By (4) we get $n - m = q^2(q^2-1)$, which is strictly less than Vm as $\lambda > V$. Hence $n < (V+1) m$. Since $m^2 \leqslant n$, we get $m < V + 1 \leqslant 4$, a contradiction. Therefore we may assume $\Omega_1 \neq \varnothing$. For $i \in \Omega_1$ we have $C_i = d_i \geqslant qf_i \geqslant q$ by (7). Now (6) yields $q(q^2+1) \geqslant 2q(q-1) + (1/V) \sum_{i \in \Omega_1} qf_i(q-1)$. Thus $q^2 - 1 + 2 = q^2 + 1 \geqslant 2(q-1) + (1/V)(q-1) \sum_{i \in \Omega_1} f_i$. After dividing by $q-1$, this becomes $q + 1 + 2/(q-1) \geqslant 2 + (1/V) \sum_{i \in \Omega_1} f_i$. Therefore

$$V(q-1) + \frac{2V}{q-1} \geqslant \sum_{i \in \Omega_1} f_i = \sum_{i=1}^{t} f_i. \tag{8}$$

Case 1. $V = 1$.

By (8), $\sum_{i=1}^t f_i \leqslant q$. By (7), $\sum_{i=1}^t d_i \leqslant q^2 + \sum_{i=1}^t r_i$. Therefore $(\lambda - 2) q^2 \leqslant \sum_{i=1}^t r_i$ by (5). Hence $m \geqslant 1 + (\lambda - 2) q^2(q^2-1)$. Suppose $\lambda > 2$. Then $Z(P)$ is planar and m is the order of $\mathrm{Fix}(Z(P))$. By (4), $n - m \leqslant 2q^2(q^2-1)$, which is strictly less than $2m$. Thus $m^2 \leqslant n < 3m$ and so $m < 3$. However, by the definition of m, $\lambda > 2$, and $q \geqslant 3$, we get $m > 3$. This contradiction shows that $\lambda = 2$ as $\lambda \geqslant 2$. By (4), $n = 1 + 2q^2(q^2-1)$. If $Z(P)$ is not planar, then $n \equiv 1 \pmod{q^3(q^2-1)}$ by the action of P on $P^*\mathscr{C}(\alpha)$ and (3). This contradiction shows that $\mathrm{Fix}(Z(P))$ is a subplane of order m. Thus $n - m \equiv 0 \pmod q$. Since $n - m = (2q^2 - \sum_{i=1}^t r_i)(q^2-1)$, $\sum_{i=1}^t r_i \equiv 0 \pmod q$. By the definition of m, we get $m \geqslant 1 + q(q^2-1)$. Hence $m^2 > n$, a contradiction. Therefore case 1 cannot occur.

Case 2. $V = 3$.

Since $V = (q+1, 3)$, $q \neq 3$. Thus $q \geqslant 5$. So $2V/(q-1) \leqslant 2V/(q+1) \leqslant 1$. By (8), $3q - 2 \geqslant \sum_{i=1}^t f_i$. By (7), $\sum_{i=1}^t d_i \leqslant q(3q-2) + \sum_{i=1}^t r_i$. Therefore, by (5), $\sum_{i=1}^t r_i \geqslant (\lambda - 6) q^2 + 2q$. By the definition of m, we now have $m \geqslant 1 + [(\lambda - 6) q^2 + 2q]((q^2-1)/V)$. Hence, by (4), $n - m \leqslant 2q(3q-1)((q^2-1)/V)$.

Suppose $\lambda > 6$. Then Fix$(Z(P))$ is a subplane of order m and $m \geqslant 1 + [q^2 + 2q]((q^2 - 1)/V) > 7$. Thus $n - m < 6m$. Hence $m^2 \leqslant n < 7m$, and so $m < 7$. This contradiction shows that $4 \leqslant \lambda \leqslant 6$.

Assume that $Z(P)$ is planar. Then Fix$(Z(P))$ has order m. Since $n - m \equiv 0 \pmod q$, $\sum_{i=1}^{t} r_i \equiv 0 \pmod q$. Thus $m \geqslant 1 + q((q^2 - 1)/V)$. Since $\lambda \leqslant 6$, we have $1 + 6q^2((q^2 - 1)/V)\, n \geqslant m^2 \geqslant 1 + q^2((q^2 - 1)/V)^2$. This contradicts $(q^2 - 1)/V \geqslant 8$. Therefore $Z(P)$ is not planar. As before we get $n \equiv 1 \pmod{q^3((q^2 - 1)/V)}$. By (4), q divides $\lambda > 6$. This forces $q = \lambda = 5$, and $n = 1001 \equiv 0 \pmod 7$. Table I shows that if $|G_X|$ is even then $7 \mid (q^2 - q + 1)$ must divide $|X^G|$. Hence $7 \nmid |G_X|$. However, $n^2 + n + 1 \equiv 1 \pmod 7$ implies that an element of order 7 of G cannot act semi-regularly on the points of π. This contradiction shows that Case 2 cannot occur either, and the proof of the theorem is complete.

We prove Theorem A in the rest of this section. By Theorem 4.1, it suffices to treat Case E. By Proposition 3.2, π_1 is a subplane, whose order will be denoted by m. Let $\hat{H}$ be the collineation group of H induced on π_1.

If $\sigma \in P \setminus Z(P)$, Fix$(\hat{\sigma})$ is a triangle. Since $\hat{H}$ is strongly irreducible, $\hat{P}$ acts transitively on Fix$(\hat{\sigma})$. Let the vertex of Fix$(\hat{\sigma})$ be denoted by $\{R, S, T\}$. Let $I(\sigma) = \{i \in H \mid i^2 = 1, \sigma^i = \sigma^{-1}\}$. Then the three centers of involutions of $I(\sigma)$ are on one side of Fix$(\hat{\sigma})$ and different from its vertex. Let this line be ST. Then the three axes of the involutions of $I(\sigma)$ are in $[R]$. Let $\mathscr{A} = \{$axes of involutions in $G\}$. Then $|[X] \cap \mathscr{A}| \geqslant 3$ for $X \in \{S, T\}$. Since an involution commutes with $9 - 3 = 6$ other involutions, we have $|([S] \cup [T] \cup (\bigcup_{i \in I(\sigma)} [\mathscr{C}(i)])) \cap \mathscr{A}| \geqslant 3 + 3 + 3 \cdot 6 = 24$. Since $|\mathscr{A}| = 9(9 - 3 + 1) = 63$, there are at most 39 axes in $\mathscr{A}$ intersecting ST not in S, T or $\mathscr{C}(i)$ for $i \in I(\sigma)$. Since $|G_V| =$ even for any point V, $n + 1 - 5 \leqslant 39$. Thus $n \leqslant 43$. By Proposition 3.2, $m \equiv 1 \pmod 3$. Since $m^2 < n$, $m = 4$. Therefore an involution will induce an elation on π_1, which contradicts the fact that it is a homology. The proof of Theorem A is now complete.

5. Theorem B

Conditions and notations of Section 3 are kept in this section.

5.1. Theorem. *We assume the following additional conditions:* (1) $(f_G, (q + 1)/V) = 1$, *where* $f_G =$ l.c.m.$\{|G_{ABCD}| \mid A, B, C, D$ *form a quadrangle in* $\pi\}$, *and* $V = (q + 1, 3)$; (2) $|G_V|$ *is either even or not less than* q, *where* q *is a prime. Then* π *contains a Desarguesian subplane of order* q^2, *where* G *acts naturally on, except possibly in Case* E *of Section* 4.

Proof. Using the same notations as in the proof of Theorem 4.1. Following the proof of Theorem 4.1 with slight modification we get a contradiction except $n = 1001$ and $q = 5$. As 1001 is not a sum of two squares we can apply The Bruck–Ryser Theorem [1].

5.2. In the rest of this paper we prove Theorem B. Define f_G as in the statement of Theorem 4.1. Theorem: Since $(q+1)/V$ is a power of 2, $(f_G, (q+1)/V) = 1$. As in the proof of Theorem 4.1, it suffices, by Theorem 5.1, to consider $X \in P^*\mathscr{C}(\alpha)$ such that $|G_X|$ is odd. Thus X belongs to the orbits of V's.

Assume $|G_X| < q$. Since $|X^G|$ is divisible by $(q^2-1)/V \cdot q$, $|G_X|$ divides $q^3+1 = 4 \cdot 7$ if $q = 3$ and $8 \cdot 43$ if $q = 7$. Therefore $|G_X| = 7$ if $q = 3$ and $|G_X| = 43$ if $q = 7$ as $|G_X|$ is odd. This contradicts $|G_X| < q$.

In the rest of the proof we assume that $q = 5$ and $|G_X| < 5$. Since $q^3 + 1 = 3^2 \cdot 2 \cdot 7$, $|G_X| = 3$. This shows that we may assume that $|G_X| \geqslant 3$ for any point X such that $|G_X|$ is odd. As in the proof of Theorem 4.1, we get $q(q^2+1) \geqslant q(q-1) + q(q-1) + (1/V)\sum_{i=1}^{t} d_i(C_i - 1)$. We just show that $C_i \geqslant 3$. Since $d_i \geqslant qf_i$, the last equation yields $(q^2 - 2q + 3) \geqslant (2/V)\sum_{i=1}^{t} f_i$. As $q = 5$, we get $9V \geqslant \sum_{i=1}^{t} f_i$. Since $9 = 2q - 1$ and $V = 3$, the last equation yields $6q^2 - 3q = q(2q-1)\, V \geqslant \sum_{i=1}^{t} qf_i$. From the proof of Theorem 4.1 we get $\sum_{i=1}^{t} d_i = (\lambda - 3)\, q^2$ as $V = 3$. Hence $\sum_{i=1}^{t} r_i \geqslant (\lambda - 9)\,(q^2+1)\, 3q$, and $m = 1 + (\sum_{i=1}^{t} r_i)((q^2-1)/V) \geqslant 1 + ((\lambda - 9)\, q^2 + 3q)((q^2-1)/V)$. Recall $n = 1 + \lambda q^2((q^2-1)/V)$ in the proof of Theorem 4.1. If $\lambda \geqslant 9$, then $m \geqslant 1 + 3q((q^2-1)/V)$. Suppose $\lambda > 9$. Then $Z(P)$ is planar and $m > (q^2+3q)((q^2-1)/V) > 9$. Now $n - m \leqslant (9q^2 - 3q)((q^2-1)/V) < 8m$. Since $m^2 \leqslant n$, the last inequality yields $m < 9$, a contradiction. Suppose $\lambda = 9$. Then $n = 1 + 9(q^2(q^2-1)/V$ and $m \geqslant 1 + 3q((q^2-1)/V)$. Thus $Z(P)$ is planar. Hence $m^2 < n$, which is impossible. Therefore $3 \leqslant \lambda \leqslant 8$ as $\lambda \geqslant V = 3$. Now $n \leqslant 1 + 8q^2((q^2-1)/V)$ and $m \geqslant 1 + q(q^2-1)/V$. Since $(q^2-1)/V \geqslant 8$, $m^2 > n$. Hence $Z(P)$ cannot be planar. This implies $n \equiv 1 \pmod{q^3((q^2-1)/V)}$. Since $n = 1 + \lambda q^2((q^2-1)/V)$, q divides λ. As $3 \leqslant \lambda \leqslant 8$ and $q = 5$, $\lambda = q = 5$ and $n = 1001$. This has been shown to be impossible in the last part of the proof of Theorem 5.1. The proof of 5.2 is now complete.

Acknowledgment

The second author wishes to express his gratitude to the hospitality of the University of Toronto, where this research has been developed.

References

1. P. Dembowski, "Finite Geometries," Springer-Verlag, Berlin, 1968.
2. D. Gorenstein, "Finite Groups," Harper & Row, New York, 1968.

3. C. HERING, On the structure of finite collineation groups of projective planes, *Abh. Math. Sem. Univ. Hamburg*, (1979), 154–182.
4. C. HERING AND M. WALKER, Perspectivities in irreducible collineation groups of projective planes, II, *J. Statist. Plann. Inference*, (1979), 151–177.
5. C. Y. HO, Finite projective planes that admit a strongly irreducibly collineation group, *Canad. J. Math.* **37** (1985), 579–611.
6. A. R. HOFFER, On unitary collineation groups, *J. Algebra* **22** (1972), 211–218.
7. W. KANTOR, On unitary polarities of finite projective planes, *Canad. J. Math.* **23**, No. 6 (1971), 1060–1077.
8. W. KANTOR, On homologies of finite projective planes, *Israel J. Math.* **16** (1973), 351–361.
9. W. KANTOR, On the structure of collineation groups of finite projective planes, *Proc. London Math. Soc.*, (1976), 385–402.
10. M. SEIB, Unitäre Polaritäten endlicher projectiven Ebenen, *Arch. Math.* **21** (1970), 103–112.