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The Riemann Problem for a Class of Conservation Laws of Mixed Type

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1. INTRODUCTION

Let $\sigma: \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. If $\sigma'(u) \geqslant 0$ for all u, the system

$$\begin{aligned} u_t - v_x &= 0, \\ v_t - \sigma(u)_x &= 0, \end{aligned} \qquad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

is the quasilinear wave equation $w_{tt} - \sigma(w_x)_x = 0$, written as a system by setting $v = w_t$, $u = w_x$. The Riemann problem for Eq. (1.1) consists of solving (1.1) subject to initial step data of the form

$$\begin{aligned} (u(x, 0), v(x, 0)) &= (u_l, v_l) \qquad \text{if} \quad x < 0, \\ &= (u_r, v_r) \qquad \text{if} \quad x > 0, \end{aligned} \tag{1.2}$$

for given (u_l, v_l), (u_r, v_r) in $\mathbb{R}^2$. The initial value problem (1.1), (1.2) has been studied under various hypotheses on σ [4, 7, 12] and has served as a prototype for studies of more general 2×2 hyperbolic conservation laws [3, 5, 8, 11]. In particular , Keyfitz and Kranzer [5] recently solved the Riemann problem for a class of 2×2 non-strictly hyperbolic conservation laws, for arbitrary initial data (1.2). This class includes Eq. (1.1) if (among other conditions) $\sigma'(u) > 0$ except at one point.

For conservation laws of mixed type, very little is known about the Riemann problem. Mock [9], in studying shock waves for such systems, demonstrates that properties useful for solving the Riemann problem in the hyperbolic case fail to hold for systems of mixed type. James [4] considers Eq. (1.1) when $\sigma'(u) \geqslant 0$ except in an interval (α, β) (as in Fig. 1), and studies possible shock wave solutions.

In this paper, we solve the Riemann problem for a class of systems (1.1)

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of mixed type, for arbitrary initial data (1.2). The graph of σ is shown in Fig. 1, and the conditions required of σ are given in Section 2. Our approach is based upon that of Keyfitz and Kranzer [5] for non-strictly hyperbolic systems; the relationship between the analysis here and that in [5] is indicated in Remark 3.1.

There is an unusual and somewhat unsatisfactory feature of our solution of the Riemann problem. For any given (u_l, v_l) in (1.2), there are values of (u_r, v_r) for which our solution involves stationary shock waves. In Section 5 we show that these shock waves do not satisfy analogues of the viscosity criteria commonly associated with the quasilinear wave equation, while all other shock waves in our solution satisfy the viscosity criteria associated with a model from viscoelastic bar theory. A difficulty of this nature is discussed by James [4] in connection with the application of (1.1) to a study of elastic bars exhibiting plastic properties. James indicates that in order to explain experimental observations of phase transitions in bars, it may be appropriate to allow solutions of the Riemann problem to be non-unique. We do not pursue this application here, and indeed formulate admissibility criteria for shock waves that guarantee uniqueness of the solution of the Riemann problem. The relationship between admissibility and viscosity criteria for shock wave solutions of (1.1) is also explored by Slemrod [10] in the context of propagating phase boundaries in a van der Waals fluid.

In Section 2 we describe the set of points (u_2, v_2) which may be joined to a given point (u_1, v_1) by an admissible shock wave or rarefaction wave, with (u_1, v_1) being the value of the corresponding weak solution of (1.1) to the left of the wave, and (u_2, v_2) being the value on the right. Section 4 contains proofs of propositions stated in Section 3 in connection with the construction of the solution of the Riemann problem.

Throughout, the term *smooth* will be used to mean continuously differentiable, and whenever $\sigma'(u) \geqslant 0$, it will be convenient to use the notation

$$c(u) = (\sigma'(u))^{1/2} \geqslant 0. \tag{1.3}$$

2. Shock Curves and Rarefaction Curves

Consider the system of conservation laws

$$\begin{aligned} u_t - v_x &= 0, \\ v_t - \sigma(u)_x &= 0, \end{aligned} \tag{2.1}$$

where $\sigma\colon \mathbb{R} \to \mathbb{R}$ is a given function with the following properties.

(i) σ is twice continuously differentiable,

(ii) $\sigma'(u) \to +\infty$ as $|u| \to \infty$,

(iii) there exist numbers $\alpha < \beta$ such that

$$\sigma'(u) > 0 \quad \text{and} \quad \sigma''(u) < 0 \quad \text{for } u < \alpha,$$
$$\sigma'(\alpha) = 0, \quad \sigma''(\alpha) < 0,$$
$$\sigma'(u) < 0 \quad \text{for } \alpha < u < \beta \quad \text{and} \quad \sigma''(u) = 0$$

for exactly one value of u in this range

$$\sigma'(\beta) = 0, \quad \sigma''(\beta) > 0,$$
$$\sigma'(u) > 0 \quad \text{and} \quad \sigma''(u) > 0 \quad \text{for } u > \beta.$$

A typical σ is graphed in Fig. 1.
The values γ and δ given by $\sigma(\gamma) = \sigma(\beta)$, $\sigma(\delta) = \sigma(\alpha)$ will play an important role in the analysis.

The characteristic values for Eq. (2.1) are $\lambda_{\pm}(u) = \pm c(u)$, with associated eigenvectors $r_+(u) = (1, -c(u))$ and $r_-(u) = (1, c(u))$, respectively. Equation (2.1) is strictly hyperbolic in the regions D_1, D_2 of (u, v)-space defined, respectively, by $u > \beta$, $u < \alpha$, where the characteristic values are real and distinct, hyperbolic in the closure of these regions, and elliptic in the region D_0 defined by $\alpha < u < \beta$.

A point $U_1 \in \mathbb{R}^2$ may be joined to a point $U_2 \in \mathbb{R}^2$ by a (centered) shock wave with (constant) shock speed s if the function

$$(u, v)(x, t) = U_1 \qquad x < st,$$
$$= U_2 \qquad x > st, \qquad t > 0,$$

is a weak solution of Eq. (2.1). This is the case if and only if $U_1 = (u_1, v_1)$, $U_2 = (u_2, v_2)$ and s are related by the Rankine–Hugoniot conditions [6],

$$v_2 - v_1 = -s(u_2 - u_1),$$
$$\sigma(u_2) - \sigma(u_1) = -s(v_2 - v_1). \tag{2.2}$$

In order to solve the Riemann problem uniquely, it is necessary to restrict

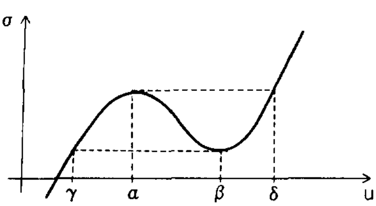

FIG. 1. Graph of σ.

attention to a subclass of such shock waves, the *admissible* shock waves. If U_1, U_2 and s satisfy (2.2), we say that U_1 is joined to U_2 on the right by an admissible shock wave if at least one of the following *admissibility criteria* is satisfied.

For all w between u_1 and u_2, either

$$(\sigma(w)-\sigma(u_1))/(w-u_1) \leqslant (\sigma(u_2)-\sigma(u_1))/(u_2-u_1) \qquad \text{if} \quad s \leqslant 0, \tag{2.3}$$

or

$$(\sigma(w)-\sigma(u_1))/(w-u_1) \geqslant (\sigma(u_2)-\sigma(u_1))/(u_2-u_1) \qquad \text{if} \quad s \geqslant 0, \tag{2.4}$$

or

$$s=0. \tag{2.5}$$

Admissible shock waves satisfying (2.3) will be called 1-shocks, while those satisfying (2.4) will be called 2-shocks. Shock waves with $s=0$ will be called *stationary shocks.*

Conditions (2.3), (2.4) are analogues of Oleinik's Condition E for a single conservation law, and have also been used elsewhere [3, 8, 12]. In Section 5, we discuss the relationship between the admissibility criteria (2.3)–(2.5) and other admissibility criteria that might be used for this problem.

From (2.2), we have

$$s^2=(\sigma(u_2)-\sigma(u_1))/(u_2-u_1) \tag{2.6}$$

so that no two points within the elliptic region D_0 may be joined by a shock. For a fixed U_0 in the hyperbolic region $\bar{D}_1 \cup \bar{D}_2$, let $S_1(U_0)$ be the connected component containing U_0, of those points to which U_0 may be joined by a 1-shock, with U_0 on the left. $S_1^*(U_0)$ will denote the set of those points not in $S_1(U_0)$ to which U_0 may be joined by a 1-shock with U_0 on the left, and $S_2(U_0)$ will be the set of points to which U_0 may be joined by a 2-shock, with U_0 on the left. As we shall see, $S_2(U_0)$ has exactly one component, so we have no need of the notation $S_2^*(U_0)$. If $U_0 \in D_0$, let $S_1(U_0)$, $S_1^*(U_0)$ be the curves of points in D_1, D_2, respectively, to which U_0 may be joined by a 1-shock, with U_0 on the left. There are no points to which U_0 may be joined by a 2-shock with U_0 on the left.

To describe the shock curves $S_1(U_0)$, $S_1^*(U_0)$ and $S_2(U_0)$ precisely, we introduce some notation. Let $U_0=(u_0, v_0)$. If $(\sigma(u_0)-\sigma(u_1))/(u_0-u_1) \geqslant 0$, define

$$\begin{aligned} s_+(u_0, u_1) &= \{(\sigma(u_0)-\sigma(u_1))/(u_0-u_1)\}^{1/2}; \\ s_-(u_0, u_1) &= -s_+(u_0, u_1) \end{aligned} \tag{2.7}$$

(taking the positive square root).

Let $U_0 \in \bar{D}_1 \cup \bar{D}_2$. If $u_0 \notin [\gamma, \delta]$, define $u_3 = u_3(u_0)$ by $\sigma'(u_3) = \{s_+(u_0, u_3)\}^2$, and if $u_0 \in [\gamma, \delta]$, define u_3 by $\sigma(u_3) = \sigma(u_0)$ and $\sigma'(u_3) \leqslant 0$. Note that for $u_0 \notin [\gamma, \delta]$, the line joining the points $(u_0, \sigma(u_0))$ and $(u_3, \sigma(u_3))$ is tangent at $(u_3, \sigma(u_3))$ to the graph of σ, whereas for $u_0 \in [\gamma, \delta]$, this line is horizontal. Similarly, define $u_4 = u_4(u_0)$ by $\sigma'(u_0) = \{s_+(u_0, u_4)\}^2$, so that the line joining $(u_0, \sigma(u_0))$ to $(u_4, \sigma(u_4))$ is tangent at $(u_0, \sigma(u_0))$ to the graph of σ.

Next, define points $U_3 = (u_3, v_3)$ and $U_4 = (u_4, v_4)$ by $u_3 = u_3(u_0)$, $v_3 = v_0 - s_+(u_0, u_3)(u_3 - u_0)$ and $u_4 = u_4(u_0)$, $v_4 = v_0 - s_-(u_0, u_4)(u_4 - u_0)$. U_3 and U_4 are end points of the curves $S_2(U_0)$ and $S_1^*(U_0)$, respectively, as emphasized in the following representation of the shock curves.

If $U_0 \in \bar{D}_1$,

$$S_1(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_-(u_0, u_1)(u_1 - u_0),\ u_1 \geqslant u_0\},$$

$$S_1^*(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_-(u_0, u_1)(u_1 - u_0),\ u_1 \leqslant u_4(u_0)\},$$

$$S_2(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_+(u_0, u_1)(u_1 - u_0),\ u_3(u_0) \leqslant u_1 \leqslant u_0\}.$$

If $U_0 \in \bar{D}_2$,

$$S_1(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_-(u_0, u_1)(u_1 - u_0),\ u_1 \leqslant u_0\},$$

$$S_1^*(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_-(u_0, u_1)(u_1 - u_0),\ u_1 \geqslant u_4(u_0)\},$$

$$S_2(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_+(u_0, u_1)(u_1 - u_0),\ u_0 \leqslant u_1 \leqslant u_3(u_0)\}.$$

If $U_0 \in D_0$, the shock curves may be described as follows. Define u_3, u_3^* by $\sigma(u_0) = \sigma(u_3) = \sigma(u_3^*)$, $\beta < u_3 < \delta$, $\gamma < u_3{}^* < \alpha$, and set $U_3 = (u_3, v_0)$, $U_3^* = (u_3^*, v_0)$. Then U_3, U_3^* are end points of the curves $S_1(U_0)$, $S_1^*(U_0)$, respectively:

$$S_1(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_-(u_0, u_1)(u_1 - u_0),\ u_1 \geqslant u_3\},$$

$$S_1^*(U_0) = \{U_1 = (u_1, v_1) \colon v_1 = v_0 - s_-(u_0, u_1)(u_1 - u_0),\ u_1 \leqslant u_3^*\}.$$

Fixing U_0, a simple calculation shows that along each shock curve,

$$dv_1/du_1 = -(s^2 + \sigma'(u_1))/2s, \tag{2.8}$$

where $s = s_\pm(u_0, u_1)$, depending on which shock curve is being considered. Since $\sigma'(u_1) \geqslant 0$ on $S_1(U_0)$ and $S_1^*(U_0)$, these curves are monotonic, whereas $S_2(U_0)$ may have zero, one or two turning points in the elliptic region D_0. In the regions D_1 and D_2, however, v_1 is a decreasing function of u_1 along $S_2(U_0)$.

If $U_0 = (u_0, v_0) \in D_1 \cup D_2$ and $u_0 \in (\gamma, \delta)$, then U_0 may be joined by a stationary shock to the point $U_0^* = (u_0^*, v_0)$, where $u_0^* \neq u_0$ is defined by $\sigma'(u_0^*) > 0$ and $\sigma(u_0^*) = \sigma(u_0)$.

From (2.8), we note that, for $U_0 \in D_0$, $S_1(U_0)$, $S_1^*(U_0)$ are vertical (in the (u, v)-plane) at U_3, U_3^*, respectively.

We next describe rarefaction curves for Eq. (2.1). These are sections of the integral curves of the eigenvectors $r_-(u)$ and $r_+(u)$. For a state $U_0 = (u_0, v_0)$ to be joined to a state $U_1 = (u_1, v_1)$ by a (centered) rarefaction wave, with U_1 the state on the right of the wave, it is necessary and sufficient that U_1 lie on the integral curve of $r_\pm(u)$ through U_0 and that $\lambda_\pm(u)$ (respectively) be increasing along this curve from U_0 to U_1 [6]. For a fixed point U_0, the set of points U_1, to which U_0 may be so joined by a rarefaction wave, forms two curves $R_1(U_0)$ (corresponding to $r_-(u)$) and $R_2(U_0)$ (corresponding to $r_+(u)$). The corresponding rarefaction waves are called 1-rarefactions and 2-rarefactions, respectively. $R_1(U_0)$, $R_2(U_0)$ are given explicitly by the following.

If $U_0 \in \bar{D}_1$,

$$R_1(U_0) = \left\{(u_1, v_1): v_1 = v_0 - \int_{u_1}^{u_0} c(w)\, dw, \beta \leqslant u_1 \leqslant u_0\right\},$$

$$R_2(U_0) = \left\{(u_1, v_1): v_1 = v_0 - \int_{u_0}^{u_1} c(w)\, dw, u_1 \geqslant u_0\right\}.$$

If $U_0 \in \bar{D}_2$,

$$R_1(U_0) = \left\{(u_1, v_1): v_1 = v_0 + \int_{u_0}^{u_1} c(w)\, dw, u_0 \leqslant u_1 \leqslant \alpha\right\},$$

$$R_2(U_0) = \left\{(u_1, v_1): v_1 = v_0 + \int_{u_1}^{u_0} c(w)\, dw, u_1 \leqslant u_0\right\}.$$

If $U_0 \in D_0$, it can be joined to no other state by a rarefaction wave. For $U_0 \in \bar{D}_1 \cup \bar{D}_2$, $R_1(U_0)$, $R_2(U_0)$ are smooth monotonic convex curves lying entirely in the same region $\bar{D}_1$ or $\bar{D}_2$ as U_0.

For a fixed $U_0 = (u_0, v_0) \in \bar{D}_1 \cup \bar{D}_2$, the *wave locus* $W(U_0)$ of U_0 is the set of points U_1 to which U_0 may be joined by an admissible shock or rarefaction wave. We have shown that if $U_0 \in \bar{D}_1 \cup \bar{D}_2$, $W(U_0)$ consists of the curves $S_i(U_0)$, $R_i(U_0)$ $(i = 1, 2)$, $S_1^*(U_0)$ and the point U_0^*, if $\gamma < u_0 < \delta$. A typical wave locus is shown in Fig. 2, for the case $u_0 > \delta$.

If $U_0 \in D_0$, the wave locus $W(U_0)$ consists only of the curves $S_1(U_0)$, $S_1^*(U_0)$.

The following result was proved in [5] for the non-strictly hyperbolic case. The corresponding proof here is only sightly complicated by the non-monotonicity of $S_2(U_0)$.

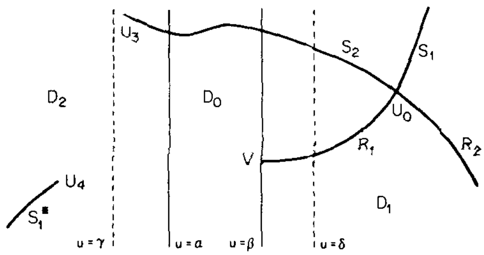

FIG. 2. The wave locus $W(U_0)$.

PROPOSITION 2.1. *For a fixed U_0, consider the parameterized family of curves $S_2(U_1)$, or $R_2(U_1)$, where U_1 is constrained to lie on $S_1(U_0)$, $S_1^*(U_0)$, or $R_1(U_0)$. Each of these families smoothly fills a sector of $\mathbb{R}^2$, without gaps or self-intersections.*

Proof. Each curve in each family is the solution of a smooth ordinary differential equation, with initial values U_1 varying smoothly. Consequently, there are no gaps. The ordinary differential equation determining $R_2(U_1)$ is in fact autonomous, so the curves $R_2(U_1)$ have no self-intersections. To show that the families of curves $S_2(U_1)$ are not self-intersecting, it is necessary to consider each case separately. The calculation, however, is similar for each case, so we give the details only for the family given by $U_1 \in S_1(U_0)$.

Suppose $U_1 = (u_1, v_1)$ and $\bar{U}_1 = (\bar{u}_1, v_1)$ lie on $S_1(U_0)$ and are such that $S_2(U_1)$ and $S_2(\bar{U}_1)$ intersect at some point $U = (u, v)$. Without loss of generality, we shall assume $U_0 \in \bar{D}_1$ and $\bar{v}_1 > v_1$. Then, by the monotonicity of $S_1(U_0)$, we have $\bar{u}_1 < u_1$. Now U_1 and $\bar{U}_1$ must both be joined to U by a shock wave with positive shock speed, with U on the left (these shock waves are not admissible, however). The set of points to which U may be joined by a shock wave with positive shock speed is a curve, monotonically decreasing in D_1 (formual (2.8) applies to this curve). Since $\bar{U}_1$ lies in the first quadrant of the U-plane, with respect to U_1, it is not possible for this curve to pass through both U_1 and $\bar{U}_1$. This contradicts the initial assumption that $S_2(U_1)$ and $S_2(\bar{U}_1)$ intersect at U.

3. SOLUTION OF THE RIEMANN PROBLEM

In this section we construct solutions of the Riemann problem consisting of Eq. (2.1), together with initial step data of the form

$$\begin{aligned}(u(x,0), v(x,0)) &= (u_l, v_l), \qquad x < 0,\\ &= (u_r, v_r), \qquad x > 0.\end{aligned} \tag{3.1}$$

The principal element in the construction is a division of the (u, v)-plane into several regions, the division depending upon the location of (u_l, v_l). The solution of the Riemann problem consists of a combination of admissible shock waves and rarefaction waves, the combination determined by the region in which (u_r, v_r) lies.

If $U = (u, v) \in R^2$ and $\gamma \leqslant u \leqslant \alpha$ or $\beta \leqslant u \leqslant \delta$, we shall henceforth denote by U^* the point (u^*, v), with $u^* \neq u$ given by $\sigma(u^*) = \sigma(u)$ and $\sigma'(u^*) \geqslant 0$.

Suppose U_0 may be joined to U_1 by a shock or rarefaction wave with U_1 on the right. We say U_1 may be further joined to a state U_2 on the right by a *faster wave* if U_1 may be joined to U_2 by a shock or rarefaction wave, and the speed of the shock, or minimum speed of the rarefaction wave, joining U_1 to U_2 is greater than the speed of the shock, or maximum speed of the rarefaction, joining U_0 to U_1. Indeed, for U_0 to be joined to U_2 by a combination of two shock or rarefaction waves yielding a weak solution of (2.1), it is necessary that U_0 be joined to an intermediate state U_1 on the right and U_1 to U_2 on the right, with the latter wave being faster in the sense specified above.

For a fixed $U_0 = (u_0, v_0)$, the set of points U_1 in the wave locus $W(U_0)$ that may be joined to some other state U_2 on the right by a faster wave constitutes the *continuable set* $C(U_0)$. It is readily seen that if $U_0 \in \bar{D}_1 \cup \bar{D}_2$, then $C(U_0)$ consists of $S_1(U_0)$, $R_1(U_0)$, $S_1^*(U_0)$, the point $U_3(U_0)$ and the point $U_0{}^*$ if $\gamma < u_0 < \delta$. If $U_0 \in D_0$, however, then $C(U_0)$ consists of the entire wave locus $W(U_0) = S_1(U_0) \cup S_1^*(U_0)$.

The next stage is to consider the set of points to which each $U_1 \in C(U_0)$ can be joined, with U_1 on the left, by a wave faster than that joining U_0 to U_1 with U_1 on the right. This will include $S_2(U_1)$, $R_2(U_1)$ and U_1^*, where appropriate, but will also include $U_3(U_1)$ and $U_4(U_1)$. It is therefore necessary to describe the loci of the points $U_3(U_1)$ and $U_4(U_1)$, as U_1 varies along $C(U_0)$.

Let $U_0 \in \bar{D}_1$. As U_1 moves along $R_1(U_0) \cup S_1(U_0)$ from V to infinity, $U_3(U_1)$ describes a composite curve $J \cup \tilde{J}$, $\tilde{J}$ being the part of the curve where $v_3 = v_1$, corresponding to stationary shock waves. Similarly, $U_4(U_1)$ describes a curve E as U_1 moves along $R_1(U_0)$ from U_0 to V. Note that $U_1 \in S_1(U_0)$ is *not* joined to $U_4(U_1)$ by a faster wave. We shall need the properties of J, $\tilde{J}$ and E described in the following proposition, whose proof we leave to Section 4. Let $U_5 = (u_5, v_5) \in S_1(U_0) \cup R_1(U_0)$ be the point for which $u_5 = \delta$.

PROPOSITION 3.1. *Let $U_0 \in \bar{D}_1$. The curve $J = \{U_3(U_1)$: $U_1 \in S_1(U_0) \cup R_1(U_0)$ between U_5 and infinity$\}$ is a smooth monotonic curve joining U_5^* to infinity, and is vertical at U_5^*. For each $U_3 = U_3(U_1)$ in J, $R_2(U_3)$ lies to the left of J, whereas $S_2(U_1)$ lies to the right of J.*

The curve $\tilde{J} = \{U_3(U_1)$: $U_1 \in S_1(U_0) \cup R_1(U_0)$ between U_5 and $V\}$ is a *smooth monotonic curve joining U_5^* to V, is horizontal at U_5^* and vertical at V.*

The curve $E = \{U_4(U_1): U_1 \in R_1(U_0)\}$ is a smooth monotonic curve joining $U_4(U_0)$ to V^*, and is vertical at V^*.

Each point $U_1 \in S_1^*(U_0) \cup E$ may be further joined to points on $S_2(U_1)$ by faster shock waves. However, $S_2(U_1)$ extends only from $U_1$to $U_3(U_1)$. The curve $J_1 = \{U_3(U_1): U_1 \in S_1^*(U_0) \cup E\}$ has the properties described in the next proposition.

PROPOSITION 3.2. *Let $U_0 \in \bar{D}_1$. J_1 is a smooth monotonic curve joining V to infinity, and is vertical at V. For each $U_3 = U_3(U_1) \in J_1$, $R_2(U_3)$ lies to the right of J_1, whereas $S_2(U_1)$ lies to the left of J_1.*

We also require a description of the locus K of points U_1^*, as U_1 moves along $R_1(U_0) \cup S_1(U_0)$ from V to U_5:

PROPOSITION 3.3. *Let $U_0 \in \bar{D}_1$. $K = \{U_1{}^*: U_1 \in R_1(U_0) \cup S_1(U_0)$ between V and $U_5\}$ is a smooth monotonic curve joining V^* to U_5^*. K is vertical at V^* and horizontal at U_5^*.*

If $U_0 \in D_0$, we require a different construction. Let $V^* = (u^*, v^*) \in D_2$ be the point on $S_1^*(U_0)$ for which $u^* = \gamma$, and set $V = (\beta, v^*)$. As U_1 moves along $S_1(U_0)$ from U_3 to U_5 and from U_5 to infinity, the locus of points $U_3(U_1)$ forms a composite curve $\tilde{J} \cup J$, $\tilde{J}$ being the section lying in D_0. Similarly, as U_1 moves along $S_1^*(U_0)$ from U_3^* to V^* and from V^* to infinity, the locus of points $U_3(U_1)$ forms a curve $\tilde{J}_1 \cup J_1$, $\tilde{J}_1$ being the section lying in D_0.

PROPOSITION 3.4. *Let $U_0 \in D_0$. J, $\tilde{J}$, J_1, $\tilde{J}_1$ are smooth monotonic curves. J joins U_5^* to infinity and is vertical at U_5^*. $\tilde{J}$ joins U_5^* to U_0, is horizontal at U_5^* and vertical at U_0. $\tilde{J}_1$ joins U_0 to V, is horizontal at V and vertical at U_0. J_1 joins V to infinity and is vertical at V.*

For each $U_3 = U_3(U_1) \in J$ (respectively, J_1), $R_2(U_3)$ lies entirely to the left of J (the right of J_1), while $S_2(U_1)$ lies entirely to the right of J (the left of J_1).

As U_1 moves along $S_1^*(U_0)$ between V^* and U_3^*, the points $U_1^* \in D_1$ form a curve K_1. Similarly, we denote by K the locus of points U_1^*, for $U_1 \in S_1(U_0)$ between U_3 and U_5.

PROPOSITION 3.5. *Let $U_0 \in D_0$. K is a smooth monotonic curve joining U_3^* to U_5^*, is vertical at U_3^* and horizontal at U_5^*. K_1 is a smooth monotonic curve joining U_3 to V, is vertical at U_3 and horizontal at V.*

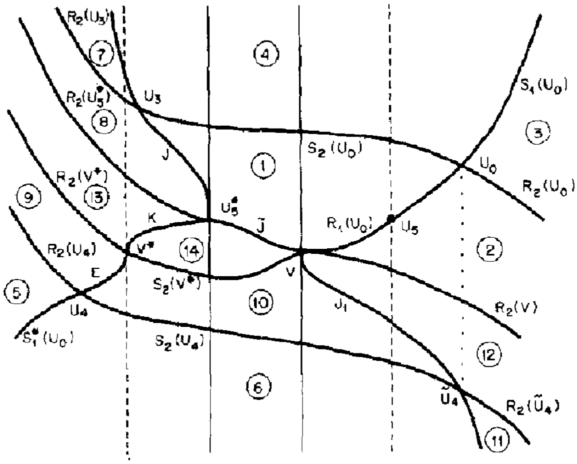

FIG. 3. Division of the U-plane for $u_0 > \delta$.

Note that $\tilde{J}$, $\tilde{J}_1$ are also the loci of points $U_3(U_1)$ as U_1 moves along K, K_1, respectively.

Propositions 3.1–3.5 are proved in Section 4.

We now divide the (u, v)-plane into several regions, depending upon the location of $U_0 = (u_0, v_0)$.

If $u_0 > \delta$, the wave locus $W(U_0)$, together with the curves J, $\tilde{J}$, E, K, $R_2(V)$, $S_2(V^*)$, $R_2(U_5^*)$, $R_2(U_4)$, $S_2(U_4)$ (which terminates at $\tilde{U}_4 = U_3(U_4) \in J_1$) and $R_2(\tilde{U}_4)$, divide the plane into fourteen regions, as shown in Fig. 3.

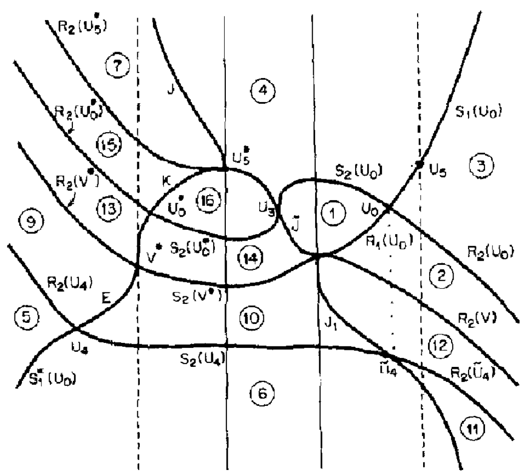

FIG. 4. Division of the U-plane for $\beta < u_0 < \delta$.

If $\beta < u_0 < \delta$, then $U_0^* \in K$ is defined and $U_3 \in \tilde{J}$. Moreover, $S_2(U_0^*)$ joins U_0^* to U_3, but $R_2(U_3)$ is no longer present. Consequently, there is no region corresponding to region 8 of Fig. 3, but we include two new regions 15 and 16, created by $S_2(U_0^*)$ and $R_2(U_0^*)$. The division of the plane is shown in Fig. 4.

If $\alpha < u_0 < \beta$, the wave locus $S_1(U_0) \cup S_1^*(U_0)$, together with the curves J, $\tilde{J}$, J_1, $\tilde{J}_1$, K, K_1, $R_2(U_3)$, $R_2(U_3^*)$, $S_2(U_3)$, $S_2(U_3^*)$, $R_2(V)$ and $R_2(U_5^*)$, divide the plane into ten regions, as shown in Fig. 5. The numbering of the regions is consistent with Fig. 3 and 4, and we note the inclusion of two additional regions, 17, 18, while several other regions of Figs. 3 and 4 do not have analogues in Fig. 5.

We can now construct a solution of the Riemann problem depending upon the location of (u_r, v_r) in relation to (u_l, v_l). Set $U_0 = (u_l, v_l)$. The construction is slightly different, depending upon the location of U_l.

I.

If $\delta < u_l$, the solution is constructed with the aid of Fig. 3. If $U_r = (u_r, v_r)$ lies in any of the regions 1 to 12, the solution is constructed in a manner similar to that described in [5], which we reproduce below for completeness. If U_r lies in regions 13 or 14, the solution includes stationary shocks, which do not appear in [5].

Regions 1–6. There is a single intermediate state U_1 and $U_r = U_2$. In 1 and 2, $U_1 \in R_1(U_0)$ and $U_2 \in S_2(U_1)$ and $R_2(U_1)$, respectively. In 3 and 4, $U_1 \in S_1(U_0)$ and $U_2 \in R_2(U_1)$ and $S_2(U_1)$, respectively. In 5 and 6, $U_1 \in S_1^*(U_0)$ and $U_2 \in R_2(U_1)$ and $S_2(U_1)$, respectively.

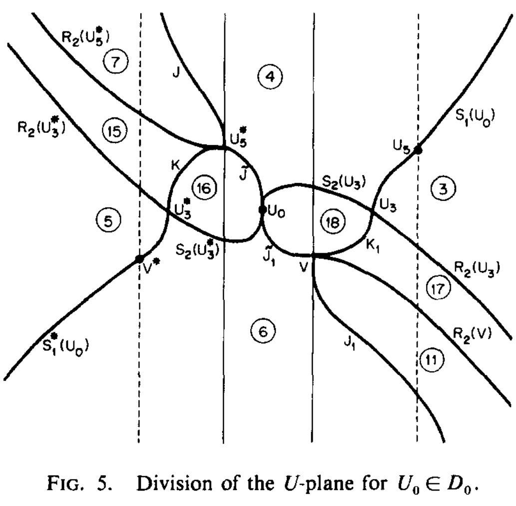

FIG. 5. Division of the U-plane for $U_0 \in D_0$.

Regions 7 *and* 8. There are two intermediate states $U_1 \in S_1(U_0)$ and $R_1(U_0)$, respectively, and $U_2 = U_3(U_1) \in J$. U_2 may be joined to U_r on the right by a 2-rarefaction wave whose slowest speed is $\lambda_+(u_2)$. Thus $U_r \in R_2(U_2)$.

Regions 9 *and* 10. There are two intermediate states $U_1 \in R_1(U_0)$ and $U_2 = U_4(U_1) \in E$. In 9, $U_r \in R_2(U_2)$, whereas in 10, $U_r \in S_2(U_2)$. The speed of the shock joining U_1 to U_2 on the right is $\lambda_-(U_1)$, which is the fastest speed of the rarefaction wave joining U_0 to U_1.

Region 11. There are two intermediate states $U_1 \in S_1^*(U_0)$ and $U_2 = U_3(U_1) \in J_1$. $U_r \in R_2(U_2)$. The speed of the shock joining U_1 to U_2 is also the slowest speed of the rarefaction joining U_2 to U_r.

Region 12. There are three intermediate states $U_1 \in R_1(U_0)$, $U_2 = U_4(U_1) \in E$, $U_3 = U_3(U_2) \in J_1$. $U_r \in R_2(U_3)$. The speed of the shock joining U_1 and U_2 is the fastest speed of the rarefaction joining U_0 to U_1. Similarly, the speed of the shock joining U_2 to U_3 is the slowest speed of the rarefaction joining U_3 to U_r.

Regions 13 *and* 14. There are two intermediate states $U_1 \in R_1(U_0)$ and $U_2 = U_1^* \in K$, while $U_r \in R_2(U_2)$ and $S_2(U_2)$, respectively. The shock joining U_1 to U_2 is stationary.

II.

If $\beta < u_0 < \delta$, the construction proceeds according to Fig. 4.

Regions 1–7 *and* 9–14. The construction of the solution is exactly as described in Case I.

Regions 15 *and* 16. There are two intermediate states $U_1 \in S_1(U_0)$ and $U_2 = U_1^* \in K$, while $U_r \in R_2(U_2)$ and $S_2(U_2)$, respectively.

Next, we consider the limiting cases $u_0 = \delta$ and $u_0 = \beta$. If $u_0 = \delta$, we have $U_3 = U_3^* = U_5^*$. The corresponding diagram is obtained from Fig. 3 by shrinking region 8 (as $u_0 \to \delta+$), or from Fig. 4 by shrinking regions 15 and 16 (as $u_0 \to \delta-$). The solution is constructed as for Case I, for U_r in regions 1 to 7 or 9 to 14.

If $u_0 = \beta$, we have $U_0 = V = U_3 = \tilde{U}_4$ and $U_0^* = V^* = U_4$. The curves $R_1(U_0)$ and E, and the regions 1, 2, 9, 10, 12–14 do not appear. The solution is constructed as in Case II for any U_r in the remaining regions (which of course fill the plane).

If $U_l \in \bar{D}_2$, the solution is constructed in a similar manner by first drawing the diagrams corresponding to Figs. 3 or 4 (depending on whether $u_0 \leqslant \gamma$ or $\gamma \leqslant u_0 \leqslant \alpha$).

III.

If $\alpha < u_l < \beta$, the solution is constructed with the aid of Fig. 5. Set $U_0 = U_l$. If U_r lies in any of the regions other than 17, 18, the solution is constructed precisely as described in Case II.

Regions 17 *and* 18. There are two intermediate states $U_1 \in S_1^*(U_0)$ between U_3^* and V^*, and $U_2 = U_1^* \in K_1$. $U_r \in R_2(U_2)$ and $S_2(U_2)$, respectively. U_1 is joined to U_2 by a stationary shock wave.

The solution we have constructed for each U_l, U_r is unique within the class of admissible solutions we have chosen. That this is so may be demonstrated by a standard argument. Indeed, starting at $U_l = U_0$ there is only one way of joining U_0 to a given U_r by progressively faster admissible waves. This is verified routinely by considering U_r in each region separately, but we omit the details.

Remark. In Figs. 3 and 4 we have used a notation largely consistent with that of Keyfitz and Kranzer [5]. In fact, we can recover the solution of the Riemann problem in the non-strictly hyperbolic case, given in [5], by letting $\alpha \to \beta$ in Fig. 1 as follows. Let $\sigma_\varepsilon: \mathbb{R} \to \mathbb{R}$, $0 \leqslant \varepsilon \leqslant 1$, be a parameterized family of functions satisfying, for each $\varepsilon > 0$, the conditions of Section 2, with corresponding $\alpha = \alpha_\varepsilon$, $\beta = \beta_\varepsilon$. We also require:

1. The mappings $(\varepsilon, u) \to \sigma_\varepsilon(u)$ and $(\varepsilon, u) \to \sigma_\varepsilon'(u)$ are continuous.
2. $\sigma_1 = \sigma$.
3. $\lim_{\varepsilon \to 0} \alpha_\varepsilon = \alpha_0 = \lim_{\varepsilon \to 0} \beta_\varepsilon$.
4. σ_0 is three times continuously differentiable, and $\sigma_0'(u) > 0$, $\sigma''(u) = \operatorname{sgn}(u - \alpha_0)$ for all $u \neq \alpha_0$.
5. $\sigma_0'(\alpha_0) = \sigma_0''(\alpha_0) = 0$, $\sigma_0'''(\alpha_0) > 0$.

Then as $\varepsilon \to 0$, Fig. 3 deforms continuously into a diagram corresponding to Fig. 7 in [5]. Figure 4 also deforms continuously, but since $\gamma = \gamma_\varepsilon$ and $\delta = \delta_\varepsilon$ both tend to α_0 as $\varepsilon \to 0$, in the limit U_0 lies on the line $u = \alpha_0$ of parabolic degeneracy for Eq. (2.1), with σ replaced by σ_0. As ε approaches zero, the points U_5, U_5^* and V^* in Fig. 3 approach V, and the curves K, $\tilde{J}$, $S_2(V^*)$, together with regions 13, 14, disappear in the limit.

4. Details of the Curves Used in Section 3

The proofs of Propositions 3.1–3.5 are by direct calculation of derivatives, and for each of J, $\tilde{J}$, J_1, $\tilde{J}_1$, E, K and K_1 are very similar. We give the details for J, and omit most of the details for the other curves. For definiteness, we fix $U_0 = (u_0, v_0) \in \bar{D}_1 \cup D_0$.

If $u_0 > \delta$, J consists of the points $U_3(U_1) = (u_3, v_3)$, as U_1 moves from U_5 along $R_1(U_0)$ to U_0, and then from U_0 along $S_1(U_0)$ to infinity. If $\alpha \leqslant u_0 \leqslant \delta$, however, J is defined by points $U_1 \in S_1(U_0)$ between U_5 and infinity. We need to show that $dv_3/du_3 \to -\infty$ as u_3 approches u_5^* from below, and in the first case, that dv_3/du_3 is continuous at $u_3 = u_3(u_0)$.

Now (u_3, v_3) is defined by

$$v_3 = v_1 - c(u_3)(u_3 - u_1) \tag{4.1}$$

and

$$\sigma'(u_3) = (\sigma(u_3) - \sigma(u_1))/(u_3 - u_1), \tag{4.2}$$

where $c(u) = (\sigma'(u))^{1/2} \geqslant 0$ if $u \notin (\alpha, \beta)$. From (4.1), and the chain rule,

$$\frac{dv_3}{du_3} = \frac{dv_1}{du_1} u_1'(u_3) - c'(u_3)(u_3 - u_1) - c(u_3)(1 - u_1'(u_3)), \tag{4.3}$$

From (4.2),

$$u_1'(u_3) = (u_3 - u_1)\,\sigma''(u_3)/(\sigma'(u_3) - \sigma'(u_1)). \tag{4.4}$$

If $(u_1, v_1) \in R_1(U_0)$, we have

$$dv_1/du_1 = c(u_1), \tag{4.5}$$

whereas for $(u_1, v_1) \in S_1(U_0)$, dv_1/du_1 is given by (2.8), with $s = s_-(u_0, u_1) < 0$:

$$dv_1/du_1 = -(s^2 + \sigma'(u_1))/2s. \tag{4.6}$$

Thus, $dv_1/du_1 \geqslant 0$, with equality only at $u_1 = \beta$.

Now, $u_3 < u_1$, $\sigma''(u_1) > 0$ and $0 \leqslant \sigma'(u_3) < \sigma'(u_1)$, so that $u_1'(u_3) < 0$. Combining (4.3)–(4.6), we have $dv_3/du_3 < 0$ as desired. Moreover, since dv_1/du_1 is continuous at U_0 (when $U_0 \in D_1$) [6], dv_3/du_3 is aso continuous, except at $u_3 = \alpha$, corresponding to the right-hand end U_5^* of J. In fact, as $u_3 \to \alpha-$, we have $dv_3/du_3 \to -\infty$, so that J is vertical at U_5^*.

Specifically, we obtain, after some calculation,

$$\begin{aligned} dv_3/du_3 = -c(u_3) + (u_3 - u_1)\,\sigma''(u_3) \\ \times (c(u_3) + c(u_1))/2c(u_3)(c(u_3) - c(u_1)). \end{aligned} \tag{4.7}$$

In particular, $dv_3/du_3 < -c(u_3)$, so that $R_2(U_3)$, which has slope $-c(u_3)$ at U_3, emanates from U_3 to the left of J. If $R_2(U_3)$ should intersect J at some first point $U = (u, v)$, say, then the slope of $R_2(U_3)$ at U would be at most dv_3/du_3, evaluated at u. But $R_2(U_3)$ has slope $-c(u)$ at U, and $dv_3/du_3 <$

$-c(u)$ at U. This contradiction shows that $R_2(U_3)$ must lie entirely to the left of J. The curve $S_2(U_1)$ also has slope $-c(u_3)$ at $U_3 = U_3(U_1)$, and a similar argument shows that $S_2(U_1)$ lies entirely to the right of J. Moreover, $R_2(U_3) \cup S_2(U_1)$ is smooth at $U_3 = U_3(U_1)$.

The calculation for $\tilde{J}$ is simpler than that for J. We have $v_3 = v_1$, $\sigma(u_3) = \sigma(u_1)$, so that

$$\frac{dv_3}{du_3} = \frac{dv_1}{du_1}\frac{du_1}{du_3} \qquad \text{and} \qquad \frac{du_1}{du_3} = \frac{\sigma'(u_3)}{\sigma'(u_1)}.$$

As for J, $dv_1/du_1 \geqslant 0$ along $\tilde{J}$, with equality only when $u_1 = \delta$. Since $\alpha \leqslant u_3 \leqslant \beta$, we also have $du_1/du_3 \leqslant 0$, with equality when $u_3 = \alpha$ or $u_3 = \beta$. Hence $dv_3/du_3 < 0$ for $\alpha < u_3 < \beta$ and is zero at U_5^*, where $u_3 = \alpha$. Let $U_0 \in \bar{D}_1$. As U_1 approaches V, either $U_1 \in R_1(U_0)$ or $u_0 = \beta$. In either case, dv_1/du_1 approaches zero, while a elementary argument shows that $\sigma'(u_3)/\sigma'(u_1)$ tends to -1, as U_1 approaches V. Hence $\tilde{J}$ is horizontal as V. Similarly, if $U_0 \in D_0$, as U_1 approaches U_3 along $S_1(U_0)$, dv_1/du_1 tends to $-\infty$, while $\sigma'(u_3)$ and $\sigma'(u_1)$ remain finite and bounded away from zero. Hence $\tilde{J}$ is vertical at U_0. The same calculation shows that $\tilde{J}_1$ is also vertical at U_0, and horizontal at V.

Let $U_4 = (u_4, v_4) \in E$. Then $U_4 = U_4(U_1)$ for some $U_1 \in R_1(U_0)$. We have

$$v_4 = v_1 + c(u_1)(u_4 - u_1) \tag{4.8}$$

and

$$\sigma'(u_1) = (\sigma(u_4) - \sigma(u_1))(u_4 - u_1). \tag{4.9}$$

Thus,

$$\frac{dv_4}{du_4} = \frac{dv_1}{du_1} u_1'(u_4) + c'(u_1)\, u_1'(u_4)(u_4 - u_1) + c(u_1)(1 - u_1'(u_4)), \tag{4.10}$$

where dv_1/du_1 is given by (4.5). As for J, it is elementary to show from (4.9), (4.10) that $dv_4/du_4 > 0$, and tends to infinity as u_4 approaches γ, so that E is vertical at V^*.

The calculation for J_1 is a little more complicated. If $U_3 = (u_3, v_3) \in J_1$, we have $U_3 = U_3(U_2)$ for some $U_2 \in S_1^*(U_0) \cup E$. If $U_2 \in S_1^*(U_0)$, the calculation proceeds as for J. If $U_2 = (u_2, v_2) \in E$, we have $U_2 = U_4(U_1)$ for some $U_1 = (u_1, v_1) \in R_1(U_0)$. Moreover, $u_3 = u_1$, and U_1, U_2, U_3 are related by

$$v_3 = v_2 - c(u_3)(u_3 - u_2), \qquad \sigma'(u_1) = \sigma'(u_3) = (\sigma(u_3) - \sigma(u_2))/(u_3 - u_2),$$

$$v_2 = v_1 + c(u_1)(u_2 - u_1) \qquad \text{and} \qquad dv_1/du_1 = c(u_1).$$

From these, it is straightforward to show that

$$dv_3/du_3 < -c(u_3) < 0 \tag{4.11}$$

along J_1, is continuous at $\tilde{U}_4$, and $dv_3/du_3 \to -\infty$ as U_3 tends to V. That $R_2(U_3)$ lies entirely to the right of J_1 and $S_2(U_1)$ lies entirely to the left for each $U_3 \in J_1$ follows from (4.11) by the same argument used for J.

The properties of K, K_1 in Propositions 3.3 and 3.5 are verified by a calculation very similar to that given above for $\tilde{J}$, J_1.

We next discuss the behavior of $S_2(V^*)$ at V when $U_0 \in D_1$. If $V^* = (u^*, v^*)$, we have from (4.6), if $U_1 = (u_1, v_1) \in S_2(V^*)$,

$$dv_1/du_1 = -(s^2 + \sigma'(u_1))/2s, \qquad s > 0,$$

where $s^2 = (\sigma(u_1) - \sigma(\beta))/(u_1 - u^*)$ (note $\sigma(u^*) = \sigma(\beta)$). By an elementary argument,

$$dv_1/du_1 \to (\sigma''(\beta)(\beta - u^*)/2)^{1/2} \qquad \text{as} \quad u_1 \to \beta.$$

Hence $S_2(V^*)$ has positive slope at V, as indicated in Fig. 3 and 4.

The other qualitative features of Figs. 3–5 are easily verified. For instance, if $U_0 \in D_0$, then (4.8) implies that $S_1(U_3)$ and $S_1^*(U_3^*)$ are vertical at U_0. In fact, if U_1 lies on K, K_1, $S_1(U_0)$, $S_1^*(U_0)$ or $R_1(U_0)$, and is such that $U_3(U_1) \in \tilde{J} \cup \tilde{J}_1$, then $S_2(U_1)$ is vertical at $U_3(U_1)$.

5. Concluding Remarks

We conclude this paper with a discussion of the admissibility conditions (2.3)–(2.5).

Consider the system

$$\begin{aligned} u_t - v_x &= 0, \\ v_t - \sigma(u)_x &= \varepsilon u_{xt}. \end{aligned} \tag{5.1}$$

This system has been studied in connection with viscoelastic bar theory [2, 4], the term εu_{xt} being dissipative. Indeed, (5.1) corresponds to the single equation

$$w_{tt} - \sigma(w_x)_x = \varepsilon w_{xxt}. \tag{5.2}$$

Let $U_1 = (u_1, v_1)$, $U_2 = (u_2, v_2)$ and $s \neq 0$ satisfy the Rankine–Hugoniot conditions (2.2). We seek a travelling wave solution of (5.1), of the form $u = u((x-st)/\varepsilon)$, $v = v((x-st)/\varepsilon)$ and satisfying, where $U = (u, v)$,

$$\lim_{\xi \to -\infty} U(\xi) = U_1, \qquad \lim_{\xi \to +\infty} U(\xi) = U_2. \tag{5.3}$$

Substituting $(u, v)((x-st)/\varepsilon)$ into (5.1) and integrating once, we obtain

$$-s(u-u_1) - (v-v_1) = 0, \tag{5.4}$$

$$-s(v-v_1) - (\sigma(u) - \sigma(u_1)) = -su', \tag{5.5}$$

where $(u, v) = (u, v)(\xi)$, and a prime denotes $d/d\xi$. We have also supposed u' and v' tend to zero as $|\xi| \to \infty$. Substituting (5.4) into (5.5), we obtain

$$\sigma(u) - \sigma(u_1) - s^2(u-u_1) = su' \tag{5.6}$$

with the conditions (5.3),

$$\lim_{\xi \to -\infty} u(\xi) = u_1, \qquad \lim_{\xi \to +\infty} u(\xi) = u_2. \tag{5.7}$$

There is a solution of (5.6) satisfying (5.7) if and only if the admissibility conditions (2.3) or (2.4) hold. As $\varepsilon \to 0$, the travelling wave solution of (5.1) converges pointwise (except at $x = st$) to the shock wave travelling at speed s, joining U_1 on the left to U_2 on the right. This observation is also made by Slemrod [10].

We remark in passing that the Conley–Smoller criterion [1] used by Keyfitz and Kranzer [5] for the non-strictly hyperbolic case ($\sigma'(u) > 0$ except at one point) does not appear to apply to the present problem. The Conley–Smoller criterion consists of studying the limit as $\varepsilon \to 0$ of travelling wave solutions of the system

$$\begin{aligned} u_t - v_x &= \varepsilon u_{xx}, \\ v_t - \sigma(u)_x &= \varepsilon v_{xx}. \end{aligned} \tag{5.8}$$

The non-monotonicity of σ makes the study of (5.8) somewhat harder than in [5], but a preliminary calculation indicates that there are admissible shocks not satisfying the Conley–Smoller criterion, and also shocks that satisfy the Conley–Smoller criterion but fail to satisfy the admissibility criteria (2.3)–(2.5). The latter result has been established independently by Slemrod [10].

For certain values of U_r, our solution of the Riemann problem involves stationary shocks that do not satisfy (2.3) or (2.4). Specifically, this is the case if U_r lies in either of the regions 13 or 14 of Fig. 3, or in any of the regions 13–16 of Fig. 4, or in any of the regions 15–18 of Fig. 5. For such a state U_r, there is no solution of the Riemann problem satisfying only (2.3) or (2.4) across shocks, as the curve K may no longer be used in the construction of such a solution. For this reason alone, we have admitted all stationary shocks. Indeed, no stationary shock can be the limit as $\varepsilon \to 0$ of travelling wave solutions of (5.1) satisfying (5.3), and such a shock satisfies the Conley–Smoller criterion only in the special case

$$\int_{u_1}^{u_2} (\sigma(u) - \sigma(u_1))\, du = 0.$$

It is possible that some other admissibility criterion will give rise to Figs. 3 and 4, but with K replaced by a different curve joining V^* to U_5^*. We have been unable to formulate such a condition.

James [4] suggests that uniqueness of the solution of the Riemann problem may not be appropriate for the theory of elastic bars with non-monotone stress strain relations (given by the graph of σ). However, it is not at all clear how to characterize solutions not necessarily satisfying (2.3)–(2.5), or whether some weaker admissibility criteria would be appropriate.

In summary, we have solved the Riemann problem (2.1), (3.1) uniquely in a class of admissible solutions that includes all stationary shock waves.

References

1. C. C. Conley and J. A. Smoller, Viscosity matrices for two-dimensional nonlinear hyperbolic systems, *Comm. Pure Appl. Math.* **23** (1970), 867–884.
2. C. M. Dafermos, The mixed initial-boundary value problem for the equations of nonlinear one-dimensional visco-elasticity, *J. Differential Equations* **6** (1969), 71–86.
3. C. M. Dafermos, The entropy rate admissibility criterion for solutions of hyperbolic conservation laws, *J. Differential Equations* **14** (1973), 202–212.
4. R. D. James, The propagation of phase boundaries in elastic bars, *Arch. Rat. Mech. Anal.* **73** (1980), 125–158.
5. B. L. Keyfitz and H. C. Kranzer, The Riemann problem for a class of hyperbolic conservation laws exhibiting a parabolic degeneracy, Preprint, 1981.
6. P. D. Lax, Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.
7. L. Liebovich, Solutions of the Riemann problem for hyperbolic systems of quasilinear equations without convexity conditions, *J. Math. Anal. Appl.* **45** (1974), 81–90.
8. T. P. Liu, The Riemann problem for general 2×2 conservation laws, *Trans. Amer. Math.*
9. M. S. Mock, Systems of conservation laws of mixed type, *J. Differential Equations* **37** (1980), 70–88.
10. M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Rat. Mech. Anal.*, to appear.
11. J. A. Smoller and J. L. Johnson, Global solutions for an extended class of hyperbolic systems of conservation laws, *Arch. Rat. Mech. Anal.* **32** (1969), 169–189.
12. B. Wendroff, The Riemann problem for materials with nonconvex equations of state. I. Isentropic flow. *J. Math. Anal. Appl.* **38** (1972), 454–466.