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# NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS WITH AN OPERATIONAL APPROACH TO THE TAU METHOD

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**Abstract**—We discuss the numerical solution of linear partial differential equations with variable coefficients by means of an operational approach to Ortiz' recursive formulation of the Tau method.

We discuss a procedure which makes it possible to determine the coefficients of a bivariate Tau approximant by means of a reduced set of matrix operations. It involves no discretization of the variables, approximate quadratures or the use of special trial functions.

Error surfaces exhibit a remarkable equioscillatory behaviour.

## 1. INTRODUCTION

This paper concerns the application of an operational approach to Ortiz' recursive formulation of the Tau method[6] discussed by the authors in a recent paper on the numerical solution of non-linear ordinary differential equations[9].

The approximate solution obtained with this technique is a polynomial which, as in the Tau method, satisfies the given partial differential equation, but for a small perturbation term in the right hand side; the supplementary (initial, boundary or mixed) conditions are satisfied exactly, provided they are of polynomial form.

The coefficients of the Tau approximant in two variables are determined through the use of a systematic and computationally simple technique based on the use of linear combinations of products of two matrices with only one line different from zero. They are used to set up a collection of linear conditions on the coefficients of the approximant, imposed by either the differential operator or the supplementary conditions. These are equated to the right hand side of the differential equation and of the supplementary conditions.

The approximate solution can be constructed in any bi-variate polynomial basis. In the examples given in this paper we have chosen it to be the Chebyshev product basis (see Ortiz[7]).

The examples considered here are second order partial differential equations with either constant or variable (polynomial) coefficients. We have successfully solved biharmonic and parabolic equations with a variety of supplementary conditions. A more extensive list of examples is given in [10], where we use a technique proposed by Ortiz[8], to deal with Burger's and other examples of non-linear partial differential equations. Their solution is reduced to that of a sequence of linear problems with variable coefficients, the fixed point of which is the solution of the non-linear problem. Segmentation of the domain into Tau elements, used in [4] in connection with crack problems, is also discussed in [10]. Theoretical error estimates suggest that, for a given degree, the error of a Tau approximant and the error of the best uniform approximation by polynomials are of the same order (see [2]). Further details on Ortiz' formulation of the Tau method can be found in [1] and [5].

## 2. DIFFERENTIATION, SHIFTING, AND TRACES OF BIVARIATE POLYNOMIALS

In this paper we follow the notation of Ortiz[6] for the Tau method, and that of Ortiz and Samara[9] for the operational approach. Let ( , ) stand for transposition. Matrices

$$\mu = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ldots & \end{bmatrix}; \; N = \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ 0 & & 3 & \\ & & & \ldots \end{bmatrix}, \text{ and } \eta = N\mu';$$

where introduced in [9] to discuss the effect of combined repeated differentiation and multiplication by the variable (shifting) on the coefficients of a given polynomial $y(x) = \underline{a}\underline{x}$, where $\underline{a} = (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$ and $\underline{x} = (1, x, x^2, \ldots, x^n, \ldots)'$. It is shown in [9], p. 17, that

$$x^s \frac{d^r}{dx^r} y(x) = \underline{a}\eta^r\mu^s\underline{x} = \underline{x}\mu'^s\eta'^r\underline{a}. \tag{1}$$

In the case of two independent variables we have:

Lemma 1

Let

$$a(x, y) = \sum_{i=0}\sum_{j=0} a_{ij}x^i y^j = \underline{x}a\underline{y} \tag{2}$$

where $a = ((a_{ij}))$, $i, j \in \mathbb{N} = 0, 1, 2, \ldots$; then

$$\left[x^n y^m \frac{\partial^{r+t}}{\partial x^r \partial y^t}\right] a(x, y) = \underline{x}B\underline{y}$$

where

$$B = (\mu')^n(\eta')^r a\eta^t\mu^m.$$

*Proof.* Follows by applying (1) to each of the variables.

Let $\underline{u} = U\underline{x}$, $\underline{v} = V\underline{x}$ be two polynomial basis defined by lower triangular matrices $U$, $V$ respectively. Then

$$a(u, v) = \underline{u}A\underline{v} = \underline{x}a\underline{y}, \text{ where } A = (U')^{-1}aV^{-1}, \tag{3}$$

is the expansion of (2) in the basis $\Psi = \{u, v\}$.

Lemma 2

Let $a(u, v)$ be given by (3). The combined effect of repeated partial differentiation and shifting of $a(u, v)$ with respect to $x$ and $y$ is given by

$$\left[x^n y^m \frac{\partial^{r+t}}{\partial x^r \partial y^t}\right] a(u, v) = \underline{u}E\underline{v},$$

where

$$E = (\hat{\mu}_u')^n(\hat{\eta}_u')^r A(\hat{\eta}_v)^t(\hat{\mu}_v)^m,$$

and

$$\hat{w}_u = UwU^{-1}, \quad \hat{w}_v = VwV^{-1}, \text{ for any matrix } w.$$

*Proof.* Follows from (3) and Lemma 1. Let $\underline{z}|_{z=k} = (1, k, k^2, \ldots, k^n, k^{n+1}, \ldots) := \underline{\underline{z(k)}}$, and $a(x, k) = g_k(x) = \underline{g}_k\underline{x}$ be the <u>trace of</u> $a(x, y)$ <u>on</u> $y = k$, then

$$a(x, k) = \underline{x}a\underline{y(k)} = \underline{x}U'AV\underline{y(k)},$$

and

$$U'AV\underline{y(k)} = \underline{g}_k. \tag{4}$$

Similarly, for $a(k, y) = h_k(y) = \underline{h}_k\underline{y}$, <u>trace of</u> $a(x, y)$ <u>on</u> $x = k$, we have

$$\underline{x(k)}U'AV = \underline{h}_k. \tag{5}$$

## 3. PARTIAL DIFFERENTIAL EQUATIONS

(i) *The differential operator*

Let $\mathscr{D}$ (see Ortiz[6]) be the class of linear partial differential operators $L$ in the two variables $x, y$, and with polynomial coefficients. Let

$$p_{rt}(x, y) = \sum_{ij} p_{ijrt} x^i y^j,$$

be the polynomial coefficient of the partial derivative of order $r$ in $x$ and $t$ in $y$. Then, for all

$$L \in \mathscr{D} : L \equiv \sum_{rt}^{v_x v_y} p_{rt}(x, y) \frac{\partial^{r+t}}{\partial x^r \partial y^t}, \quad i = i(r, t), \quad j = j(r, t); \tag{6}$$

$v_x$ and $v_y$ are the maximum orders of differentiation in $x$ and in $y$ respectively.

THEOREM 1

The effect of a linear differential operator $L \in \mathscr{D}$ on the coefficients of a bivariate polynomial $a(x, y) = \underline{x} a \underline{y} = \underline{\underline{u}} A \underline{v}$ is given by

$$L\, a(x, y) = \underline{\underline{u}}\, \mathrm{d}(A) \underline{v},$$

where

$$\mathrm{d}(A) = \sum_{ijrt}^{v_x v_y} p_{ijrt} \hat{\alpha}'_{ir} A \hat{\beta}_{jt}, \tag{7}$$

and

$$\hat{\alpha}_{ir} = U \eta^r \mu^i U^{-1}, \quad \hat{\beta}_{jt} = V \eta^t \mu^j V^{-1}.$$

*Proof.* Follows from Lemma 2 applied to (6).

*Remark* 1. The only line of non-zero elements of matrix $\alpha_{ir}$ (or $\beta_{jt}$) is obtained by multiplying the element $I_{ss}$, $s \in \mathbb{N}$, of the unit matrix $I$ by $(r+s)!/s!$ (or $(t+s)!/s!$) and adding $r$ (or $t$) rows and $i$ (or $j$) column of zeros.

(ii) *The supplementary conditions*

The supplementary conditions of a partial differential equation can be regarded as defined by differential operators acting on $a(x, y)$ along specified sections of the boundary $\Gamma$ of the domain $\Omega$ in which the solution is required. If we assume that $\Omega$ is a rectangle with sides parallel to the coordinate axis, these operators will depend on only one variable:

$$D_{xj} a(x, y) = \left[ \sum \xi_{irt} x^i \frac{\partial^{rt}}{\partial x^r \partial y^t} \right] a(x, y)|_{y = y_j} = g_j(x),$$
$$D_{iy} a(x, y) = \left[ \sum \theta_{jtr} y^j \frac{\partial^{tr}}{\partial y^t \partial x^r} \right] a(x, y)|_{x = x_i} h_i(y), \tag{8}$$

where $i$ and $j$ stand for indices related to different sectors of the boundary $\Gamma$ of the domain $\Omega$; $\xi_{irt}$, $\theta_{jtr}$ are coefficients related to the differential operators $D_{xj}$ and $D_{iy}$ defined on sections of $\Gamma$.

THEOREM 2

The effect of the supplementary conditions on the coefficients of a bivariate polynomial $a(x, y) = \underline{x} a \underline{y} = \underline{\underline{u}} A \underline{v}$ is given by

$$D_{xj} a(x, y) = \underline{\underline{u}} \underline{R}_j(A)$$

and

$$D_{iy} a(x, y) = \underline{\underline{Le}}_i(A) \underline{v},$$

where

$$\underline{R}_j(A) = \sum_{irt} \xi_{irt}\hat{\alpha}'_{ir}A\underline{\gamma}_t \qquad \begin{cases} \hat{\alpha}_{ir} = U\eta^r\mu^i U^{-1} \\ \underline{\gamma}_t = V\eta^t \underline{y}_j \end{cases},$$

$$\underline{\underline{Le}}_i(A) = \sum_{jrt} \theta_{jtr}A\hat{\beta}_{jt} \qquad \begin{cases} \hat{\beta}_{jt} = V\eta^t\mu^j V^{-1} \\ \underline{\delta}_r = U\eta^r \underline{x}_i \end{cases}.$$

*Proof.* Follows from Theorem 1 and (8).

(iii) *The right hand side*

Let $f(x, y) = \underline{x}f\underline{y}$ be the right hand side of the partial differential equation we wish to solve approximately. From (7) it follows that

$$\underline{\underline{u}}\,[\mathrm{d}(A)]\underline{v} = f(x, y) = \underline{\underline{u}}\varphi\underline{v}, \text{ where } \varphi = (U')^{-1}fV^{-1}. \tag{9}$$

Similarly, the supplementary conditions lead to the following set of conditions on matrix $A$:

$$\begin{cases} \underline{R}_j(A) = \underline{G}_j \\ \underline{\underline{Le}}_i(A) = \underline{\underline{H}}_i \end{cases}, \text{ where } \begin{cases} \underline{G}_j = (U')^{-1}g_j \\ \underline{\underline{H}}_i = \underline{\underline{h}}_i V^{-1} \end{cases}. \tag{10}$$

These equations can be written as

$$R(A) = G, \quad Le(A) = H,$$

where $R$, $G$ are matrices with a number of columns equal to the number of supplementary conditions of the form $y = \text{const.}$, and $H$, $Le$ are matrices with a number of rows equal to the number of conditions with $x = \text{const.}$

(iv) *Assembly of the equations for the coefficient matrix A*

The conditions on the coefficient matrix $A$ expressed by (9) and (10), namely

$$\mathrm{d}(A) = \varphi; \quad R(A) = G; \quad Le(A) = H$$

can be given in the form of a linear matrix equation:

$$D(A) = \Phi, \text{ where } D(A) = \left[\begin{array}{c|c} 0 & Le(A) \\ \hline R(A) & \mathrm{d}(A) \end{array}\right], \text{ and } \Phi = \left[\begin{array}{c|c} 0 & H \\ \hline G & \varphi \end{array}\right]. \tag{11}$$

A Tau approximate solution $u_{nm}(x, y)$ of the partial differential equation

$$L\,a(x, y) = f(x, y) \quad (x, y)\in\Omega,$$

with boundary conditions given by (8), is obtained by given a pair $(n, m)$, where $n$ is the degree of approximation in the $x$ variable and $m$ the degree of approximation in the $y$ variable, and solving a suitable truncation of (11), which gives the approximate coefficient matrix $A_{nm} = ((a_{ij}))$, $i = 0(1)n$, $j = 0(1)m$. Then, the Tau approximation is

$$a_{nm}(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}u_i(x)v_j(y),$$

with $u_i(x)\in U\underline{x}$, and $v_j(y)\in V\underline{x}$.

## 4. NUMERICAL SOLUTION OF THE ALGEBRAIC PROBLEM $D(A) = \Phi$

Although the numerical solution of matrix equations has a substantial literature (see for instance Graham[3], and the references given there) we shall describe an *ad hoc* procedure which has been found efficient to deal with matrix equations of the general form

$$D(A) = \sum_i A_i X + XB_i + C_i XD_i;$$

where terms of the forms $A_iX$, $B_iX$ come from partial derivatives with respect of the first or second variable, and those of the form $C_iXD_i$ from mixed derivatives.

Let us consider the linear mapping $S: X \to \underline{\Upsilon} = \underline{\Upsilon}(X)$, called *stringing* where

$$X = \begin{bmatrix} x_1 & x_2 & x_5 & x_{10} \cdots \\ x_3 & x_4 & x_7 & x_{12} \cdots \\ x_6 & x_8 & x_9 & x_{14} \cdots \\ x_{11} & x_{13} & x_{15} & x_{16} \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}, \quad \text{and} \quad \underline{\Upsilon} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \cdots \end{bmatrix}$$

THEOREM 3

The mapping $S$ is such that: (i) $\underline{\Upsilon}(\Sigma\alpha A\beta) = \Sigma\underline{\Upsilon}(\alpha A\beta)$; (ii) there exist a unique matrix $K_{\alpha\beta}$ such that $\underline{\Upsilon}(\alpha A\beta) = K_{\alpha\beta}\underline{\Upsilon}(A)$; (iii) for any given vector $\underline{b}$ there exist a matrix $B$ such that

$$A\underline{b} = \underline{c} \text{ implies } B\underline{\Upsilon}(A) = \underline{c}.$$

*Proof.* Let us consider only the case $\underline{\Upsilon}(\alpha A) = (\alpha_{1h}a_{h1}, \alpha_{1h}a_{h2}, \ldots)$, where the repetition of an index indicates summation over it. We can write

$$\alpha_{ih}a_{hj} = \underline{l}_i\underline{\Upsilon}(A),$$

where $\underline{l}$ is clearly unique. If we call $Le = (\underline{l}_i)$, $i = 1, 2, \ldots$, it follows that $\underline{\Upsilon}(\alpha A) = Le\ \underline{\Upsilon}(A)$. The same arguments lead to $\underline{\Upsilon}(A\beta) = R\underline{\Upsilon}(A)$, hence (ii).

If we write $\underline{A}_i$ for the $i$th row of $A$, and $a_{ij}$ for the general element of $A$, we have

$$c_i = \underline{A}_i\underline{b} = a_{in} = \underline{B}_i\underline{\Upsilon}(A)$$

and the same argument as before completes the proof of (iii). We notice that matrix $B$ is such that there exists an integer $k: b_{ij} = 0$ for all $j - i = k$. We will refer to such matrices as *banded from below*; its transpose is then *banded from above*.

*Remark* 2. The linearity of the mapping $S$ makes it possible that the operations of generation of the equation $D(A) = \Phi$ and its stringing be carried out simultaneously, which is computationally convenient. Let

$$\lambda = p_{ijrt}x^iy^j\frac{\partial^{r+t}}{\partial x^r\partial y^t}$$

be a representative of the terms in the differential operator. Such terms will be used to generate a matrix

$$\mathrm{d}(\lambda) = p_{ijrt}\hat{\alpha}'_{ir}A\hat{\beta}_{jt}, \text{ where } \hat{a}_{ir} = U\eta^r\mu^iU^{-1}, \quad \hat{\beta}_{jt} = V\eta^t\mu^jV^{-1};$$

the process of stringing can be applied to the matrix $\mathrm{d}(\lambda)$ to obtain $\Lambda$ such that $\underline{\Upsilon}(\mathrm{d}(\lambda)) = \Lambda\ \underline{\Upsilon}(A)$.

Let us assume that each of the individual terms of $D(A)$, and also of the supplementary conditions, are treated in this way, and that each of them is accumulated in a bi-dimensional array. We then obtain a linear algebraic system for the unknown vector $\underline{\Upsilon}(A)$. The coefficients of a Tau approximate solution are obtained by solving a suitable truncation of that system, and re-defining the indices to transform the elements of the solution vector back into the $a_{ij}$ form. That is, applying $S^{-1}$ to the vector to get elements $a_{ij}$ of $A$.

## 4. NUMERICAL EXAMPLES

The numerical solution of a linear partial differential equation by means of the present approach to the Tau method requires following the steps listed below:

(1) Choice of the degrees $(n, m)$ of approximation in each of the variables.

(2) Choice of the basis $\Psi = \{u, v\}$.

(3) Construction and simultaneous stringing of the matrices $d(A)$, $R_j(A)$, $Le_i(A)$, $G$, $H$ and $\varphi$, that is of the elements of the matrix equation $D(A) = \Phi$, up to the given $(n, m)$.

(4) Inversion of the linear system of algebraic equations defined by the strang out problem for (n, m).

(5) Redefinition of indices to get the required coefficient matrix $A = ((a_{ij}))$, $i = 0(1)n$, $j = 0(1)m$.

We shall consider two numerical examples.

*Example* 1. Poisson's equation in the square.

$$\begin{aligned} &\nabla^2 a(x, y) = f(x, y), \text{ for } (x, y) \in \Omega = [-1, 1] \times [-1, 1], \text{ with} \\ &a(x, -1) = g_1(x);\ a(x, 1) = g_2(x), \\ &a(-1, y) = h_1(y);\ a(1, y) = h_2(y). \end{aligned} \tag{12}$$

Thus, $L \equiv \nabla^2$ is Laplace's operator and $D_{x1}$, $D_{x-1}$, $D_{1y}$, $D_{-1y}$ are point evaluation functionals applied to the variables $x$ or $y$.

We shall assume that $f(x, y) = \underline{x} f \underline{y} = \underline{u} \varphi \underline{v}$, $g_i(x) = \underline{x} g_i = \underline{u} G_i$, $h_i(y) = \underline{h}_i \underline{y} = \underline{H}_i \underline{v}$, $i = 1, 2$, are polynomials (or polynomial approximations of given functions) of degree at most equal to $n$ in $x$ and $m = n$ in $y$; we are interested in obtaining an approximate solution of degree $\geq (n, n)$. Let us choose as $\Psi = \{u, v\}$ the double Chebyshev basis defined in $[-1, 1] \times [-1, 1]$, with $\underline{u} = U\underline{x}$, $\underline{v} = V\underline{y}$, where $U = V$ is the Chebyshev coefficient matrix in $[-1, 1]$: $T_r(t) = \cos(r \arccos t) = U\underline{t} = V\underline{t}$, $r \in \mathbb{N}$. On account of Theorem 1, Laplace's operator applied to $a(x, y)$ leads to the condition

$$d(A) = \underline{u}[\hat{\eta}'^2 A + A\hat{\eta}^2]\underline{v} = \underline{u}\varphi\underline{v} \tag{13}$$

The boundary conditions, because of Theorem 2, lead, in turn, to the conditions

$$\begin{aligned} &\underline{u}R_1(A) = \underline{u}A\underline{v(-1)} = \underline{u}G_1; \qquad \underline{u}R_2(A) = \underline{u}A\underline{v(1)} = \underline{u}G_2 \\ &\underline{\underline{Le}}_1(A)\underline{v} = \underline{\underline{u(-1)}}A\underline{v} = \underline{H}_1\underline{v}; \qquad \underline{\underline{Le}}_2(A)\underline{v} = \underline{\underline{u(1)}}A\underline{v} = \underline{H}_2\underline{v}, \end{aligned} \tag{14}$$

where $\underline{v(\pm 1)}$, $\underline{\underline{u(\pm 1)}}$ have the meaning indicated in (4), (5). As the elements of $\Psi$ are linearly independent, from (12), (13) we deduce the following matrix equation

$$\begin{cases} \hat{\eta}'^2 A + A\hat{\eta}^2 = \varphi \\ A\underline{v(-1)} = G_1 \\ A\underline{v(1)} = G_2 \\ \underline{\underline{u(-1)}}A = \underline{H}_1 \\ \underline{\underline{u(1)}}A = \underline{H}_2, \end{cases}$$

which, as in (11) we write in the following form

$$D(A) = \alpha X + X\beta = \Phi \tag{15}$$

where

$$\alpha = \begin{bmatrix} 0 & 0 & \underline{\underline{u(-1)}} \\ 0 & 0 & \underline{\underline{u(1)}} \\ 0 & 0 & \hat{\eta}'^2 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \underline{v(-1)} & \underline{v(1)} & \hat{\eta}^2 \end{bmatrix}, \ X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \ \Phi = \begin{bmatrix} 0 & 0 & \underline{H}_1 \\ 0 & 0 & \underline{H}_2 \\ G_1 & G_2 & \varphi \end{bmatrix}$$

Application of the stringing mapping $S$ to (15) leads to a linear system of algebraic

Table 1. Numerical approximation of the solution of $\nabla^2 a(x, y) = -2$ in $\Omega$, with homogenous boundary conditions

| Degree of approximation in $x$ | in $y$ | Maximum Absolute error |
|---|---|---|
| 4 | 4 | $1.6 \times 10^{-2}$ |
| 6 | 6 | $2.5 \times 10^{-3}$ |
| 8 | 8 | $4.0 \times 10^{-4}$ |

equations of the form

$$M\underline{\Upsilon}(X) = \underline{\Upsilon}(\Phi),$$

for the column coefficient vector $\underline{\Upsilon}(X)$. A re-definition of indices gives us the required approximate coefficients for the Tau approximant $a_{nn}(x, y)$. In Table 1 we give the maximum absolute error of approximations of degree 4, 6 and 8 in each of the variables for the case of Saint Venant's torsion problem for a prismatic bar of section $\Omega$, for which the exact solution is known. The problem solved is:

$$\nabla^2 a(x, y) = -2, \ (x, y) \in \Omega = [-1, 1] \times [-1, 1],$$

with

$$a(-1, y) = a(1, y) = a(x, -1) = a(x, 1) = 0.$$

Figure 1 displays the normalized error surfaces

$$\frac{u(x, y) - u_{nm}(x, y)}{\max\limits_{\Omega} |u(x, y) - u_{nm}(x, y)|}$$

for $n = m = 4$, 6 and 8. They show a remarkable equioscillatory behaviour.

*Remark* 3. If equation $M\underline{\Upsilon}(X) = \underline{\Upsilon}(\Phi)$ is solved parametrically in terms of the coefficients $\underline{G}_1$, $\underline{G}_2$, $\underline{H}_1$ and $\underline{H}_2$ a system of linear relations is obtained which gives the values of the coefficients of the Tau approximate solution $a_{nm}(x, y)$ in terms of those of the coefficients of the boundary conditions. Thus the solution of the problem, for fixed $n, m$ and changing boundary conditions is expressed in a closed algebraic form. Furthermore, for a given equation, say Poisson's or the biharmonic, and a fixed degree of

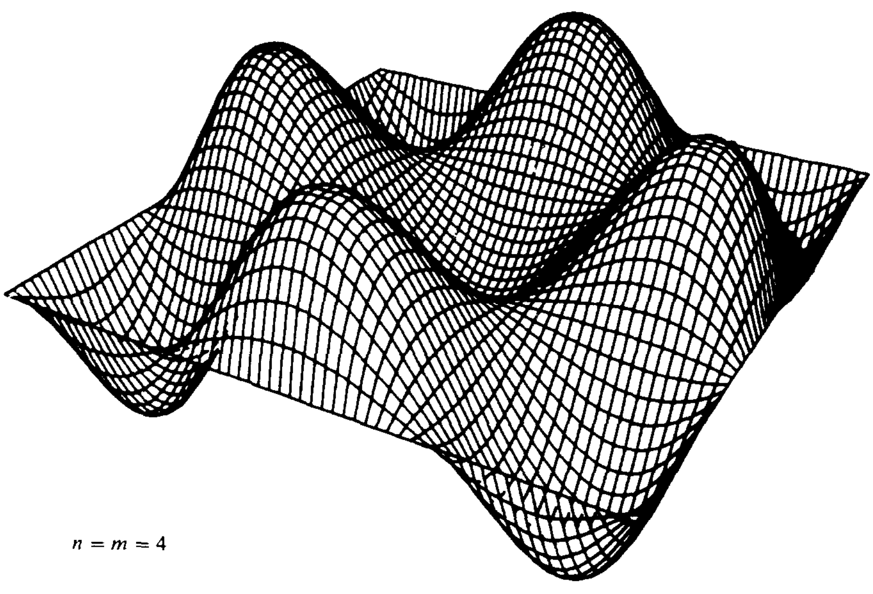
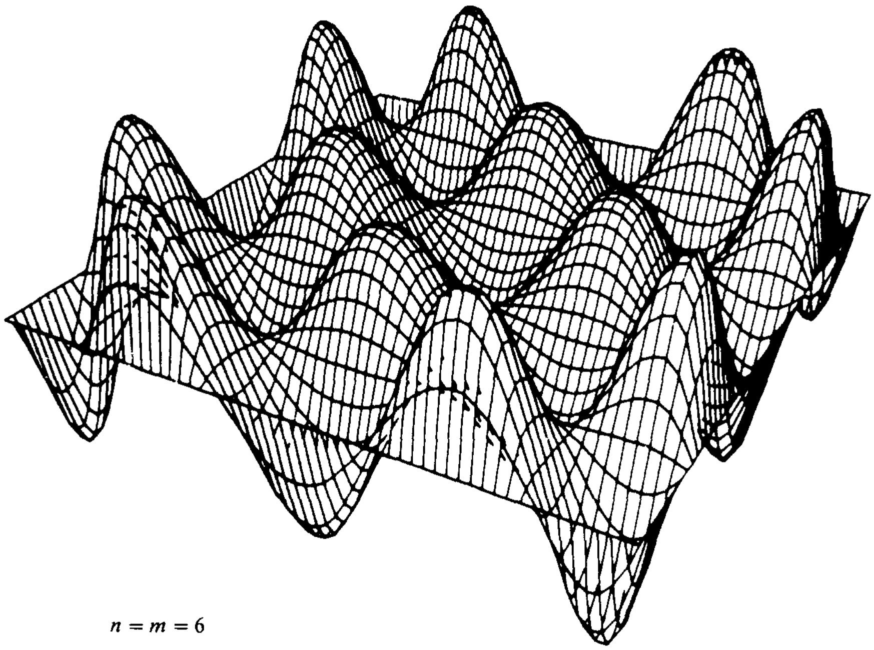
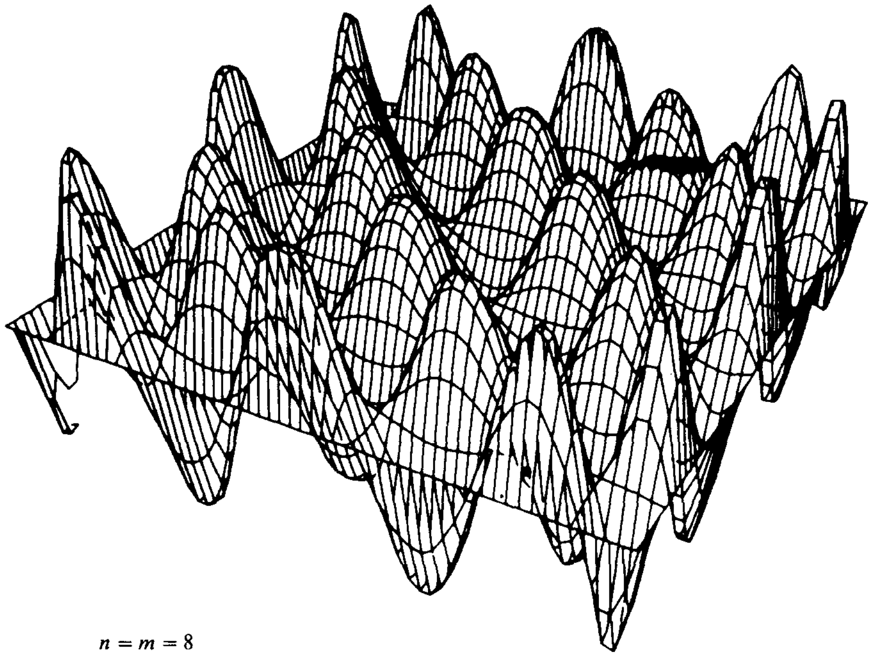



Fig. 1(a).



$n = m = 6$

(b)



$n = m = 8$

(c)

Fig. 1. Normalized error surfaces for Tau approximations $u_{nm}(x, y)$, $n = m = 4(2)8$, of the solution of Saint Venant's torsion problem for a prismatic bar with a square section.

approximation $(n, m)$ it is possible to store $M_{nm}^{-1}$ and find $u_{nm}(x, y)$ by simply doing the product of a matrix and a vector.

*Example* 2. Let us consider the following second order partial differential equation with variable coefficients:

$$(1 + x + x^2)\frac{\partial^2}{\partial x^2}a(x, y) - \left(x + \frac{1}{2}y\right)\frac{\partial^2}{\partial x \partial y}a(x, y) + y(2 - y)\frac{\partial^2}{\partial y^2} = 2y^2 \qquad (16)$$

$$(x, y) \in \Omega = [-1, 1] \times [-1, 1]$$

with either of the two sets of boundary conditions:

$$\text{(i)}\ \begin{cases} a(x,y)|_{y=\pm 1} = x^2 \\ a(x,y)|_{x=\pm 1} = y^2 \end{cases} \qquad \text{or (ii)}\ \begin{cases} \dfrac{\partial}{\partial x} a(x,y)|_{y=-1} = 2x \\ \dfrac{\partial}{\partial y} a(x,y)|_{y=1} = 2x^2 \\ a(x,y)|_{x=\pm 1} = y^2. \end{cases}$$

It is remarked in [6] that a Tau approximation of degree $n$ should be identical to the solution if the latter is a polynomial of degree $\leq n$. In the case of (16), $a(x,y) = x^2y^2$. Approximations of degree $n, m \geq 2$ give exactly $u_{nm}(x,y) = x^2y^2$.

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