JOURNAL OF COMBINATORIAL THEORY 11, 258–267 (1971)

On the Maximum Genus of a Graph

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Communicated by Frank Harary

Received August 25, 1970

We define the *maximum genus*, $\gamma_M(G)$, of a connected graph G to be the largest genus $\gamma(N)$ for compact orientable 2-manifolds N in which G has a 2-cell imbedding. Several general results are established concerning the parameter $\gamma_M(G)$, and the maximum genus of the complete graph K_n with n vertices is determined:

$$\gamma_M(K_n) = \left[\frac{(n-1)(n-2)}{4}\right]$$

1. INTRODUCTION

Denote the genus of a compact orientable 2-manifold N by $\gamma(N)$. The *genus*, $\gamma(G)$, of a graph G is the smallest number $\gamma(N)$ for compact orientable 2-manifolds N in which G can be imbedded. An imbedding of G in N is *minimal* if $\gamma(N) = \gamma(G)$. A 2-*cell imbedding* of G in N has each component of the complement of G in N homeomorphic to an open unit disk. Youngs [7] has shown that every minimal imbedding is necessarily a 2-cell imbedding. Define the *maximum genus*, $\gamma_M(G)$, of a connected graph G to be the largest number $\gamma(N)$ for compact orientable 2-manifolds N in which G has a 2-cell imbedding. Note that, if G is not connected, then no imbedding will be 2-cell. The restriction of consideration to 2-cell imbeddings is essential, as otherwise arbitrarily many handles may be added to any compact orientable 2-manifold in which a graph is imbedded.

The purpose of this paper is to establish some results about the parameter $\gamma_M(G)$ in general and then to determine $\gamma_M(K_n)$ exactly, where K_n is the complete graph with n vertices.

* Research supported in part by NSF Grant GZ 1222.

2. A Result on 2-Cell Imbeddings

Let S_k denote a compact orientable 2-manifold of genus k (a sphere with k handles). Duke [2] has shown that, if a connected graph G has 2-cell imbeddings in S_m and S_n, where $m \leqslant n$, then G also has a 2-cell imbedding in S_k for every integer k, $m \leqslant k \leqslant n$. The following theorem is now immediate:

THEOREM 1. *A connected graph G has a 2-cell imbedding in S_k if and only if $\gamma(G) \leqslant k \leqslant \gamma_M(G)$.*

3. Edmonds' Permutation Technique

Several of the proofs to follow employ Edmonds' permutation technique, described in [3] (see also Youngs [7]) and which we now review. Let the n vertices of G be specified by $V(G) = \{1,..., n\}$ with $V(i)$ being the set of vertices adjacent to vertex i, for $i = 1,..., n$. Let $P_i : V(i) \to V(i)$ be a cyclic permutation of $V(i)$. Then each choice $(P_1 ,..., P_n)$, together with a fixed orientation, uniquely determines a 2-cell imbedding of G in a compact orientable 2-manifold N. Conversely, each such imbedding uniquely determines a collection $(P_1 ,..., P_n)$ which gives rise to that imbedding. Now, corresponding to an undirected edge ab in G, let (a, b) be the edge directed from vertex a to vertex b. Then, defining $D = \{(a, b) : ab \in E(G)\}$ and the permutation $P : D \to D$ by $P((a, b)) = (b, P_b(a))$, it follows that the orbits under P correspond to the (2-cell) faces of the imbedding. As a matter of notational convenience, we will abbreviate an orbit

$$\cdots (a, b), (b, P_b(a)) \cdots \text{ by } \cdots a, b, P_b(a),... .$$

4. Bounds for the Maximum Genus

In this section we establish upper and lower bounds for the maximum genus of a connected graph. We first present the following theorem:

THEOREM 2. *If H is a connected subgraph of a connected graph G, then $\gamma_M(H) \leqslant \gamma_M(G)$.*

Proof. Let H be 2-cell imbedded in $S_{\gamma_M(H)}$. It is clear that any path in G having only one vertex in common with H may be added to H so that the modified graph H^1 is also 2-cell imbedded in $S_{\gamma_M(H)}$. Now let Q be a path of length k in G with end-vertices i and j such that $Q \cap H = \{i\} \cup \{j\}$. It is not difficult to prove, with the aid of Edmonds' permutation technique that the addition of Q to the 2-cell imbedding of H so as to achieve a 2-cell imbedding of $H \cup Q$ either leaves the genus unchanged or increases it by one. We omit the details. The theorem then follows, since G may be constructed from H by a sequence of operations of the types described above.

COROLLARY 2a. *If G is a connected graph with $\gamma(G) = 1$, then $\gamma_M(G) \geqslant 2$.*

Proof. Since G is non-planar, then, by Kuratowski's Theorem, G must contain a connected subgraph H homeomorphic to the complete bipartite graph $K_{3,3}$ or to the complete graph K_5. By use of Edmonds' permutation scheme, it is readily established that $\gamma_M(K_{3,3}) = 2$ and $\gamma_M(K_5) = 3$. Since graphs which are homeomorphic (isomorphic to within vertices of degree 2) have the same maximum and minimum genus numbers, it follows from Theorem 2 that $\gamma_M(G) \geqslant \gamma_M(H) \geqslant 2$. If G has no subgraph homeomorphic to $K_{3,3}$, we have the stronger result that $\gamma_M(G) \geqslant 3$.

We conjecture that, for any connected graph G, the inequality $\gamma_M(G) \geqslant 2\gamma(G)$ is valid. The truth of this conjecture at once implies the truth of a conjecture made by Duke [2] that $\beta(G) \geqslant 4\gamma(G)$, since $\beta(G) \geqslant 2\gamma_M(G)$, as shown below.

We next establish an upper bound for $\gamma_M(G)$, in terms of the first Betti number, $\beta(G)$, of the connected graph G. Recall that $\beta(G) = E - V + 1$, where G has V vertices and E edges. The notation $[x]$ indicates the greatest integer less than or equal to x.

THEOREM 3. *An upper bound for the maximum genus of an arbitrary connected graph G is given by:*

$$\gamma_M(G) \leqslant \left[\frac{\beta(G)}{2}\right].$$

Equality holds if and only if the imbedding has one or two faces according as $\beta(G)$ is even or odd, respectively.

Proof. For any 2-cell imbedding of a connected graph G in a compact orientable 2-manifold S_γ, the Euler Formula $F + V = E + 2(1 - \gamma)$ holds, where F is the number of 2-cells. Equivalently, $\beta(G) = 2\gamma + F - 1$. Thus γ will be a maximum when F is a minimum, and the theorem follows.

To illustrate the above theorem, note that, in the sphere, trees and cycles have 2-cell imbeddings with $F = 1$ and $F = 2$, respectively. Edmonds [4] has shown that every connected graph can be imbedded in some compact 2-manifold N (perhaps non-orientable) so that $N - G$ is a single 2-cell. In the orientable case, however, the inequality of Theorem 3 may be arbitrarily inaccurate, as the graph G_n of Figure 1 indicates. It is clear that $\frac{1}{2}\beta(G_n) = n$, but it is easy to show that $\gamma_M(G_n) = 0$. The inequality of Theorem 3 may be strict even if G is a block, as the graph G of Figure 2 shows, since

$$\gamma_M(G) = 1 < \left[\frac{\beta(G)}{2}\right].$$

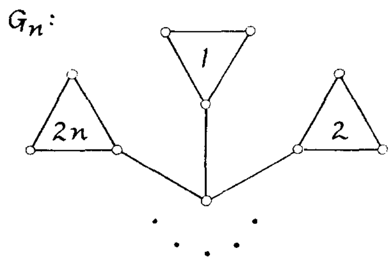

FIGURE 1.

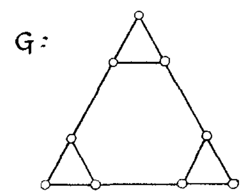

FIGURE 2.

We remark that the Euler Formula $V + F = E + 2 - \tilde{\gamma}$ for compact non-orientable 2-manifolds shows that $\tilde{\gamma}_M(G) \leqslant \beta(G)$, with equality if and only if $F = 1$, but we will not pursue the non-orientable case further in this paper.

In [1], Battle, Harary, Kodama, and Youngs have shown that the genus of a graph is the sum of the genera of its blocks. The following theorem gives a lower bound for the maximum genus of a connected graph in terms of the maximum genera of its blocks.

THEOREM 4. *If G is a connected graph with blocks H_i, $i = 1,..., n$, then*

$$\gamma_M(G) \geqslant \sum_{i=1}^{n} \gamma_M(H_i).$$

Proof. We use induction on n. The result is trivial for $n = 1$. Now let G_{n-1} be connected with $n - 1$ blocks H_i, $i = 1,..., n - 1$, and assume that

$$\gamma_M(G_{n-1}) \geqslant \sum_{i=1}^{n-1} \gamma_M(H_i).$$

In $G = G_n$, let $G_{n-1} \cap H_n = \{v\}$. Let G_{n-1} and H_n have 2-cell imbeddings in $N_1 = S_{\gamma_M(G_{n-1})}$ and $N_2 = S_{\gamma_M(H_n)}$, respectively. Now, following Battle, Harary, Kodama, and Youngs, we take an open 2-cell C_1 in N_1 with simple closed boundary curve J_1 and an open 2-cell C_2 in N_2 with simple closed boundary curve J_2 such that $(C_1 \cup J_1) \cap G_{n-1} = \{v\}$, and $(C_2 \cup J_2) \cap H_n = \{v\}$. Now identify the boundary curve J_1 of $(N_1 - C_1)$ with the boundary curve J_2 of $(N_2 - C_2)$ so that the two copies of vertex v coincide. The result is a 2-cell imbedding of G in S_γ, where $\gamma_M(G) \geqslant \gamma = \gamma_M(G_{n-1}) + \gamma_M(H_n) \geqslant \sum_{i=1}^n \gamma_M(H_i)$.

That the inequality of Theorem 4 may be strict is illustrated by the graph G of Figure 3; each block has maximum genus zero, yet $\gamma_M(G) = 1$.

G :

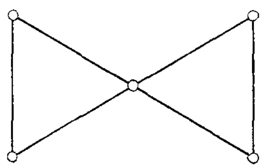

FIGURE 3

COROLLARY 4a. *If G is a connected graph with n blocks H_i, $i = 1,..., n$, then*

$$\sum_{i=1}^{n} \gamma_M(H_i) \leqslant \gamma_M(G) \leqslant \left[\frac{\beta(G)}{2}\right].$$

COROLLARY 4b. *If G is a connected graph with n blocks H_i, such that $\gamma_M(H_i) = \frac{1}{2}\beta(H_i)$, $i = 1,..., n$, then*

$$\gamma_M(G) = \sum_{i=1}^{n} \gamma_M(H_i) = \frac{1}{2}\beta(G).$$

Proof. We have

$$\sum_{i=1}^{n} \gamma_M(H_i) = \sum_{i=1}^{n} \frac{1}{2}\beta(H_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n} (E_i - V_i + 1) = \frac{1}{2}(E - (V + (n - 1)) + n)$$

$$= \frac{1}{2}(E - V + 1) = \frac{1}{2}\beta(G).$$

Then, by Corollary 4a, $\frac{1}{2}\beta(G) \leqslant \gamma_M(G) \leqslant [\frac{1}{2}\beta(G)]$, so that $\gamma_M(G) = \frac{1}{2}\beta(G)$. Note that the proof shows that, in general, $\beta(G) = \sum_{i=1}^{n} \beta(H_i)$.

As an application of Corollary 4b, let G_n be any graph with n blocks, each of which is isomorphic to the graph H formed by removing one edge from K_4. Since $\gamma_M(H) = 1 = \frac{1}{2}\beta(H)$, it follows that $\gamma_M(G_n) = n$, and we have a family of graphs for which the two bounds of Corollary 4a coincide.

Youngs [6] defined a graph G to be *irreducible of genus n* if $\gamma(G) = n$, but $\gamma(G - e) < n$ for every edge e of G. Similarly, we say that G is *irreducible of maximum genus n* if $\gamma_M(G) = n$, but $\gamma_M(G - e) < n$ for every edge e of G. The graph G_n defined above is irreducible of maximum genus n, for:

$$\gamma_M(G_n - e) \leqslant \left[\frac{\beta(G_n - e)}{2}\right] = \left[\frac{\beta(G_n) - 1}{2}\right]$$

$$= \frac{\beta(G_n)}{2} - 1 = \gamma_M(G_n) - 1,$$

for all $e \in E(G_n)$.

The *regional number*, $\delta(G)$, of a graph has been defined in [7] to be the maximum number of components of $(N - G)$ for all possible 2-cell imbeddings of G. Battle, Harary, Kodama, and Youngs [1] have shown that, for a connected graph G having n blocks H_i, $i = 1,..., n$,

$$\delta(G) = 1 - n + \sum_{i=1}^{n} \delta(H_i).$$

Let $\delta_M(G)$ be the minimum number of components of $(N - G)$ for all possible 2-cell imbeddings of G; $\delta_M(G)$ may be interpreted as a regional number for maximal 2-cell imbeddings.

THEOREM 5. *If G is a connected graph having n blocks H_i, $i = 1,..., n$, then*

$$\delta_M(G) \leqslant 1 - n + \sum_{i=1}^{n} \delta_M(H_i).$$

Proof. From the Euler formula, we have that

$$\delta_M(H_i) = 2 + E_i - V_i - 2\gamma_M(H_i),$$

and

$$\begin{aligned}\delta_M(G) &= 2 + E - V - 2\gamma_M(G)\\ &= 2 + \sum_{i=1}^{n} (E_i - V_i) + n - 1 - 2\gamma_M(G)\\ &= n + 1 + \sum_{i=1}^{n} [\delta_M(H_i) - 2 + 2\gamma_M(H_i)] - 2\gamma_M(G)\\ &= 1 - n + \sum_{i=1}^{n} \delta_M(H_i) + 2\left[\sum_{i=1}^{n} \gamma_M(H_i) - \gamma_M(G)\right]\\ &\leqslant 1 - n + \sum_{i=1}^{n} \delta_M(H_i), \qquad \text{by Theorem 4.}\end{aligned}$$

5. The Maximum Genus of the Complete Graph

In this section we show that the upper bound for the maximum genus given in Theorem 3 is attained by the complete graph K_n. This leads to a determination of all compact orientable 2-manifolds on which a given complete graph has a 2-cell imbedding.

THEOREM 6. *The maximum genus of the complete graph on n vertices is given by*

$$\gamma_M(K_n) = \left[\frac{(n-1)(n-2)}{4}\right].$$

Proof. Let the n vertices of K_n be specified by $V(K_n) = \{1,..., n\}$, with

$$V(i) = \{i + 1, i + 2,..., i + n - 1\}$$

being the set of vertices adjacent to vertex i, for $i = 1,..., n$. All arithmetic in connection with this labeling of the vertices of K_n is modulo n, with n written instead of 0. We use Edmonds' permutation technique to produce a 2-cell imbedding of K_n for which $F = 1$ or 2, according as $n \equiv 1$ or 2 (mod 4) or $n \equiv 0$ or 3 (mod 4), respectively. The result will then follow directly from Theorem 3, as

$$\beta(K_n) = \frac{(n-1)(n-2)}{2}.$$

The presentation of the appropriate imbeddings falls most naturally into the two following cases:

Case (i). *n even*. Imbed K_n by selecting:

$$P_i : (i+1, i+2, ..., i+n-1), i = 1, ..., n-2,$$

$$P_{n-1} : (n; 1, n-2, 3, n-4, 5, ..., 4, n-3, 2),$$

$$P_n : (n-1; 1, n-2, 3, n-4, 5, ..., 4, n-3, 2).$$

Now, form the orbit $S = \{P^k(2, n-1): k = 0, 1, ...\}$; the length of S is the minimum positive integer r such that $P^r(2, n-1) = (2, n-1)$. It is helpful to study the occurrences of edges of the form $(2h, n-1)$ in orbits under the permutation P (see Section 3). For $h = 1$, we have as successive vertices in the boundary of the face corresponding to the orbit containing $(2, n-1)$:

$$2, (n-1), n, 1, 2, 3, ..., (n-3), (n-2), (n-1), ... ;$$

for $h = 2$, we have:

$$4, (n-1), (n-3), n, 2, 1, 3, 2, ..., (n-3), (n-1), 2, n, (n-1), ... ;$$

for $h = 3, ..., (n/2) - 1$, we have:

$$2h, (n-1), (n-2h+1), n, (2h-2), 1, (2h-1), 2, 2h, 3, ...,$$
$$(n-2h+1), (n-1), (2h-2), n, (n-2h+3), 1, (n-2h+4), 2, ...,$$
$$(2h-4), (n-1), ... ;$$

finally, for $h = n/2$, we have:

$$n, (n-1), 1, n, (n-2), 1, (n-1), (n-2), n, 3, 1, 4,$$
$$2, ..., (n-4), (n-1), ...$$

It is now clear that, in S, directed edges of the form $(2h, n-1)$ appear, in order, for $2h = 2, n-2, n-6, ..., x, 0 < x \leqslant 4$; where $x = 2$ if $n \equiv 0 \pmod 4$, and $x = 4$ if $n \equiv 2 \pmod 4$. In the former case, we have completed one orbit (of length $r = (n^2/2) - n$), and a second orbit (of length $r = n^2/2$) occurs with edges of the form $(2h, n-1)$ appearing, in order, for $2h = n, n-4, ..., 8, 4, n$. If, however, $n \equiv 2 \pmod 4$, the orbit containing edge $(2, n-1)$ continues, with $2h = n, n-4, ..., 6, 2$. In either case, the orbit(s) we have computed contain all

$$n + \left(\frac{n}{2} - 1\right)(2n) = n(n-1) = 2E$$

directed edges, so that our imbeddings have $F = 2$ and $F = 1$ for $n \equiv 0$ and $n \equiv 2 \pmod 4$, respectively.

Case (ii). *n odd.* Imbed K_n as follows:

$$P_i : (i + 1, i + 2, \ldots, i + n - 1),\ i = 1, \ldots, n - 1,$$

$$P_n : (1, 3, \ldots, n - 2; n - 1, n - 3, \ldots, 2).$$

We proceed as in case (i), applying P to directed edges of the form $(2h, n)$. For $h = 1$, we have:

$$2, n, 1, 2, 3, \ldots, (n - 2), (n - 1), n, \ldots ;$$

for $h = 2$:

$$4, n, 2, 1, 3, 2, 4, \ldots, (n - 2), n, (n - 1), 1, n, 3,$$

$$1, 4, 2, 5, \ldots, (n - 3), n, \ldots ;$$

for $h = 3, \ldots, (n - 1)/2$:

$$2h, n, (2h - 2), 1, (2h - 1), 2, 2h, \ldots, (n - 2h + 2), n,$$

$$(n - 2h + 4), 1, (n - 2h + 5), 2, \ldots, (2h - 4), n, \ldots .$$

Here, we see that edges of the form $(2h, n)$ appear, in order, for $2h = 2$, $n - 1, n - 5, \ldots, x, 0 < x \leqslant 4$, where $x = 2$ if $n \equiv 3 \pmod 4$, and $x = 4$ if $n \equiv 1 \pmod 4$. In the former case, we have completed one orbit (of length $r = [(n - 1)^2/2] + 1$) and a second orbit (of length $r = (n^2 - 3)/2$) includes edges $(2h, n)$, in order, for $2h = 4, n - 3, n - 7, \ldots, 8, 4$. For $n \equiv 1 \pmod 4$, the orbit containing edge $(2, n)$ continues, with $2h = n - 3, n - 7, \ldots, 6, 2$ in succession. In either case, the orbit (s) we have computed contain all

$$n + (4n - 5) + \left(\frac{n - 5}{2}\right)(2n - 2) = n(n - 1) = 2E$$

directed edges of K_n, so that these imbeddings have $F = 2$ and $F = 1$ for $n \equiv 3$ and $n \equiv 1 \pmod 4$, respectively. This completes the proof.

Since Ringel and Youngs [5] have shown that

$$\gamma(K_n) = \left\{\frac{(n - 3)(n - 4)}{12}\right\},$$

for $n \geqslant 3$ (where $\{x\}$ denotes the least integer greater than or equal to x), we have, using Theorems 1 and 6:

COROLLARY 6a. *The complete graph K_n, $n \geqslant 3$, has a 2-cell imbedding in S_k if and only if*

$$\left\{\frac{(n-3)(n-4)}{12}\right\} \leqslant k \leqslant \left[\frac{(n-1)(n-2)}{4}\right].$$

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