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Existence of Solutions for Generalized Vector Equilibrium Problems*

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The concepts of a $F_{(C,D)}$-pseudomonotone mapping and of a (C, D)-pseudomonotone pair of mappings are introduced. By employing Fan's lemma, we establish several existence results for generalized vector equilibrium problems. The new results extend and modify various existence theorems for similar problems.

Key Words: $F_{(C,D)}$-pseudomonotone mapping; generalized vector equilibrium problem; existence results; multivalued mappings.

1. INTRODUCTION

Let K be a nonempty convex subset of a topological vector space X and let $f\colon K \times K \to R$ be a scalar bifunction such that $f(x, x) \geq 0$ for each $x \in K$. The scalar equilibrium problem (in short, EP) is the problem of finding

$$\bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0 \qquad \forall y \in K.$$

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This problem has a number of applications in mathematical physics, economics and operations research and it is extensively investigated; e.g., see [1, 2]. Recently, some extensions of EP were introduced for vector-valued functions and the corresponding existence theorems were proved for extended problems in [3–5]. Most of them were established under certain monotonicity type assumptions on the main mapping.

In this paper, we consider a generalization of EP. Namely, let Y be a topological vector space and let $\Pi(Y)$ denote the set of all nonempty subsets of Y. Let F be a mapping from $K \times K$ to $\Pi(Y)$. Recall that a set $P \subseteq Y$ is said to be solid if int $P \neq \emptyset$ and P is said to be proper if $P \subset Y$. Let C: $K \to \Pi(Y)$ be a mapping such that $C(x)$ is a closed, convex, proper, and solid cone for each $x \in K$. The *generalized vector equilibrium problem* (in short, GVEP) is to find

$$\bar{x} \in K \quad \text{such that} \quad F(\bar{x}, y) \not\subseteq -\operatorname{int} C(\bar{x}) \qquad \forall y \in K.$$

We will denote by $K^*_{F,C}$ the solution set of this problem. Note that a similar GVEP was introduced in [5].

The main purpose of the present paper is to establish the existence results under certain monotonicity assumptions on F, which strengthen and modify those in [5]. Also, we strengthen and extend the results from [3, 4] for single-valued and simultaneous problems.

2. PRELIMINARIES

In this section we give some definitions and properties, which are intended to be used in the sequel.

Firstly we define the dual form of GVEP. Namely, we shall consider the problem of finding

$$\bar{x} \in K \quad \text{such that} \quad F(y, \bar{x}) \not\subseteq \operatorname{int} C(\bar{x}) \qquad \forall y \in K.$$

We denote by $K^d_{F,C}$ the solution set of this problem.

DEFINITION 2.1. Let F: $K \times K \to \Pi(Y)$ and C: $K \to \Pi(Y)$ be mappings such that, for each $x \in K$, $C(x)$ is a closed, convex, and solid cone.

(i) F is C_x-*quasiconvex* if, for all $x \in K$, $y' \in K$, $y'' \in K$, and $\alpha \in [0, 1]$, we have either

$$F(x, y') \subseteq F(x, \alpha y' + (1 - \alpha) y'') + C(x)$$

or

$$F(x, y'') \subseteq F(x, \alpha y' + (1 - \alpha) y'') + C(x).$$

(ii) F is explicitly $\delta(C_x)$*-quasiconvex* if, for all $y' \in K$, $y'' \in K$, and $\alpha \in (0,1)$, we have either

$$F(y_\alpha, y') \subseteq F(y_\alpha, y_\alpha) + C(y')$$

or

$$F(y_\alpha, y'') \subseteq F(y_\alpha, y_\alpha) + C(y'),$$

and, in case $F(y_\alpha, y') - F(y_\alpha, y'') \subseteq \operatorname{int} C(y')$ for all $\alpha \in (0,1)$, we have

$$F(y_\alpha, y') \subseteq F(y_\alpha, y_\alpha) + \operatorname{int} C(y'),$$

where $y_\alpha = \alpha y' + (1-\alpha)y''$.

Remark 2.1. When $C(x) \equiv C$, the concept of C_x-quasiconvexity of F reduces to that of C-quasiconvexity of $F(x,\cdot)$ for each $x \in K$, which, in turn, extends that of C-convexity; e.g., see [6].

DEFINITION 2.2. Let $F\colon K \to \Pi(Y)$ be a mapping. F is said to be *u-hemicontinuous,* if for any $x \in K$, $y \in K$ and $\alpha \in [0,1]$, the mapping $\alpha \to F(\alpha x + (1-\alpha)y)$ is upper semicontinuous at 0^+.

We now give some relationship between $K^*_{F,C}$ and $K^d_{F,C}$.

LEMMA 2.1. *Let K be a nonempty convex subset of X. Let $C\colon K \to \Pi(Y)$ be a mapping such that $C(x)$ is a closed, convex, proper, and solid cone for each $x \in K$. Let $F\colon K \times K \to \Pi(Y)$ be explicitly $\delta(C_x)$-quasiconvex, $F(y,y) \subseteq C(x)$ for all $x, y \in K$. Let $F(\cdot, y)$ be u-hemicontinuous for each $y \in K$. Then, $K^d_{F,C} \subseteq K^*_{F,C}$.*

Proof. Let $\bar{x} \in K^d_{F,C}$. Assume, for contradiction, that $\bar{x} \notin K^*_{F,C}$. Then there exists $y \in K$ such that

$$F(\bar{x}, y) \subseteq -\operatorname{int} C(\bar{x}).$$

By u-hemicontinuity of $F(\cdot, y)$, it follows that, for some $\alpha \in (0,1)$,

$$F(x_\alpha, y) \subseteq -\operatorname{int} C(\bar{x}) \tag{1}$$

where $x_\alpha = \alpha y + (1-\alpha)\bar{x}$. By explicit $\delta(C_x)$-quasiconvexity of F, we now have either

$$F(x_\alpha, y) \subseteq F(x_\alpha, x_\alpha) + C(\bar{x}) \subseteq C(\bar{x})$$

or

$$F(x_\alpha, \bar{x}) \subseteq F(x_\alpha, x_\alpha) + C(\bar{x}) \subseteq C(\bar{x}).$$

The first relation contradicts (1). Thus, we must hav $F(x_\alpha, \bar{x}) \subseteq C(\bar{x})$, hence,

$$F(x_\alpha, \bar{x}) - F(x_\alpha, y) \subseteq \operatorname{int} C(\bar{x}).$$

By explicit $\delta(C_x)$-quasiconvexity of F, we then have

$$F(x_\alpha, \bar{x}) \subseteq F(x_\alpha, x_\alpha) + \operatorname{int} C(\bar{x}) \subseteq \operatorname{int} C(\bar{x}),$$

which contradicts our assumption. The proof is complete. ∎

DEFINITION 2.3. Let $F\colon K \times K \to \Pi(Y)$ and $C\colon K \to \Pi(Y)$ be the same as in Definition 2.1. Let $G\colon K \times K \to \Pi(Y)$ and $D\colon K \to \Pi(Y)$ be mappings such that, for each $x \in K$, $D(x)$ is a closed convex solid cone.

(i) G is $F_{(C,D)}$-*pseudomonotone* if, for all $x, y \in K$,

$$G(x, y) \not\subseteq -\operatorname{int} D(x) \quad \text{implies} \quad F(y, x) \not\subseteq \operatorname{int} C(x).$$

(ii) F is C_x-*pseudomonotone* if it is $F_{(C,C)}$-pseudomonotone.

(iii) The pair of mappings F and G is (C, D)-*pseudomonotone* if G is $F_{(C,D)}$-pseudomonotone and F is $G_{(D,C)}$-pseudomonotone.

The following property can be viewed as an extension of the corresponding results in [2, 7, 3].

COROLLARY 2.1. *Let K, C, and F be the same as in Lemma* 2.1. *Suppose also that F is C_x-pseudomonotone. Then, $K^*_{F,C} = K^d_{F,C}$.*

Proof. From Lemma 2.1 we have $K^d_{F,C} \subseteq K^*_{F,C}$. By C_x-pseudomonotonicity, we obtain $K^*_{F,C} \subseteq K^d_{F,C}$ and the result follows. ∎

The following well-known Fan lemma [8, Lemma 1] will play a crucial role in proving the existence results of solutions for GVEP in our paper.

THEOREM 2.1 (Ky Fan). *In a Hausdorff topological vector space, let Y be a convex set and X a nonempty subset of Y. For each $x \in X$, let $F(x)$ be a closed subset of Y such that the convex hull of every finite subset $\{x_1, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. If there is a point $x_0 \in X$ such that $F(x_0)$ is compact, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

DEFINITION 2.4 (see [9]). A set-valued mapping $F\colon X \to 2^X$ is called KKM-map if

$$\operatorname{conv}\{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$$

for any finite subset $\{x_1, \dots, x_n\}$ of X, where "conv A" denotes the convex hull of the set A.

3. EXISTENCE RESULTS FOR GVEP

We now establish the main result of this paper.

THEOREM 3.1. *Let X and Y be Hausdorff topological vector spaces. Let K be a nonempty compact convex subset of X. Let C: $K \to \Pi(Y)$ and D: $K \to \Pi(Y)$ be such that for each $x \in K$, $C(x)$ and $D(x)$ are closed, convex, proper, and solid cones. Let F: $K \times K \to \Pi(Y)$ be explicitly $\delta(C_x)$-quasiconvex, $F(y, y) \subseteq C(x)$ for all $x, y \in K$. Let $F(\cdot, y)$ be u-hemicontinuous for each $y \in K$. Let G: $K \times K \to \Pi(Y)$ be $F_{(C,D)}$-pseudomonotone and D_x-quasiconvex, $G(y, y) \subseteq D(x)$ for all $x, y \in K$. Suppose also that the set*

$$\tilde{F}_2(y) = \{x \in K | F(y, x) \subseteq \operatorname{int} C(x)\}$$

is open for each $y \in K$. Then, there exists $\bar{x} \in K^*_{F,C}$.

Proof. Define set-valued mappings G_1, F_2: $K \to 2^K$ by

$$G_1(y) = \{x \in K | G(x, y) \not\subseteq -\operatorname{int} D(x)\}$$

and

$$F_2(y) = K \setminus \tilde{F}_2(y).$$

We divide the proof of the theorem into several steps.

(i) G_1 is a KKM-map on K.
Let $u \in \operatorname{conv}\{y_1, \dots, y_n\}$ for any finite subset $\{y_1, \dots, y_n\}$ of K. Assume, for contradiction, that

$$u \notin \bigcup_{i=1}^{n} G_1(y_i).$$

Then, $u \in K$ and

$$G(u, y_i) \subseteq -\operatorname{int} D(u) \qquad \forall i = 1, \dots, n.$$

Since G is D_x-quasiconvex, we have

$$G(u, y_i) \subseteq G(u, u) + D(u) \subseteq D(u)$$

for some i, which is a contradiction.

(ii) $G_1(y) \subseteq F_2(y)$ for all $y \in K$ and F_2 is a KKM-map on K.
By $F_{(C,D)}$-pseudomonotonicity of G, we have $G_1(y) \subseteq F_2(y)$ for all $y \in K$. Since G_1 is a KKM-map, so is F_2.

(iii) For each $y \in K$, $F_2(y)$ is a closed subset of K.
Indeed, since $\tilde{F}_2(y)$ is open, $F_2(y)$ is closed.

(iv) There exists $\bar{x} \in K^*_{F,C}$.

From Step (iii), as K is compact and $F_2(y) \subseteq K$, we see that $F_2(y)$ is a compact subset of K, for each $y \in K$. Thus, by Theorem 2.1, we have

$$K^d_{F,C} = \bigcap_{y \in K} F_2(y) \neq \emptyset.$$

Besides from Lemma 2.1 we have $K^d_{F,C} \subseteq K^*_{F,C}$. Consequently, there exists $\bar{x} \in K^*_{F,C}$, as required.

The proof is complete. ∎

We now establish the similar result for GVEP in the unbounded case.

We note that F is said to be *v-coercive* on K if there exists a compact subset B of X and $\bar{y} \in B \cap K$, such that

$$K \setminus B \subseteq \tilde{F}_2(\bar{y}). \tag{2}$$

THEOREM 3.2. *Let X, Y, C, D, F, and G be the same as in Theorem* 3.1. *Let K be a nonempty closed convex subset of X. In addition, suppose that F is v-coercive on K. Then, there exists $\bar{x} \in K^*_{F,C}$.*

Proof. We first define set-valued mappings $G_1, F_2: K \to 2^K$ as those in the proof of Theorem 3.1. Choose compact subset B of X and $\bar{y} \in B \cap K$ such that (2) holds.

To prove this theorem it is sufficient to follow Steps (i)–(iii) in the proof of Theorem 3.1 and the following step.

(iv)′ There exists $\bar{x} \in K^*_{F,C}$.
Since $K \setminus B \subseteq \tilde{F}_2(\bar{y})$, we have $F_2(\bar{y}) \subseteq K \cap B$. Hence $F_2(\bar{y})$ is a compact subset of K. Thus, by Theorem 2.1, we have

$$K^d_{F,C} = \bigcap_{y \in K} F_2(y) \neq \emptyset.$$

Besides, from Lemma 2.1 we have $K^d_{F,C} \subseteq K^*_{F,C}$. Consequently, there exists $\bar{x} \in K^*_{F,C}$, as required.

The proof is complete. ∎

Remark 3.1. (i) The assertions of Theorems 3.1 and 3.2 remain valid if we replace the condition of D_x-quasiconvexity of G with the following:

$$\{ y \in K | G(x, y) \subseteq -\operatorname{int} D(x) \}$$

is convex for each $x \in K$.

(ii) The topologies on X and Y need not be equivalent. For instance, if X and Y are normed spaces, we can use the weak topology on X and the norm topology on Y.

Remark 3.2. (i) In [3], several existence results were established for the case where F is a single valued mapping, C is a constant mapping, and Y is locally convex. Thus, Theorems 3.1 and 3.2 can be viewed as extensions of Theorems 3.1 and 3.2 from [3].

(ii) Note that our existence results are different from those in [5]. In fact, Theorem 1 in [5] contains two relationships between the pairs (f, C) and (g, D), respectively (see assumptions (iv) and (v) in Theorem 1), assumption (iv) corresponding to $g_{(D,C)}$-pseudomonotonicity of f. However, Theorem 1 in [5] contains also the "reverse" assumption (v), whereas, in Theorems 3.1 and 3.2 we make use of $F_{(C,D)}$-pseudomonotonicity of G which allows us to exclude any additional joint condition. Therefore, our existence results also modify and strengthen those in [5].

We now establish existence results for simultaneous GVEP's.

THEOREM 3.3. *Let X, Y, K, C, and D be the same as in Theorem* 3.1. *Let $F\colon K \times K \to \Pi(Y)$ and $G\colon K \times K \to \Pi(Y)$ be a (C, D)-pseudomonotone pair of mappings. Let F (respectively, G) be explicitly $\delta(C_x)$-quasiconvex (respectively, explicitly $\delta(D_x)$-quasiconvex and D_x-quasiconvex), $F(y, y) \subseteq C(x)$ for all $x, y \in K$ (respectively, $G(y, y) \subseteq D(x)$ for all $x, y \in K$). Let $F(\cdot, y)$ and $G(\cdot, y)$ be u-hemicontinuous for each $y \in K$. Suppose also that the set $\tilde{F}_2(y)$ is open for each $y \in K$. Then,*

(i) $K^d_{F,C} = K^*_{F,C} = K^d_{G,D} = K^*_{G,D}$;

(ii) *there exists* $\bar{x} \in K^*_{F,C}$.

Proof. From Lemma 2.1 we have $K^d_{F,C} \subseteq K^*_{F,C}$ and $K^d_{G,D} \subseteq K^*_{G,D}$. On the other hand, since the pair F, G is (C, D)-pseudomonotone, we have $K^*_{F,C} \subseteq K^d_{G,D}$ and $K^*_{G,D} \subseteq K^d_{F,C}$. Hence, (i) holds. Assertion (ii) follows from Theorem 3.1. The proof is complete. ∎

THEOREM 3.4. *Let X, Y, C, D, F, and G be the same as in Theorem* 3.3. *Let K be a nonempty closed convex subset of X. In addition, suppose that F is v-coercive on K. Then,*

(i) $K^d_{F,C} = K^*_{F,C} = K^d_{G,D} = K^*_{G,D}$;

(ii) *there exists* $\bar{x} \in K^*_{F,C}$.

Proof. Assertions (i) and (ii) are true due to the same argument as that in Theorems 3.3 and 3.2, respectively. ∎

The assertions of Theorems 3.3 and 3.4 can be viewed as extensions of Theorem 3.1 in [4] to the set-valued and "moving" case.

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