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Chebyshev series expansion of inverse polynomials

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Abstract

The Chebyshev series expansion $\sum'^{\infty}_{n=0} a_n T_n(x)$ of the inverse of a polynomial $\sum_{j=0}^{k} b_j T_j(x)$ is well defined if the polynomial has no roots in $[-1, 1]$. If the inverse polynomial is decomposed into partial fractions, the a_n are linear combinations of simple functions of the polynomial roots. Also, if the first k of the coefficients a_n are known, the others become linear combinations of these derived recursively from the b_j's. On a closely related theme, finding a polynomial with minimum *relative* error towards a given $f(x)$ is approximately equivalent to finding the b_j in $f(x)/\sum_0^k b_j T_j(x) = 1 + \sum_{k+1}^{\infty} a_n T_n(x)$; a Newton algorithm produces these if the Chebyshev expansion of $f(x)$ is known.

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1. Introduction

1.1. Scope

The Chebyshev polynomials of the first kind $T_n(x)$ are even or odd functions of x defined as [1, (22.3.6)] [4, (3.6)] [10, p. 51]

$$T_0(x) = 1, \quad T_n(x) = \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}, \quad n = 1, 2, 3, \ldots \tag{1}$$

where the Gauss bracket $\lfloor . \rfloor$ denotes the largest integer not greater than the number it embraces. The reverse formula is [6, p. 412] [10, p. 52] [11],

$$x^n = 2^{1-n} \sum_{\substack{j=0 \\ n-j \text{ even}}}^{n}{}' \binom{n}{(n-j)/2} T_j(x), \tag{2}$$

where the prime at the sum symbol means the first term (at $j = 0$) is to be halved—unless it is skipped.

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The expansion of an inverse polynomial of degree k in a power series is [1, (3.6.16)]

$$\frac{1}{\sum_{j=0}^{k} d_j x^j} = \sum_{n=0}^{\infty} c_n x^n, \tag{3}$$

with recursively accessible c_n [14, 0.313] [20,18]. The topic of this script is the equivalent arithmetic expansion of the inverse polynomial in a Chebyshev series,

$$\frac{1}{\sum_{j=0}^{k} d_j x^j} = \frac{1}{\sum_{j=0}^{k} b_j T_j(x)} = \sum_{n=0}^{\infty}{}' a_n T_n(x), \tag{4}$$

i.e., computation of the coefficients

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{T_n(x)}{\sum_{j=0}^{k} b_j T_j(x)} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{5}$$

given the sets $\{b_j\}$ or $\{d_j\}$ that define the original function. The expansion (4) exists if the inverse polynomial is bound in the interval $[-1, 1]$, i.e., if $\sum d_j x^j$ has no roots in $[-1, 1]$.

Characteristic approximate methods of evaluating (5) [22] are not reviewed here: (i) Fourier transform methods [6, (4.7)] [5,9,8], (ii) sampling the inverse polynomial with Gauss-type quadratures [1, (25.4.38)] [10, Section 1.8] [16,21,28,32,34,35], (iii) approximation by truncation of (3), then insertion of (2) [11,3,2], (iv) using the near-minimax properties of the Chebyshev series [26,24].

Chapter 2.1 explains how the a_n of (5) could be computed suppose the inverse polynomial has been decomposed into partial fractions. Chapter 2.2 provides a recursive algorithm to derive high-indexed $a_{n\geqslant k}$ suppose the low-indexed $a_{n<k}$ are given by other means. Chapter 2.3 recalls a (standard) integral-free method to compute approximate low-indexed a_n, which builds the framework for a specific inverse problem of Chapter 3—that is finding the b_j from partially known a_n—related to polynomial approximants with minimum relative error.

1.2. Basic properties

We will refer to the well-known product rule [1, (22.7.24)],

$$T_n(x)T_m(x) = \tfrac{1}{2}\left(T_{|m-n|}(x) + T_{m+n}(x)\right), \quad n, m \geqslant 0. \tag{6}$$

From the case $\gamma = -\frac{1}{2}$ in [19, (13)] we derive

$$\frac{\mathrm{d}^s}{\mathrm{d}x^s} T_n(x) = 2^s n \sum_{\substack{k=0 \\ n-s-k \text{ even}}}^{n-s}{}' \binom{\frac{n+s-k}{2} - 1}{\frac{n-s-k}{2}} \frac{((n+s+k)/2-1)!}{((n-s+k)/2)!} T_k(x), \quad s = 1, 2, 3, \ldots . \tag{7}$$

2. Accesses to the expansion coefficients

2.1. The case of known partial fractions

A compact, exact way of computing the Chebyshev series (4) decomposes $1/\sum d_j x^j$ into partial fractions [14, 2.102], which reduces (4) to the calculation of the $a_{n,s}$ in

$$\frac{1}{(z-x)^s} \equiv \sum_{n=0}^{\infty}{}' a_{n,s}(z) T_n(x), \tag{8}$$

where z is a root of the polynomial, $\sum_{j=0}^{k} d_j z^j = 0$.

Sign flips of z and x in (8) show that

$$a_{n,s}(-z) = (-1)^{n+s} a_{n,s}(z). \tag{9}$$

Lemma 1. *For multiplicity $s = 1$, the expansion coefficients are*

$$a_{n,1}(z) = \frac{2}{(z^2-1)^{1/2}} \frac{1}{w^n}, \quad w \equiv z + (z^2-1)^{1/2}, \quad z \notin [-1, 1]. \tag{10}$$

The branch of $(z^2-1)^{1/2}$ must be chosen such that $|w| > 1$.

Lemma 2. *For general $s \geqslant 0$, $n \geqslant 0$ the coefficient is*

$$\begin{aligned} a_{n,s+1}(z) = \binom{s+n}{s} \frac{2^{s+1} w^{3s-n-2}}{(z^2-1)^{1/2}(w^2-1)^{2s}} \Bigg[(w^2-1) {}_2F_1 \left({1-s, n+1-s \atop n+1} \Big| \frac{1}{w^2} \right) \\ + \frac{2s}{n+1} {}_2F_1 \left({1-s, n+1-s \atop n+2} \Big| \frac{1}{w^2} \right) \Bigg], \end{aligned} \tag{11}$$

in terms of two hypergeometric series—which terminate if $s \geqslant 1$ or sum up to (10) *if $s = 0$.*

Proof. Eq. (10) has already been demonstrated [15, (A.6), 31,33] based on [1, (22.9.9)] [36, (18)]. Eq. (11) is a transformation of the Legendre Function $P_{s-1}^{-n}(z/\sqrt{z^2-1})$ in Elliott's equations [7, (18)+(26)] to a unified formula for arbitrary signs of $\Re z$ and $\Im z$. An independent derivation starts from the derivative

$$\frac{\mathrm{d}^s}{\mathrm{d}x^s} \frac{1}{z-x} = s! \frac{1}{(z-x)^{s+1}} = \frac{\mathrm{d}^s}{\mathrm{d}x^s} \sum_{n=0}^{\infty}{}' a_{n,1}(z) T_n(x) = s! \sum_{n=0}^{\infty}{}' a_{n,s+1}(z) T_n(x).$$

Insertion of (10) for $a_{n,1}(z)$ and use of (7) builds at the r.h.s.

$$\begin{aligned} a_{n,s+1} &= \frac{2^s}{s!} \sum_{\substack{m=n+s \\ m-n-s \text{ even}}}^{\infty} \frac{2m}{(z^2-1)^{1/2}} \frac{1}{w^m} \left({\frac{m+s-n}{2} - 1 \atop \frac{m-s-n}{2}} \right) \frac{((m+s+n)/2-1)!}{((m-s+n)/2)!} \\ &= -\frac{2^{s+1} w}{s!(s-1)!(z^2-1)^{1/2}} \frac{\partial}{\partial w} \left[\frac{1}{w^{n+s}} \sum_{k=0,2,4,\ldots}^{\infty} \frac{(k+s-1)!}{(k)!} \frac{(k+s+n-1)!}{(k+n)! w^{2k}} \right] \\ &= -\binom{s+n-1}{s} \frac{2^{s+1}}{(z^2-1)^{1/2}} \frac{w}{n} \frac{\partial}{\partial w} \left[\frac{1}{w^{n+s}} {}_2F_1 \left({s, s+n \atop n+1} \Big| \frac{1}{w^2} \right) \right]. \end{aligned} \tag{12}$$

The product rule for derivatives is applied, then both ${}_2F_1()$ are converted from infinite to terminating series with [1, (15.3.3)]. One of the two is re-formatted with an intermediate variable $\Omega \equiv 1/w^2$,

$$\begin{aligned} &\frac{\partial}{\partial w} \left[\left(1 - \frac{1}{w^2}\right)^{1-2s} {}_2F_1 \left({1-s, n+1-s \atop n+1} \Big| \frac{1}{w^2} \right) \right] \\ &\quad = -2 \frac{1}{w^3} \frac{\partial}{\partial \Omega} \left[(1-\Omega)^{1-2s} {}_2F_1 \left({1-s, n+1-s \atop n+1} \Big| \Omega \right) \right], \end{aligned} \tag{13}$$

and reaches (11) facilitated by [1, (15.2.6)]. □

Note 1. The coefficients for the polynomial roots of multiplicity 2 are

$$a_{n,2}(z) = \frac{4}{(z^2-1)^{1/2} w^{n-3}(w^2-1)} \left[\frac{n-1}{w^2} + \frac{2}{w^2-1} \right], \quad n \geqslant 0, \tag{14}$$

where $w^2 - 1 = 2w(z^2-1)^{1/2}$ with $s = 1$ has been used in (11). The Laurent series of the w-terms

$$\frac{1}{w^2-1}\left[\frac{n-1}{w^2} + \frac{2}{w^2-1} \right] = \frac{n+1}{w^4} + \frac{n+3}{w^6} + \frac{n+5}{w^7} + \cdots \tag{15}$$

transforms (14) with (10) into

$$a_{n,2}(z) = 2\sum_{k=1}^{\infty} (n+2k-1) a_{n+2k-1,1}(z), \tag{16}$$

which is the case $\gamma = -\frac{1}{2}$, $q = 1$ of [19, (5)].

In practice, one will often be interested in generating all $a_{n,s}$ from $n = 0$, $s = 1$ up to some pair of maximum indices. As an alternative to (11), one can generate the coefficients of (10), (17) and (18), and secure all coefficients in (8) for a particular z with the forward recurrence (19):

Lemma 3. *The coefficient at $n = 0$ for general multiplicity is*

$$a_{0,s+1}(z) = 2^{2-s} \sum_{l=0}^{\lfloor s/2 \rfloor} (-1)^l \binom{s-l}{l} \binom{2s-2l-1}{s-l-1} \frac{z^{s-2l}}{(z^2-1)^{s-l+1/2}}, \quad s \geqslant 0. \tag{17}$$

Lemma 4. *The recurrence from $n = 0$ to $n = 1$ is*

$$a_{1,s+1}(z) = -a_{0,s}(z) + z a_{0,s+1}(z). \tag{18}$$

Lemma 5. *A mixed index recurrence for the expansion coefficients is*

$$a_{n+1,s}(z) = a_{n-1,s}(z) - \frac{2n}{s-1} a_{n,s-1}(z), \quad n \geqslant 1, \quad s \geqslant 2. \tag{19}$$

Proof. Higher second indices s of the $a_{n,s}$ are obtained by repeated derivation of (8) w.r.t. z,

$$(-1)^s s! \frac{1}{(z-x)^{s+1}} = {\sum_{n=0}^{\infty}}' \frac{\mathrm{d}^s}{\mathrm{d}z^s} a_{n,1}(z) T_n(x). \tag{20}$$

Considering only the term at $n = 0$ means with (10),

$$a_{0,s+1}(z) = \frac{2}{s!}(-1)^s \frac{\mathrm{d}^s}{\mathrm{d}z^s} \frac{1}{(z^2-1)^{1/2}}, \tag{21}$$

which generates (17) using [14, 0.432.1]. Eq. (18) follows immediately from

$$a_{1,s+1}(z) = \frac{2}{\pi} \int_{-1}^{1} \frac{T_1(x)}{(z-x)^{s+1}} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{22}$$

with the decomposition $T_1(x) = -(z-x) + z$ plus the definition (8). (19) is the case $\gamma = -\frac{1}{2}$ in [19, (4)] post-processed by [19, (22)]. □

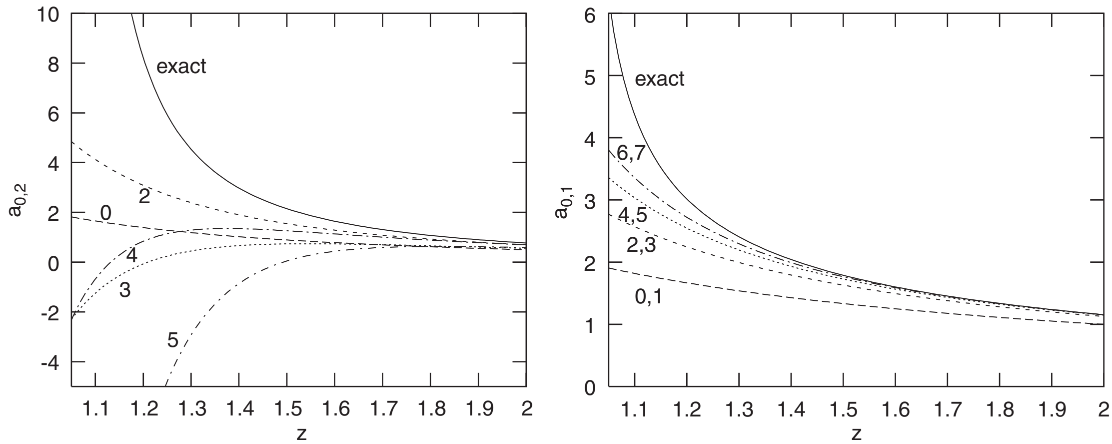

Fig. 1. Estimated $a_{0,s}(z)$ by projection of partial sums $X_0^{-1}(x)\sum_{j=0}^{N}(-1)^j D^j(x)$ of (23) on $T_0(x)$ for initial estimates $X_0^{-1}(x)=1/z^s$, $s=2$ (left) or $s=1$ (right). The exact curves $2/(z^2-1)^{1/2}$ and $2z/(z^2-1)^{3/2}$ are drawn, and the other lines are labeled with the successive numbers N (pair-wise equal at the right).

Note 2. The Broucke algorithm [3, (13)] proposes to approximate (4) by a geometric series

$$\frac{1}{\sum_{j=0}^{k} d_j x^j} \approx X_0^{-1}\left(1 - D + D^2 - D^3 + \cdots\right), \quad D \equiv X_0^{-1}\left(\sum_{j=0}^{k} d_j x^j\right) - 1, \tag{23}$$

for some initial estimate $X_0^{-1}(x)$. One could generate a sequence of associated a_n by projection of the partial sums onto $T_n(x)$, but these do not well approach the analytic structure of the $a_{n,s}(z)$ near the singularities $z=\pm 1$: There is a problem with divergence for roots of multiplicity $s>1$ (which are not matched by the geometric series), and for roots of multiplicity $s=1$ convergence becomes slow for z near ± 1. These two aspects are illustrated in the left part of Fig. 1 for $\sum_j d_j x^j = (z-x)^2$, and in the right part for $\sum_j d_j x^j = z - x$. Obviously the cases of $s>1$ must be treated separately, which requires some form of partial fraction decomposition anyway: the apparent benefit of (23) of handling the general polynomial without prior analysis is deceptive.

Note 3. Eq. (18) may be generalized to

$$\frac{2}{\pi}\int_{-1}^{1} \frac{x^l}{(z-x)^n}\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sum_{m=0}^{l}(-1)^m \binom{l}{m} z^{l-m} a_{0,n-m}, \quad l<n, \tag{24}$$

and with (2) and (6) to

$$\frac{2}{\pi}\int_{-1}^{1} \frac{x^l}{(z-x)^n}\frac{T_s(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{2^l}\sideset{}{'}\sum_{\substack{i=0\\ l-i\ \text{even}}}^{l} \binom{l}{\frac{l-i}{2}} [a_{|i-s|,n} + a_{i+s,n}]. \tag{25}$$

The Chebyshev expansion of a polynomial quotient could therefore be based on “incomplete” partial fractions decomposition (4) for numerator equal to one.

Note 4. From (10)

$$\frac{\partial a_{n,1}(z)}{\partial z} = -a_{n,1}\left[\frac{z}{z^2-1} + \frac{n}{(z^2-1)^{1/2}}\right], \tag{26}$$

so the (linear) propagation of the absolute relative error in the root z to the error in the coefficient $a_{n,1}$ *is*

$$\left|\frac{\Delta a_{n,1}}{a_{n,1}}\right| \approx \left|\frac{\Delta z}{z}\right| \cdot \left|\frac{z^2}{z^2-1} + \frac{nz}{(z^2-1)^{1/2}}\right|. \tag{27}$$

2.2. Recurrence of expansion coefficients

The T_n in (5) may be decomposed into a unique product of a polynomial by the denominator plus a remainder of polynomial degree less than k. [The argument x is omitted at all $T_n(x)$ for brevity.]

$$\begin{aligned} T_n = {} & (d_0^{(n)} T_0 + d_1^{(n)} T_1 + \cdots + d_{n-k}^{(n)} T_{n-k})(b_0 T_0 + b_1 T_1 + \cdots + b_k T_k) \\ & + \frac{c_0^{(n)}}{2} T_0 + c_1^{(n)} T_1 + c_2^{(n)} T_2 + \cdots + c_{k-1}^{(n)} T_{k-1}. \end{aligned} \tag{28}$$

Expansion with (6) yields a system of linear equations for the vector of the unknowns $d_j^{(n)}$ and $c_j^{(n)}$:

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & \cdots & 0 & 2b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & b_1 & b_0 + \frac{b_2}{2} & \frac{b_1+b_3}{2} & \frac{b_2+b_4}{2} & \cdots \\ \vdots & 0 & \ddots & \ddots & \vdots & b_2 & \frac{b_1+b_3}{2} & b_0 + \frac{b_4}{2} & \frac{b_1+b_5}{2} & \cdots \\ \vdots & \vdots & \ddots & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & b_{k-1} & \frac{b_{k-2}+b_k}{2} & \frac{b_{k-3}}{2} & \cdots & \cdots \\ \hline 0 & \cdots & \cdots & \cdots & 0 & b_k & \frac{b_{k-1}}{2} & \frac{b_{k-2}}{2} & \frac{b_{k-3}}{2} & \cdots \\ \vdots & \cdots & \cdots & \cdots & \vdots & 0 & \frac{b_k}{2} & \frac{b_{k-1}}{2} & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \frac{b_{k-1}}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & \frac{b_k}{2} \end{array}\right) \cdot \left(\begin{array}{c} c_0^{(n)} \\ c_1^{(n)} \\ \vdots \\ \vdots \\ c_{k-1}^{(n)} \\ \hline d_0^{(n)} \\ d_1^{(n)} \\ \vdots \\ d_{n-k-1}^{(n)} \\ d_{n-k}^{(n)} \end{array}\right) = \left(\begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{array}\right). \tag{29}$$

The $(n+1) \times (n+1)$ coefficient matrix $A_{r,c}$ (row index r and column index c from 0 to n) is an upper triangular matrix. It hosts a $k \times k$ unit matrix in the upper left corner, and is symmetric w.r.t. the minor diagonal that stretches from $A_{0,k}$ to $A_{n-k,n}$:

$$A_{r,c} = \delta_{r,c}, \quad 0 \leqslant c \leqslant k-1. \tag{30}$$

$$A_{r,k+c} = A_{c,k+r} = \begin{cases} 2b_0, & r = c = 0, \\ b_c, & r = 0, \quad 1 \leqslant c \leqslant k, \\ \frac{1}{2}(b_{|r-c|} + b_{r+c}), & r \neq c, \\ b_0 + b_{2c}/2, & r = c = 1, 2, \ldots, n-k. \end{cases} \tag{31}$$

This works with the auxiliary definition

$$b_i = 0, \quad i > k \quad \text{or } i < 0. \tag{32}$$

Insertion of (28) into (5) yields the format

$$a_n = 2d_0^{(n)} + \sum_{i=0}^{k-1}{}' c_i^{(n)} a_i, \quad n \geqslant k, \tag{33}$$

which means that entire sequence a_n can be generated recursively from its first k terms, if the $d_0^{(n)}$ and $c_i^{(n)}$ are generated at the same time via (29) or an equivalent method. Iterated full solution of (29) can be avoided through recursive generation of the set $\{d_i^{(n+1)}, c_i^{(n+1)}\}$ from $\{d_i^{(n)}, c_i^{(n)}\}$ and $\{d_i^{(n-1)}, c_i^{(n-1)}\}$ as follows:

Proposition 6. *The set of coefficients in* (33) *obeys*

$$d_0^{(n+1)} = d_1^{(n)} + \frac{c_{k-1}^{(n)}}{b_k} - d_0^{(n-1)}, \tag{34}$$

$$d_1^{(n+1)} = 2d_0^{(n)} + d_2^{(n)} - d_1^{(n-1)}, \tag{35}$$

$$d_j^{(n+1)} = d_{j-1}^{(n)} + d_{j+1}^{(n)} - d_j^{(n-1)}, \quad j = 2, 3, \ldots, n-k+1. \tag{36}$$

$$\frac{c_0^{(n+1)}}{2} = c_1^{(n)} - \frac{b_0 c_{k-1}^{(n)}}{b_k} - \frac{c_0^{(n-1)}}{2}, \tag{37}$$

$$c_j^{(n+1)} = c_{j-1}^{(n)} + c_{j+1}^{(n)} - \frac{b_j c_{k-1}^{(n)}}{b_k} - c_j^{(n-1)}, \quad j = 1, 2, \ldots, k-1, \tag{38}$$

where the auxiliary definitions

$$c_j^{(n)} = 0, \quad j \geqslant k \quad \text{or } j < 0, \tag{39}$$

$$d_j^{(n)} = 0, \quad j > n-k \quad \text{or } j < 0, \tag{40}$$

are made to condense the notation.

Proof. Multiplying (28) by $2T_1$ and using (6) we have

$$2T_1 \sum_{j=0}^{n-k} d_j^{(n)} T_j = d_1^{(n)} T_0 + (2d_0^{(n)} + d_2^{(n)}) T_1 + \sum_{j=2}^{n-k-1} (d_{j-1}^{(n)} + d_{j+1}^{(n)}) T_j + d_{n-k-1}^{(n)} T_{n-k} + d_{n-k}^{(n)} T_{n-k+1}, \tag{41}$$

$$2T_1 {\sum_{j=0}^{k-1}}' c_j^{(n)} T_j = c_1^{(n)} T_0 + \sum_{j=1}^{k-2} (c_{j-1}^{(n)} + c_{j+1}^{(n)}) T_j + c_{k-2}^{(n)} T_{k-1} + c_{k-1}^{(n)} T_k. \tag{42}$$

The last term in the previous equation is rewritten

$$c_{k-1}^{(n)} T_k = \frac{c_{k-1}^{(n)}}{b_k} \sum_{j=0}^{k} b_j T_j - \frac{c_{k-1}^{(n)}}{b_k} b_0 T_0 - \cdots - \frac{c_{k-1}^{(n)}}{b_k} b_{k-1} T_{k-1}. \tag{43}$$

We construct

$$\begin{aligned} 2T_1 T_n = {} & \left[\left(d_1^{(n)} + \frac{c_{k-1}^{(n)}}{b_k} \right) T_0 + (2d_0^{(n)} + d_2^{(n)}) T_1 \right. \\ & \left. + \sum_{j=2}^{n-k-1} (d_{j-1}^{(n)} + d_{j+1}^{(n)}) T_j + d_{n-k-1}^{(n)} T_{n-k} + d_{n-k}^{(n)} T_{n-k+1} \right] \cdot \left[\sum_{j=0}^{k} b_j T_j \right] \\ & + \left(c_1^{(n)} - \frac{c_{k-1}^{(n)}}{b_k} b_0 \right) T_0 + \sum_{j=1}^{k-2} \left(c_{j-1}^{(n)} + c_{j+1}^{(n)} - \frac{c_{k-1}^{(n)}}{b_k} b_j \right) T_j \\ & + \left(c_{k-2}^{(n)} - \frac{c_{k-1}^{(n)}}{b_k} b_{k-1} \right) T_{k-1}, \end{aligned} \tag{44}$$

and subtract T_{n-1} for identification of the $d_j^{(n+1)}$ and $c_j^{(n+1)}$,

$$T_{n+1} = 2T_1T_n - T_{n-1} = \left(\sum_{j=0}^{n-k+1} d_j^{(n+1)} T_j\right)\left(\sum_{j=0}^{k} b_j T_j\right) + \sum_{j=0}^{k-1}{}' c_j^{(n+1)} T_j. \qquad \square \tag{45}$$

2.3. Approximation by the truncated Chebyshev series

Approximations $\hat{a}_n$ to the a_n of (4) may be calculated assuming that the a_n beyond some index N are negligible:

$$\frac{1}{\sum_{j=0}^{k} b_j T_j(x)} \approx \sum_{n=0}^{N}{}' \hat{a}_n T_n(x). \tag{46}$$

This approach is obvious [10] and summarized here to prepare the notation for Section 3. The ansatz is multiplied by $2\sum b_j T_j$,

$$2 \approx \sum_{n=0}^{N}{}' \hat{a}_n \left(\sum_{l=n}^{k+n} b_{l-n} T_l(x) + \sum_{l=\max(n-k,0)}^{n} b_{n-l} T_l(x) + \sum_{l=1}^{k-n} b_{l+n} T_l(x)\right). \tag{47}$$

The coefficients in front of T_0 to T_N are set equal on both sides, and a system of linear equations for the $\hat{a}_n$ ensues:

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ b_1 & 2b_0+b_2 & b_1+b_3 & b_2+b_4 & \cdots \\ b_2 & b_1+b_3 & 2b_0+b_4 & b_1+b_5 & \cdots \\ b_3 & b_2+b_4 & b_1+b_5 & 2b_0+b_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \vdots \\ \vdots \\ \hat{a}_N \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \tag{48}$$

where the symmetric coefficient matrix $B_{r,c}$ has a band width of $2k+1$:

$$B_{r,c} = \begin{cases} b_c, & r=0, \\ b_r, & c=0, \\ 2b_0 + b_{2r}, & r=c\neq 0, \\ b_{|r-c|} + b_{r+c}, & r\neq c, \quad c>0, \quad r>0. \end{cases} \tag{49}$$

This gives access to a set of approximate, low-indexed a_n with no need to evaluate integrals nor with reference to the roots of $\sum b_j T_j$. The extended division problem of finding the $\hat{a}_n$ from given f_n in

$$\frac{f(x)}{\sum_{j=0}^{k} b_j T_j(x)} \approx \sum_{n=0}^{N}{}' \hat{a}_n T_n(x), \qquad f(x) \equiv \sum_{n=0}^{\infty}{}' f_n T_n(x), \tag{50}$$

has the right-hand side in (48) modified as follows:

$$\sum_{c=0}^{N} B_{r,c} \hat{a}_c = \begin{cases} f_r, & r=0, \\ 2f_r, & r=1,2,3,\ldots . \end{cases} \tag{51}$$

Note 5. The shifted Chebyshev polynomials $T^*(x) \equiv T(2x-1)$ are orthogonal over [0, 1] with weight $1/\sqrt{x(1-x)}$ [1, (22.2.8)] [29]. From (8) we get

$$\frac{1}{(z-x)^s} = 2^s \sum_{n=0}^{\infty}{}' a_{n,s}(2z-1) T_n^*(x). \tag{52}$$

The relations (50) *hold for the shifted polynomials as well*, *if all three T are substituted by* T^*.

3. Chebyshev approximation for the relative error

The maximum *absolute* error in $f(x)$ of its truncated Chebyshev series in (50) is estimated at $\sum_{n=k}^{N}|f_n|$ if terms up to k were retained; the maximum *relative* error of the polynomial approximation $\sum b_j T_j$ is estimated at $\sum_{n=0}^{\prime N}|\hat{a}_n|-1$. To optimize the approximation of $f(x)$ for the relative error

$$R(x)\equiv\frac{f(x)}{\sum_{j=0}^{k}b_jT_j(x)}-1 \tag{53}$$

in $[-1,1]$, one would rather like to find the $k+1$ coefficients b_j in (50) which force the relative error to be close to zero in the sense of

$$\hat{a}_0=2,\quad \hat{a}_1=\hat{a}_2=\hat{a}_3=\cdots=\hat{a}_k=0. \tag{54}$$

The rationale is that removal of the ripples of $T_1(x)$ to $T_k(x)$ from the quotient expansion leaves a quotient with an appropriate number of "critical" points required by the alternating maximum theorem [6,11,27,37]. The "dangling" $\hat{a}_{n>k}$ absorb these residuals similar to terms in the "τ-method" [6, p. 414] and the "telescoping" procedure [23]. As an inversion of the problem of Section 2.3, the matrix B in (51) is presumed unknown (up to some symmetry), and the first $k+1$ elements of the vector $\hat{a}_c$ and all elements of f_r are known. Contrary to the task of finding rational approximations to $f(x)$ [12,13,25], $f(x)$ is to be split into a product of a polynomial of degree k by a function close to unity.

Note 6. The case $r=0$ in (51) in conjunction with (54) mandate

$$b_0=f_0/2. \tag{55}$$

Finding the constituents b_j of B that solve the bi-linear (51) may proceed with a multivariate first-order Newton method [17]. The familiar Newton step $f^{(n)}+f'^{(n)}(x^{(n+1)}-x^{(n)})=f^{(n+1)}$ which updates $x^{(n)}\mapsto x^{(n+1)}$ in the scalar case reads $\hat{a}_l+\sum_j\frac{\partial\hat{a}_l}{\partial b_j}\Delta_j=0$ in our variables, and is executed in (59):

Algorithm 1. An iterative approach to solving (49) and (51) for known $\hat{a}_0,\hat{a}_1,\ldots,\hat{a}_k$, known r.h.s. $f_0,f_1,\ldots,f_N$, but unknown $b_1,\ldots,b_k$ and unknown $\hat{a}_{k+1},\hat{a}_{k+2},\ldots,\hat{a}_N$ is:

(1) Choose a start solution b_j, for example the obvious [10, p. 77]

$$b_j=\begin{cases}f_0/2, & j=0,\\ f_j, & j=1,2,\ldots,k.\end{cases} \tag{56}$$

(2) Compute approximate $\hat{a}_n$ (n=0, …, N) from b_j by solving the linear system of equations (51). Terminate—taking the current b_j as the result—if the $\hat{a}_0$ to $\hat{a}_k$ are sufficiently close to the desired (54).

(3) Compute an approximate $(N+1)\times k$ Jacobi matrix

$$J_{r,c}=\begin{pmatrix}\frac{\partial\hat{a}_0}{\partial b_1} & \frac{\partial\hat{a}_0}{\partial b_2} & \cdots & \frac{\partial\hat{a}_0}{\partial b_k}\\ \frac{\partial\hat{a}_1}{\partial b_1} & \frac{\partial\hat{a}_1}{\partial b_2} & \cdots & \frac{\partial\hat{a}_1}{\partial b_k}\\ \vdots & \vdots & \cdots & \vdots\\ \frac{\partial\hat{a}_N}{\partial b_1} & \frac{\partial\hat{a}_N}{\partial b_2} & \cdots & \frac{\partial\hat{a}_N}{\partial b_k}\end{pmatrix} \tag{57}$$

by partial derivation of the first $N+1$ equations off (51) w.r.t. the b_j, i.e., by solving the k systems of $N+1$ linear equations

$$\sum_{c=0}^{N} B_{r,c} J_{c,j} = -\begin{pmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \dots & \hat{a}_{k-1} & \hat{a}_k \\ \hat{a}_0+\hat{a}_2 & \hat{a}_1+\hat{a}_3 & \hat{a}_2+\hat{a}_4 & \dots & \dots & \hat{a}_{k-1}+\hat{a}_{k+1} \\ \hat{a}_1+\hat{a}_3 & \hat{a}_0+\hat{a}_4 & \hat{a}_1+\hat{a}_5 & \dots & \dots & \hat{a}_{k-2}+\hat{a}_{k+2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hat{a}_{N-1} & \hat{a}_{N-2} & \hat{a}_{N-3} & \dots & \hat{a}_{N-k-1} & \hat{a}_{N-k} \end{pmatrix} \tag{58}$$

for $r=0,\dots,N$ and $j=0,\dots,k-1$. The column $\partial\hat{a}_c/\partial b_0$ of the Jacobi matrix is not calculated, as b_0 is assumed fixed according to (55).

(4) Compute the next iterated solution $b_j \mapsto b_j + \Delta_j$ ($j=1,2,\dots,k$) of the polynomial coefficients by solving the system of k linear equations

$$\sum_{j=1}^{k} k\frac{\partial\hat{a}_l}{\partial b_j}\Delta_j = -\hat{a}_l, \quad l=1,\dots,k \tag{59}$$

for the first-order differences Δ_j. This equation is the first-order multivariate Taylor expansion of $\hat{a}_l$ as a function of the b_j set to the desired zeros (54) for this update. The $k\times k$ coefficient matrix $\partial\hat{a}_l/\partial b_j$ is a square submatrix of the Jacobi matrix J calculated in the previous step.

(5) Return to step (2) for the next cycle.

Criterion 7. *The algorithm cannot find polynomials with a uniformly convergent Chebyshev expansion of the relative error if $f(x)$ has zeros in* $[-1,1]$.

Tests run for $f(x)=\sin(\pi x/2)/x$, $\cos(\pi x/2)/(1-x^2)$, $(1/x)\arcsin(x/\sqrt{2})$, $\exp(x)$, and $J_0(\pi x/2)$ started from their estimates (56) show rapid convergence within the first update in the sense that $|\Delta_j/b_j|<4\times 10^{-19}$ ($j\leqslant k=14$; $N=3k$) for the second update of all five functions (The algorithm diverges for any of the five test functions started from the "blind" estimate $b_0=f_0/2$, $b_j=0$ ($j=1,2,\dots,k$). But initial estimates of comparable low quality are not of practical importance, because the f_r need anyway to be known for step (2) of the algorithm.).

Criterion 8. *In the region of convergence, the Newton method converges linearly or quadratically* [38]. *A test against a sufficient convergence criterion* [17, Section 3.2] *can be made after passing* (59) *by multiplying the norm of the inverse of its matrix $\partial\hat{a}_l/\partial b_j$ by the norm of the vector Δ_j and by the norm of the matrix of the second derivatives. Since the second derivatives of B w.r.t. b are all zero, each $\partial^2 a_l/(\partial b_j \partial b_r)$ follows from a linear system of equations similar to* (58) *with the r.h.s. replaced by sum of the derivative of the rth column w.r.t. b_j plus the derivative of the jth column w.r.t. b_r, both of which have already been computed via* (59).

Note 7. This algorithm involves only f_0 *to* f_N, but no higher order approximants to $f(x)$. It adapts a polynomial of degree k to an order-N representation of $f(x)$. The algorithm is "lossy" in the sense that it is only called to reduce $N>k$ inputs to k outputs (Otherwise, if $N\leqslant k$, the best and trivially exact adaptation that leads to zero absolute and zero relative error is to copy the input to the output with (56), complemented by $b_{N+1}=\dots=b_k=0$.).

A set of b_j found that way is also a starting point to calculate the solution with the minimax property of the relative error. The Remez exchange algorithm [30] applied to $R(x)$ could take advantage from the specific polynomial format of $R(x)$, which supports parallel updates of all nodes: (i) all extrema of $R(x)$ are found by searching all roots of a polynomial of degree $k+N-1$, and (ii) adjusting the $b_j \mapsto b_j+\Delta_j$ with a Newton method such that the absolute values of the new alternating extrema equal the mean of the old ones ends up in a linear system of equations for the Δ_j. If the truncated representation (56) already provides a good set of coefficients to start the Remez iterations, the Algorithm 1 as a pre-conditioner to find new b_j might as well be skipped.

4. Conclusion

The expansion coefficients of the Chebyshev series of inverse polynomials can be derived from the partial fraction decomposition of the inverse polynomial. Unlike other iterative or sampling algorithms, this algorithm is *exact*—though its incarnation in finite floating point arithmetic may be not. In the form presented here, the effort grows linearly with the polynomial degree k and also linearly with the number of terms seeked in the expansion.

We have shown how expansion coefficients with indices larger than the polynomial degree are recursively linked to those of lower order. This algorithm is *complementary* to the aforementioned one and again *exact*: it starts from the low-indexed Chebyshev expansion coefficients which have to be known by any other means (optionally the partial fraction method). Compared to the partial fractions method there is no direct access to a Chebyshev coefficient of arbitrary index n: the expense grows proportional to the square k^2 of the polynomial degree (one power to update the vector of the recursion coefficients, and one power to apply it to the recursion) and proportional to n. In practice, one is almost always interested in computing a contiguous series of Chebyshev coefficients indexed 0 to n, which reduces the expense *per coefficient* from $\propto nk^2$ to $\propto k^2$. Another inherent disadvantage is the thread of loss of precision if high indices n are obtained numerically.

An iterative algorithm has been presented which derives a polynomial of a given degree such that the first terms of the Chebyshev expansion of the *relative* error of a given function represented by this polynomial vanish. Its key achievement is to dissect known from unknown parameters in an efficient scheme solely based on solving linear equations. The solutions economize the polynomial representation of functions such that the number of valid mantissa bits in an IEEE representation is optimized over $[-1, 1]$. The output inherently differs from solutions that minimize the *absolute* error over $[-1, 1]$—to which algorithms are known in the literature—if the function values are strongly fluctuating over the interval. The disadvantage compared to a Remez algorithm is that the results provide a mere *near*-minimax solution by construction, and that a Chebyshev representation of the functions must be at hand; the advantage is that no numerical search for extrema is involved.

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