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On Variational Inequalities for Monotone Operators, I*

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DEDICATED TO MY TEACHER, PROFESSOR ERICH H. ROTHE, ON HIS EIGHTY-SECOND BIRTHDAY

1. INTRODUCTION

Let X be a topological vector space over the reals R. Let Y be a topological group, and let $\langle\cdot,\cdot\rangle: X \times Y \to R$ be a "bilinear form" in the sense: for each fixed y, $\langle\cdot, y\rangle$ maps X linearly into R, and for each fixed x, $\langle x, \cdot\rangle$ is a homomorphism of Y into the additive reals. We denote the group-operation by $(+)$, though it is not really necessary that the group be Abelian. (To fix the ideas, think of Y as a Banach space taken with the norm topology, X as its adjoint space with the weak-star topology, and $\langle x, y\rangle = x(y)$.)

Let Y^w be the set Y taken with the weakest topology in which all the functions $\langle x, \cdot\rangle$ are continuous. Let C be a subset of X (in applications, often closed and convex) and think of F as being a function mapping C into Y. Let $D \subset C$ be convex and compact, and let the restriction of $\langle\cdot,\cdot\rangle$ to $(D - D) \times Y$ be continuous.

We are interested in finding a solution in D for the (infinite) system of "variational inequalities"

$$\langle x' - x, F(x)\rangle \geqslant 0 \qquad (x' \in C). \tag{1}$$

This problem has (in essence) been treated in the literature under the assumption that F is a "pseudomonotone" operator in the sense of H. Brézis, using a Galerkin-approximation method due essentially to F. E. Browder. One first solves the problem with $C = D$ and then imposes a "coercivity condition" to treat larger sets C. The prototype of a "pseudomonotone operator" is $F(x) = f(x, x)$, where f depends in one way on first argument and in a different way on second argument (we need not go into details here, since it is precisely this prototype-operator which we shall treat in this paper). A definitive (to date) treatment of the problem (with $C = D$) is essentially a special case of the variant by Brézis *et al.* [1] on Ky Fan's minimax theorem.

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The purpose of this paper is threefold. We shall first of all give a treatment of the prototype-problem, with $F(x) = f(x, x)$, based on quite different principles—the argument is very simple and does not use the Galerkin method. (Thus our results are sufficient for most known applications, and the argument may lead to a different class of generalizations.) Secondly, we make an inquiry into the fundamental nature of "coercivity conditions." Third, we apply our results to the solution of a version of Hammerstein's nonlinear integral equation $x + KFx = 0$. In virtually all formulations of this problem to date, K is taken as a linear integral operator, but in our treatment K (as well as F) may be taken as a nonlinear operator. Our present treatment is essentially a radical improvement on the treatment given by Vainberg in [6] (which was based on the treatment by Dolph and Minty in [3]).

2. The Solution of Variational Inequalities over a Compact Set

In this section, $f(x_1, x_2)$ is a mapping from $D \times D$ into Y satisfying certain conditions, as follows:

(A) For each fixed x_2, $f(\cdot, x_2)$: $D \to Y^w$ is a monotone (relative to $\langle \cdot, \cdot \rangle$) function whose restriction to any line segment in D is continuous, i.e., a "hemicontinuous" monotone function. "Monotone," in this context, means: for any $x', x \in D$, $\langle x' - x, f(x') - f(x) \rangle \geqslant 0$.

(B) For each fixed x_1, $f(x_1, \cdot)$: $D \to Y$ is continuous. (In the Banach-space context mentioned above, this hypothesis corresponds to "continuous from the weak-star topology in X to the norm-topology in Y." On the other hand, if no second argument is present in f, it is convenient to take Y with the discrete topology, and the hypothesis is vacuous.)

This section is devoted to the proof of

Theorem 1. *The subsystem of inequalities* (1) *for which* $x' \in D$, *namely,*

$$\langle x' - x, f(x, x) \rangle \geqslant 0 \qquad (x' \in D), \tag{2}$$

has a solution in D.

Lemma 1. *Any solution of the "auxiliary inequalities,"*

$$\langle x' - x, f(x', x) \rangle \geqslant 0 \qquad (x' \in D), \tag{3}$$

satisfies inequalities (2).

Proof. Consider any $x' \in D$, and real t with $0 < t \leqslant 1$. Substitute $x + t(x' - x)$ for the x' of (3), then cancel x and t and let t tend to zero, invoking the hemicontinuity of f.

Remark. The converse statement is true, and follows easily from the monotonicity of f; we shall not, however, need this converse. It is useful for proving "the solution set is convex" in case no second argument appears in f.

LEMMA 2. *Consider a finite subsystem of* (3):

$$\langle x_i' - x, f(x_i', x)\rangle \geqslant 0 \qquad (i = 1,..., n).$$

These inequalities have a solution in D.

Proof. We shall show that the inequalities,

$$\langle x_i' - \Sigma_j \lambda_j x_j', f(x_i', \Sigma_k \lambda_k x_k')\rangle \geqslant 0 \qquad (i = 1,..., n), \tag{4}$$

$$\Sigma_i \lambda_i = 1, \qquad \lambda_i \geqslant 0 \qquad (i = 1,..., n), \tag{5}$$

can be solved for $\lambda_1 ,..., \lambda_n$.

The solutions of (5) constitute a simplex S. Let C_i be the set of all $(\lambda_1 ,..., \lambda_n) \in S$ satisfying (4), for $i = 1,..., n$. Our continuity hypotheses suffice to show each C_i is a closed subset of S.

Now, by an easy computation

$$\Sigma_i \lambda_i \langle x_i' - \Sigma_j \lambda_j x_j', f(x_i', \Sigma_k \lambda_k x_k')\rangle$$
$$= \tfrac{1}{2} \Sigma_{i,j} \lambda_i \lambda_j \langle x_i' - x_j', f(x_i', \Sigma_k \lambda_k x_k') - f(x_j', \Sigma_k \lambda_k x_k')\rangle.$$

By the monotonicity of f, the right side is nonnegative, hence so is the left side. From these considerations we see easily that S is contained in the union of $C_1 ,..., C_n$, and a little more thought (putting some of the λ_i equal to zero) shows that each face of S is contained in the union of those C_i numbered the same as the vertices of that face. Thus all the hypotheses of the well-known Knaster–Kuratowski–Mazurkiewicz Lemma are satisfied, so the intersection of the C_i is nonempty.

(The reader who is interested in the history-of-ideas of Lemma 2 can follow it through the papers [2, 4, 5, 7]; the writer has also profited from discussions with W. Oettli.)

(The Knaster–Kuratowski–Mazurkiewicz Lemma is an easy consequence of Sperner's Lemma, a purely combinatorial fact. In case the second argument is missing, many proofs of Lemma 2 can be given which do not use Sperner's Lemma—see the above papers, or the remark of Valentine [8] that the K–K–M Lemma can be proved by methods of convexity theory if the sets C_i are convex.)

LEMMA 3. *Inequalities* (3) *have a solution in D.*

Proof. The set of solutions of each inequality is a closed subset of D; the conclusion follows by the "finite intersection property" for the compact set D and Lemma 2.

Theorem 1 now follows by Lemma 1.

3. Coercivity Conditions

At this point, our need to regard $F(x)$ as $f(x, x)$ has disappeared, and $F(x)$: $D \to Y$ can be regarded in this section as any function for which the conclusion of Theorem 1 holds; if desired, one can think of the domain of F as being C.

We now introduce a sort of "universal coercivity condition":

Definition 1. Let D' be a subset of D. We shall say that "Condition 'C' holds over D'" provided: for each $x \in D'$, the convex cone K_x generated by the sets,

$$S_x = \{x' - x: x' \in D\}$$

and

$$T_x = \{h: \langle h, F(x)\rangle \geqslant 0\},$$

contains the set $\{x' - x: x' \in C \setminus D\}$.

Lemma 4. *Let x be any solution of*

$$\langle x' - x, F(x)\rangle \geqslant 0 \qquad (x' \in D). \tag{6}$$

Then x satisfies inequalities (1) if and only if Condition "C" holds over the one-element set $\{x\}$.

Proof. The necessity is obvious—note that S_x can even be dispensed with. The sufficiency is trivial—note $S_x \subset T_x$.

It is now clear that Theorem 1, together with the hypothesis "Condition 'C' holds over D," suffices for a proof of existence of a solution of inequalities (1). But a weaker condition suffices:

Lemma 5. *Necessary and sufficient for Condition "C" to hold over D is: Condition "C" holds over the set of noninternal points of D for which $S_x \subset T_x$.*

Proof. If x is an internal point of D, then $S_x = X$, hence $K_x = X$. Also, if S_x is not a subset of T_x, it is a routine matter to show $K_x = X$. (A simple two-dimensional picture guides the proof.)

Lemma 6. *Let $C \subset X$ be arbitrary and suppose D has the property: for each noninternal $x \in D$, there is a unique "half-space of support" to D, or more precisely, that there is a unique (up to a multiplicative positive constant) $y \in Y$ such that for all $x' \in D$, $\langle x' - x, y\rangle \geqslant 0$ but $\langle \cdot, y\rangle$ is not identically zero. Then sufficient for Condition "C" to hold over D is: for each noninternal $x \in D$, $F(x)$ is not proportional to y by a positive proportionality-constant.*

Lemma 5 makes Lemma 6 obvious. (Note that the "uniqueness" of y implies a certain nondegeneracy of $\langle \cdot, \cdot\rangle$. In order to give Lemma 6 a more familiar appearance, think of X as a Banach space in which the norm is Fréchet differentiable, Y as its adjoint space, and D as the unit ball.)

LEMMA 7. *Let* $C \subset X$ *be arbitrary. Then sufficient for Condition "C" to hold over* D *is*: $\mathbf{O}$ *is an internal point of* D *and, for each noninternal* $x \in D$, $\langle x, F(x) \rangle \geqslant 0$.

Proof. Under the hypotheses, for noninternal $x \in D$: either $\langle \cdot, F(x) \rangle$ is identically zero and $K_x = T_x = X$, or there exists an $x' \in D$ "close to $\mathbf{O}$" such that $\langle x', F(x) \rangle < 0$, whence $\langle x' - x, F(x) \rangle < 0$ and S_x is not a subset of T_x.

In the early literature of monotone operator theory, one saw over-strong coercivity conditions; for example: $X = Y =$ Hilbert space, D a ball of radius $M > 0$ about the origin as center, and $\langle x, F(x) \rangle \geqslant 0$ for all x with $\| x \| = M$. Under these hypotheses, one could apply either Lemma 6 or Lemma 7. Lemma 6 resembles coercivity conditions invoked by F. E. Browder in the context of "reflexive Banach spaces," while Lemma 7 appears to be a new type of coercivity condition.

In Lemma 6 the set D occupies a salient position which is occupied in Lemma 7 by $\{\mathbf{O}\}$; we leave to the reader the formulation and proof of a lemma in which an arbitrary set intermediate between $\{\mathbf{O}\}$ and D occupies that position.

One is tempted to think that Lemmas 4 and 5 are "sufficient for all purposes in coercivity theory"; in the next section we shall see an example in which somewhat more ingenuity is required.

4. ON HAMMERSTEIN'S INTEGRAL EQUATION WITH NONLINEAR K

Let X_1, X_2 be two topological vector spaces over the reals. Let X_1', X_2' be the additive groups of these spaces, endowed with structures of "topological group." Let $\langle \cdot, \cdot \rangle$: $X_1 \times X_2 \to R$ be a bilinear form continuous on $(D_1 - D_1) \times X_2'$ and on $X_1' \times (D_2 - D_2)$, where D_1, D_2 are compact convex sets in X_1, X_2, respectively. (To fix the ideas, take X_1 and X_2 as a pair of mutually dual reflexive Banach spaces taken with their weak topologies, take $\langle \cdot, \cdot \rangle$ as the usual pairing, and take X_1', X_2' as the same spaces with their norm-topologies —or possibly, one or the other or both could be taken with the discrete topology.)

Let F: $D_1 \to X_2'$ and K: $D_2 \to X_1'$ be $f(x, x)$ and $k(x, x)$, where f: $D_1 \times D_1 \to X_2'$ and k: $D_2 \times D_2 \to X_1'$, these functions being monotone (with respect to $\langle \cdot, \cdot \rangle$) and hemicontinuous in their first arguments (recall that $X_1{}^w$ and $X_2{}^w$ are introduced in the definition of "hemicontinuous") and continuous in their second arguments.

We wish to find $(x_1, x_2) \in D_1 \times D_2$ such that

$$\begin{cases} K(x_2) + x_1 = 0 \\ F(x_1) - x_2 = 0. \end{cases} \qquad (7)$$

(An "equation of Hammerstein type" $x + KFx = 0$ appears on elimination of x_2.)

All hypotheses on D_1, D_2, K, F have so far been essentially "translation invariant." We now hypothesize:

(1°) $O \in X_1$ is an internal point of D_1, and $O \in X_2$ is an internal point of D_2.

(2°) $\langle K(x_2), x_2 \rangle \geqslant 0$ for all $x_2 \in D_2$.

(3°) For each noninternal point x_1 of D_1, $\langle x_1, F(x_1) \rangle \geqslant 0$.

(4°) The image under F of the set of internal points of D_1 consists entirely of internal points of D_2.

(5°) The bilinear form $\langle \cdot, \cdot \rangle$ is "nondegenerate": if $\langle x, y \rangle \geqslant 0$ for all y then $x = 0$, and vice versa.

(Again, to fix the ideas with the "reflexive Banach space" example: let D_1 be a ball containing the origin "so large" that (3°) holds, and then let D_2 be a ball containing the origin "so large" that (4°) holds. Note that (2°) is a vestigial form of the hypothesis "K is a positive semidefinite, but not necessarily self-adjoint, linear operator" and (4°) is a less-vestigial form of "F maps bounded sets into bounded sets," since in applications one has freedom in choice of D_2. Note that there is no hypothesis of the type "K maps D_2 into D_1" which one would expect to see in arguments using fixpoint theorems.)

THEOREM 2. *Under the hypotheses listed above, equations* (7) *have a solution in* $D_1 \times D_2$.

Proof. We shall work with product spaces, all taken with the "usual" direct-sum (resp. direct-product) structures and topologies.

Define the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle : (X_1 \times X_2) \times (X_1 \times X_2) \to R$ by

$$\langle\!\langle (x_1, x_2), (x_1', x_2') \rangle\!\rangle = \langle x_1, x_2' \rangle + \langle x_1', x_2 \rangle$$

and $\mathscr{F}: D_1 \times D_2 \to X_1' \times X_2'$ by

$$\mathscr{F}(x_1, x_2) = (K(x_2) + x_1, F(x_1) - x_2).$$

It is now routinely verifiable that $\mathscr{F}$ and $D_1 \times D_2$ satisfy the hypotheses of Theorem 1; $\mathscr{F}$ is now to be interpreted as $f((x_1, x_2), (x_1, x_2))$, where f: $(D_1 \times D_2) \times (D_1 \times D_2) \to X_1' \times X_2'$ is monotone (with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$) and hemicontinuous in first argument and continuous in second argument. (Note that the "$+x_1$" and "$-x_2$" go with the first argument.)

We conclude: there exists $(x_1, x_2) \in D_1 \times D_2$ such that

$$\langle\!\langle (x_1' - x_1, x_2' - x_2), (K(x_2) + x_1, F(x_1) - x_2) \rangle\!\rangle \geqslant 0$$

or

$$\langle x_1' - x_1, F(x_1) - x_2 \rangle + \langle K(x_2) + x_1, x_2' - x_2 \rangle \geqslant 0 \tag{8}$$

(for all $(x_1', x_2') \in D_1 \times D_2$).

For later reference, we point out that by putting $x_2' = x_2$ (resp. $x_1' = x_1$) we obtain

$$\begin{aligned} &\langle x_1' - x_1, F(x_1) - x_2 \rangle \geqslant 0 \qquad (\text{all } x_1' \in D_1), \\ &\langle K(x_2) + x_1, x_2' - x_2 \rangle \geqslant 0 \qquad (\text{all } x_2' \in D_2). \end{aligned} \tag{9}$$

We shall now prove the assertion of the theorem "by contradiction." Suppose one (or both) of (7) fail(s) to be satisfied. Then by (5°) and (1°) we can choose $(x_1', x_2') \in D_1 \times D_2$ such that

$$\langle x_1', F(x_1) - x_2 \rangle \leqslant 0,$$

$$\langle K(x_2) + x_1, x_2' \rangle \leqslant 0,$$

where one or both of these is a *strict* inequality. By addition,

$$\langle x_1', F(x_1) - x_2 \rangle + \langle K(x_2) + x_1, x_2' \rangle < 0.$$

But then from (8) we obtain

$$0 > \langle x_1, F(x_1) \rangle + \langle K(x_2), x_2 \rangle.$$

Thus from (2°), $\langle x_1, F(x_1) \rangle < 0$, and from (3°), x_1 is an internal point of D_1. But then from (9), $x_2 = F(x_1)$ and from (4°), x_2 is an internal point of D_2. Thus from (9) and (5°), $K(x_2) + x_1 = 0$, and we have the desired contradiction.

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