On the Order of the Commutator Subgroup
and the Schur Multiplier of a Finite $p$-Group

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TO MY DEAR FRIEND BORIS M. SCHEIN
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Green [Proc. Roy. Soc. Math A 327 (1956), 574–581] proved that if $G$ is a finite $p$-group of order $p^n$ and $M(G)$ is its Schur multiplier of order $p^{\mu(G)}$, then $m(G) \leq \frac{1}{2}n(n-1)$. We consider, among other things, the case when $m(G) = \frac{1}{2}n(n-1)$ and then prove that if this equality holds, then $G = E(p^n)$, that is, $G$ is an elementary abelian group of order $p^n$.

We need the following:

**Lemma 1** (Schur; see Proposition 4.1.3 in [2]). Let $G = C(p^{e_1}) \times \cdots \times C(p^{e_d})$, where $e_1 \leq \cdots \leq e_d$, $e_1 + \cdots + e_d = n$, and $C(n)$ is a cyclic group of order $n$. Then $M(G) = C(p^{e_1})^{d-1} \times C(p^{e_2})^{d-2} \times \cdots \times C(p^{e_d})$, where $H^{(1)}$ is a direct product of $t$ copies of a group $H$ and $H^{(0)} = 1$.

**Corollary 2.** Let $G$ be an abelian group of order $p^n$ as in Lemma 1. Then $m(G) = (d-1)e_1 + (d-2)e_2 + \cdots + e_{d-1} = (n-e_d) + (n+e_d-e_{d-1}) + \cdots + (n-e_d-\cdots-e_2) \leq (n-1) + (n-2) + \cdots + (n-d+1) \leq \frac{1}{2}n(n-1)$. Also, $m(G) = \frac{1}{2}n(n-1) \iff G = E(p^n)$.

We use the following:

**Lemma 3** (Wiegold; see Lemma 4.1.12 in [2]). Suppose that $G/Z(G)$ is a $p$-group of order $p^n$. Then $G'$ is a $p$-group and $|G'| = p^t$ for $t \leq \frac{1}{2}n(n-1)$.

**Theorem 4.** Let $G$ be a finite $p$-group of order $p^n$. If $m(G) = \frac{1}{2}n(n-1)$ then $G = E(p^n)$.
Proof. Let $\Gamma$ be a representation group of $G$. Then $\Gamma' \cap Z(\Gamma)$ contains a subgroup $M \cong M(G)$ with $\Gamma/M \cong G$. By Lemma 3,

$$p^{(1/2)n(n-1)} = |M| \leq |\Gamma'| \leq p^{(1/2)n(n-1)},$$

and hence $M = \Gamma'$ and $G \cong \Gamma/M = \Gamma'/\Gamma'$ is abelian.

Now $G = E(p^n)$ by Corollary 2.

Theorem 6 investigates a more general situation when $|G/Z(G)| = p^n$ and $|G'| = p^{(1/2)n(n-1)}$.

Let $Z_2(G)$ be the second term of the upper central series of a $p$-group $G$.

**Lemma 5.** Let $G$ be a $p$-group with $|G/Z(G)| = p^n$. Then $|G'| = p^{(1/2)n(n-1)-s}$, where $s \geq 0$ is an integer and $|(G/Z(G))'| \leq p^{1+s}$. If $|(G/Z(G))'| = p^{1+s}$, then $Z_2(G)/Z(G)$ has exponent $p$.

Proof. Without loss of generality assume that $G$ is nonabelian. Then $G/Z(G)$ is not cyclic and $n > 1$.

Let $z_0$ be a fixed element of $Z_2(G) - Z(G)$. For $x \in G$ let $\phi(x) = [x, z_0] = x^{-1}z_0^{-1}xz_0$. Then, by Grün's Lemma, $\phi$ is a homomorphism of $G$ into $[G, z_0]$. Set $\text{Im } \phi = N$ and $|N| = p^s$. Obviously, $\text{Ker } \phi = C_G(z_0) \supseteq (z_0, Z(G)) > Z(G)$. Therefore $|G : C_G(z_0)| \leq p^{n-1}$. Since $N \cong G/C_G(z_0)$, we obtain $v \leq n - 1$. Let $p^b = |G/N : Z(G/N)|$. Since $z_0NZ(G/N)$, we have $b \leq n - 1$. Now $N = [G, z_0] \leq G'$, so that $|G'| = |N| \cdot |G'/N|$. By induction, $|G'/N| \leq p^{(1/2)b(b-1)}$. Therefore

$$|G'| \leq p^{(1/2)b(b-1)+v}. \quad (1)$$

Since $b \leq n - 1$ and $v \leq n - 1$, we obtain

$$|G'| \leq p^{(1/2)n(n-1)} \quad \text{ and } \quad |G'| = p^{(1/2)n(n-1) - s} \quad (2)$$

with an integer $s \geq 0$. By (1) we have $v \geq \frac{1}{2}n(n-1) - s - \frac{1}{2}b(b-1) \geq \frac{1}{2}n(n-1) - s - \frac{1}{2}(n-1)(n-2) = n - 1 - s$. Hence, $p^{n-1-s} \leq p^v = |N| = |G : C_G(z_0)|$. Since $G/C_G(z_0) \cong N \leq Z(G)$ is abelian, we obtain $G'/Z(G) \leq C_G(z_0)$. Therefore $|G'/Z(G)| \leq |G : C_G(z_0)| \geq p^{n-1-s}$. This inequality implies $|(G/Z(G))'| \leq |G/Z(G)|/p^{n-1-s} = p^s/p^{n-1-s} = p^{s+1}$.

Now let $|(G/Z(G))'| = p^{1+s}$. Since $G'/Z(G) \leq C_G(z_0)$, for every $z_0 \in Z_2(G) - Z(G)$, we have $|G : C_G(z_0)| \leq |G : G'/Z(G)| - |G/Z(G) : (G/Z(G))'| \leq p^{n-2}$. Since $|N| = |G : C_G(z_0)|$, we obtain $v \leq n - 1 - s$. Suppose that $z_0 \notin Z(G)$ for some $z_0 \in Z_2(G) - Z(G)$ (in other words, exp $Z_2(G)/Z(G) > p$). It follows from $z_0NZ(G/N)$, where $N = [G, z_0]$, that $|G/Z(G)| \leq p^{n-2}$. Thus, by Lemma 3 or by induction, $|G'/N| \leq p^{(1/2)n(n-2)(n-3)}$ and $|G'| = |N| \cdot |G'/N| \leq |G : G'/Z(G)| \cdot |G'/N| \leq p^{n-1-s + (1/2)(n-2)(n-3)}$, or $\frac{1}{2}n(n-1) - s \leq n - 1 - s + \frac{1}{2}(n-2)(n-3)$ and $n \leq 2$. Thus $G/Z(G)$ is abelian.
of order \( p^n \leq p^2 \), and \( G = Z_2(G) \). Since \( \exp Z_2(G)/Z(G) > p \), \( G/Z(G) \) is cyclic and \( G \) is abelian, which is a contradiction.

A \( p \)-group \( G \) is called extraspecial if its center \( Z(G) \) coincides with its commutator subgroup \( G' \) and has order \( p \). The order of an extraspecial group is \( p^{1 + 2m} \) and we denote such a group by \( ES(m, p) \).

**Theorem 6.** Let \( G \) be a \( p \)-group such that \( |G/Z(G)| = p^n \). If \( |G'| = p^{(1/2)n(n-1)} \) then either \( G/Z(G) = E(p^n) \) or \( G/Z(G) = ES(\frac{1}{2}(n-1), p) \).

**Proof.** We have \( s = 0 \) in notations of Lemma 5. Then, by Lemma 5, \( |(G/Z(G))'| \leq p \).

Let \( G/Z(G) \) be abelian. Take \( z_0 \in G - Z(G) \). If \( z_0^b \notin Z(G) \), then \( b \leq n-2 \) in notations of Lemma 5, so that \( \frac{1}{2}n(n-1) = \log_p |G'| \leq \frac{1}{2}(n-2)(n-3) + (n-2) = \frac{1}{2}(n-1)(n-2) \), which is a contradiction. In this case, \( G/Z(G) = E(p^n) \), an elementary group of order \( p^n \).

Now let \( |G/Z(G))'| = p \). Then, by Lemma 5, \( \exp Z_2(G)/Z(G) = p \). Suppose that \( Z_2(G)/Z(G) \) contains two distinct subgroups \( A/Z(G) \) and \( B/Z(G) \) of order \( p \). Let \( A = \langle z_0, Z(G) \rangle \) and \( B = \langle y_0, Z(G) \rangle \). By (1) we have \( v = n-1 = b \) for every \( z_0 \in G - Z(G) \). Therefore, \( G/C_G(y_0) \) is abelian of order \( p^{n-1} \). Thus \( G' \leq C_G(z_0) \cap C_G(y_0) = Z(G) \) and \( G/Z(G) \) is abelian, which is a contradiction. Therefore, the abelian group \( Z_2(G)/Z(G) \) of exponent \( p \) has the only subgroup of order \( p \), and hence \( |Z_2(G)/Z(G)| = p \). Since \( |(G/Z(G))'| = p \) and \( (G/Z(G))' \leq Z_2(G)/Z(G) \), we obtain \( (G/Z(G))' = Z_2(G)/Z(G) = Z((G/Z(G))) \) and these are groups of order \( p \). Thus \( G/Z(G) \) is extraspecial of order \( p^n \), that is, \( G/Z(G) = ES(\frac{1}{2}(n-1), p) \).

As a corollary to Theorem 6 we obtain the following:

**Theorem 7.** Let \( G \) be a \( p \)-group of order \( p^n \) and let \( m(G) = \frac{1}{2}n(n-1) - 1 \). Then either \( G = C(p^2) \) or \( G = ES(1, p) \) of exponent \( p > 2 \).

**Proof.** Obviously, \( G = C(p^2) \) and \( G = ES(1, p) \) of exponent \( p > 2 \) satisfy our condition (for the latter group, see Theorem 4.7.3 in [2]).

Now let \( m(G) = \frac{1}{2}n(n-1) - 1 \) and let \( \Gamma \) be the representation group of \( G \). Then \( \Gamma'/Z(\Gamma) \) contains a subgroup \( M \cong M(G) \) with \( \Gamma/M \cong G \). Obviously, \( G' \cong \Gamma'/M \) and \( |\Gamma : Z(\Gamma)| \leq |\Gamma : M| = |G| = p^n \), so that \( \log_p |\Gamma'| = \frac{1}{2}n(n-1) - s \geq m(G) = \frac{1}{2}n(n-1) - 1 \), by Lemma 5, and hence \( s \leq 1 \).

Suppose that \( M < Z(\Gamma) \). Then \( |\Gamma : Z(\Gamma)| \leq p^{n-1} \) and, by Lemma 3 or 5,

\[
\frac{1}{2}(n-1)(n-2) \geq \log_p |\Gamma'| \geq \log_p |M| = m(G) = \frac{1}{2}n(n-1) - 1
\]

and \( n \leq 2 \). In this case \( G = C(p^2) \). Now let \( M = Z(\Gamma) \) and suppose that \( G \) is not cyclic. By Lemma 1, \( G \) is not elementary. Therefore \( n > 2 \). If \( G \) is abelian as in Lemma 1, then, since \( d > 1 \) and \( e_d > 1 \), we have
$m(G) = (n - e_1) + \cdots + (n - e_a - \cdots - e_2) \leq (n - 2) + (n - 3) + \cdots + 1 = \frac{1}{2}(n - 1)(n - 2) < \frac{1}{2}n(n - 1) - 1$ as, by our assumption, $n \geq 3$ in the abelian case.

Now suppose that $G$ is not abelian. Then $M < \Gamma'$. Therefore $\frac{1}{2}n(n - 1) - s = \log_p |\Gamma'| > \log_p |M| = \frac{1}{2}n(n - 1) - 1$, and hence $s = 0$. By Theorem 6, $G \cong \Gamma/Z(\Gamma) = \Gamma/M = ES(\frac{1}{2}(n - 1), p)$. If $n > 3$ then, by Theorem 4.7.3 of [2],

$m(G) = 2\left\{\frac{1}{2}(n - 1)\right\}^2 - \frac{1}{2}(n - 1) - 1 = \frac{1}{2}(n - 1)(n - 2) - 1 < \frac{1}{2}n(n - 1) - 1$,

which is a contradiction. Thus $n = 3$ and $G$ is nonabelian of order $p^3$ and exponent $p$.

Remark. Theorem 6 solves Exercise 6.41 from the book [3] (for further information in this direction, see exercises at the end of Chapter 6 of [3]).

REFERENCES