On growth rates of sub-additive functions for semi-flows: Determined and random cases

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Dedicated to Professor Zhifen Zhang on the occasion of her 80th birthday

Abstract
Let \( \phi : R_+ \times \Omega \times M \to \Omega \times M \) be a measurable random dynamical systems on the compact metric space \( M \) over \( (\Omega, \mathcal{F}, \mathbb{P}, (\sigma(t))_{t \in R_+}) \) with time \( R_+ \). Let \( \mathcal{M}_\mathbb{P}(\phi) \) and \( \mathcal{E}_\mathbb{P}(\phi) \) denote the set of all \( \phi \)-invariant measures on \( \Omega \times M \) and the set of all ergodic \( \phi \)-invariant measures whose marginal on \( \Omega \) coincide with \( \mathbb{P} \) respectively. A function \( F : R_+ \times \Omega \times M \to R \) is sub-additive with respect to \( \phi \) if
\[
F(t+s, \omega, x) \leq F(t, \omega, x) + F(s, \sigma(t)\omega, \phi(t, \omega, x)).
\]
We define the maximal growth rate of \( F \) to be
\[
\limsup_{t \to +\infty} \frac{1}{t} \sup_{x \in M} F(t, \omega, x)
\]
for \( \mathbb{P} \) a.e. \( \omega \). It is shown that it is equal to \( \max \{ \Lambda(\mu) : \mu \in \mathcal{M}_\mathbb{P}(\phi) \} \), where
\[
\Lambda(\mu) = \lim_{t \to +\infty} \frac{1}{t} \int_{\Omega \times M} F_t(x) \, d\mu
\]
and there exists \( \nu \in \mathcal{E}_\mathbb{P}(\phi) \) such that \( \Lambda(\nu) = \max \{ \Lambda(\mu) : \mu \in \mathcal{M}_\mathbb{P}(\phi) \} \). The result may have some applications in the study of the dynamical spectrum of infinite dimension random dynamical systems and robust permanence for differential equations.

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1. Introduction
Consider a continuous-time or discrete-time random dynamical systems \( \phi \) on a compact metric space. An important class of random continuous function \( F : R_+ \times \Omega \times M \to R \) is those that satisfy a sub-additivity with respect to \( \phi \):\[
F(t+s, \omega, x) \leq F(t, \omega, x) + F(s, \sigma(t)\omega, \phi(t, \omega, x)).
\]
These functions naturally arise in various settings including the study of Birkhoff sums, maximal Lyapunov exponents of smooth random dynamical systems [1,15,16], linear skew-product semiflows [6], and principal Lyapunov exponent/spectrum for random/nonautonomous parabolic equation [11]. In all of these settings, the asymptotic behavior of these sub-additive functions as \( t \to \infty \) is considered. In [7], Kingman provided the first systematic study of the long-term behavior of sub-additive functions from an ergodic point of view. Kingman’s sub-additive ergodic theorem assures that \( \frac{1}{t} F(t, \omega, x) \) has a well-defined limit almost surely for any \( \phi \)-invariant measure. Uniform upper bounds for limiting values \( \frac{1}{t} F(t, \omega, x) \) is important in the study of uniform persistence for the systems [8,11] and robust permanence for ecological differential equations [18]. It can be used to study average Lyapunov function and prove that certain positively invariant sets are repelling (see [4,5]).

In [17], the deterministic case was considered for a semi-dynamical systems on a compact metric space \( M \). By this we mean a continuous map

\[
\phi : \mathbb{T}_+ \times M \to M, \quad \phi(t, x) = \phi_t x,
\]

that satisfies \( \phi^0 x = x \) and \( \phi^{s+t} x = \phi^s \phi^t x \) for \( s, t \in \mathbb{T}_+ \), \( \mathbb{T}_+ = \mathbb{R}_+ \), or \( \mathbb{Z}_+ \).

Discrete case associates with \( \mathbb{T}_+ = \mathbb{Z}_+ \), and let \( f = \phi^1 : M \to M \). Then \( f \) is a continuous map and \( F_n(x) \) satisfies sub-additivity with respect \( f \):

\[
F_{n+m}(x) \leq F_n(x) + F_m(f^n(x)).
\]

In [17], it was proved that

\[
\sup_{x \in M} \limsup_{n \to \infty} \frac{1}{n} F_n(x) = \lim_{n \to \infty} \frac{1}{n} \max_{x \in M} F_n(x) = \sup \{ \Lambda(\mu) : \mu \in \mathcal{E}(f) \},
\]

where \( \Lambda(\mu) = \inf_{n>0} \frac{1}{n} \int_M F_n \, d\mu \) and \( \mathcal{E}(f) \) denotes all ergodic measures of \( f \).

This result was extended to continuous case for flow \( \phi_t(x) : M \to M \) (just denote it by \( \phi \) if the meaning is clear). They proved that

\[
\sup_{x \in M} \limsup_{t \to +\infty} \frac{1}{t} F_t(x) = \lim_{t \to +\infty} \frac{1}{t} \max_{x \in M} F_t(x) = \sup \{ \Lambda_\phi(\mu) : \mu \in \mathcal{E}(\phi) \},
\]

where \( \Lambda_\phi(\mu) = \inf_{t>0} \frac{1}{t} \int_M F_t \, d\mu \) and \( \mathcal{E}(\phi) \) denotes all ergodic measures of flow \( \phi \).

The idea of their proof for continuous-time case is to consider time one map \( \phi^1 \) and use the result of discrete case. But it is well known invariant measure for the time one map is not flow invariant in general and an ergodic measure for a flow is not necessarily ergodic for the time one map. So the results of time one map cannot be used directly to the corresponding flow and the last statement in Lemma 3 of [17] is not clear.

In this paper, first we will consider discrete random dynamical systems

\[
f : \Omega \times M \to \Omega \times M, \quad f(\omega, x) = (\sigma(\omega), \phi_\omega(x)),
\]

over an abstract dynamical system \( (\Omega, \mathcal{F}, \mathbb{P}, \sigma) \), where \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a \( \mathbb{P} \)-complete, countably generated probability space with \( \mathbb{P} \) preserving ergodic invertible transformation \( \sigma : \Omega \to \Omega \).
A probability space is said to be \( P \)-complete if the \( \sigma \)-algebra \( F \) contains all subsets of \( P \) probability 0 and it is said to be countably generated if there exists a countable subset of \( F \) which generates \( F \) (mod 0) (see [1, p. 535]). We prove that there exists \( \tilde{\mu} \in E_P(f) \) such that for \( P \) a.e. \( \omega \),

\[
\Lambda(\tilde{\mu}) = \max \{ \Lambda(\mu) : \mu \in \mathcal{M}_P(f) \} = \lim_{n \to \infty} \frac{1}{n} \max_{x \in M} F_n(\omega, x),
\]

where \( \mathcal{M}_P(f) \), \( E_P(f) \) denote the set of all \( f \)-invariant probability measures and ergodic measures on \( \Omega \times M \) whose marginal on \( \Omega \) coincide with \( P \) respectively and \( \Lambda(\mu) = \lim_{t \to \infty} \frac{1}{t} \int_{\Omega \times M} F_t(\omega, x) \, d\mu \) for \( \mu \in \mathcal{M}_P(f) \).

Next we consider continuous time random dynamical systems

\[
\phi : \mathbb{R}_+ \times \Omega \times M \to \Omega \times M
\]

over \( (\Omega, F, P, (\sigma(t))_{t \in \mathbb{R}_+}) \) with time \( \mathbb{R}_+ \), where \( (\Omega, F, P) \) is a \( P \)-complete, countably generated probability space with \( P \) preserving ergodic semi-flow \( \sigma(t) \). The set of all \( \phi \) invariant measures on \( \Omega \times M \) and all ergodic \( \phi \)-invariant measures whose marginal on \( \Omega \) coincide with \( P \) are denoted by \( \mathcal{M}_P(\phi) \) and \( E_P(\phi) \), respectively. We prove that there exists a \( \nu \in E_P(\phi) \) such that for \( P \) a.e. \( \omega \)

\[
\Lambda_{\phi}(\nu) = \max \{ \Lambda_{\phi}(\mu) : \mu \in \mathcal{M}_P(\phi) \} = \lim_{t \to \infty} \frac{1}{t} \max_{x \in M} F_t(\omega, x),
\]

where \( \Lambda_{\phi}(\mu) = \lim_{t \to \infty} \frac{1}{t} \int_{\Omega \times M} F_t(\omega, x) \, d\mu \) for \( \mu \in \mathcal{M}_P(\phi) \).

In [13,14], the authors consider skew-product flow and prove that under some conditions, the upper Lyapunov exponent can be achieved by an ergodic measure. This result can be deduced directly from our results as above.

When the \( \sigma \)-invariant measure \( P \) is a Dirac-\( \delta \) measure supported on a single fixed point \( \{p\} \), the settings stated above then reduce to the determined case. As a directly consequence of random case, for discrete case, we obtain there exists an ergodic measure \( \nu \in E(f) \) such that

\[
\sup_{x \in M} \limsup_{n \to \infty} \frac{1}{n} F_n(x) = \max_{x \in M} \limsup_{n \to \infty} \frac{1}{n} F_n(x) = \Lambda(\nu) = \max \{ \Lambda(\mu) : \mu \in \mathcal{M}(f) \},
\]

and for continuous-time case, there exists \( \hat{\nu} \in \mathcal{E}(\phi) \) such that

\[
\sup_{x \in M} \limsup_{t \to +\infty} \frac{1}{t} F_t(x) = \max_{x \in M} \limsup_{t \to +\infty} \frac{1}{t} F_t(x) = \Lambda_{\phi}(\hat{\nu}) = \max \{ \Lambda_{\phi}(\mu) : \mu \in \mathcal{M}(\phi) \},
\]

which give a refinement of Schreiber’s results in [17].

The paper is organized as follows. In Section 2, the discrete random dynamical systems is considered. In Section 3, the continuous-time dynamical systems is considered. In Section 4, when \( \Omega \) has only one point \( \{p\} \), the settings stated in random dynamical systems then reduce to standard deterministic dynamical systems. Then similar results can be obtained directly.

In [17], the dynamical spectrum and the measurable spectrum of a skew-product semi-flow on Banach bundle were considered. The existence of the measurable spectrum in the infinite-dimensional setting was considered by Ruelle [16] and Mane [9]. In fact, we think that our results can be used to consider the dynamical spectrum of random dynamical systems in infinite
dynamical systems as it was considered in [1,10] and principal Lyapunov exponent/spectrum for random/nonautonomous parabolic equations in [11,12]. Random invariant manifolds in random infinite-dimensional systems have been considered in papers [2,3] and it is important to consider their persistence and attractivity. We think our results can be used to study the persistence, attractivity of invariant manifolds, and the following problem: under which conditions, is a random compact invariant manifold of an infinite random dynamical system a random attractor? We will consider these problems in the further paper.

2. Discrete case for random dynamical systems

Throughout this section we will consider discrete random dynamical systems

\[ f : \Omega \times M \to \Omega \times M, \quad f(\omega, x) = (\sigma(\omega), \phi_\omega(x)), \]

over an abstract dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\), where \((\Omega, \mathcal{F}, \mathbb{P})\) is a \(\mathbb{P}\)-complete, countably generated probability space with \(\mathbb{P}\) preserving ergodic invertible transformation \(\sigma : \Omega \to \Omega\).

Let \(\text{Pr}(\Omega \times M)\) denote the set of probability measures on \(\Omega \times M\) and \(\mathcal{M}_{\mathbb{P}}(f)\) denote all \(f\)-invariant probability measures on \(\Omega \times M\) whose marginal on \(\Omega\) coincide with \(\mathbb{P}\). Such measures can be characterized in term of their disintegrations \(\mu_\omega\), \(\omega \in \Omega\), by \(\phi_\omega(\mu_\omega) = \mu_{\sigma\omega}\) \(\mathbb{P}\)-a.s.

Let \(\mathcal{E}_{\mathbb{P}}(f) \subset \mathcal{M}_{\mathbb{P}}(f)\) denote the set of all ergodic measures.

A function \(F : \Omega \times M \to \mathbb{R}\) is called random continuous function, if \(x \mapsto F(\omega, x)\) is a bounded continuous function for \(\mathbb{P}\) almost all \(\omega \in \Omega\), \(\omega \mapsto F(\omega, x)\) is a measurable for all \(x \in M\) and \(\omega \mapsto \sup_{x \in M} |F(\omega, x)|\) is integrable with respect to \(\mathbb{P}\). The space of measures on \(\Omega \times M\) with marginal \(\mathbb{P}\) on \(\Omega\) will be equipped with the smallest topology such that

\[ \nu \mapsto \int_{\Omega} \int_{M} F(\omega, x) \, d\nu_\omega(x) \, d\mathbb{P}(\omega) = \int_{\Omega \times M} F(\omega, x) \, d\nu(\omega, x) \]

is continuous for every random continuous function \(F\).

\(\{F_n(\omega, x)\}\) is called random continuous sub-additive function sequence on \(\Omega \times M\) if

\[ F_{n+m}(x, \omega) \leq F_n(x, \omega) + F_m(f^n(x, \omega)). \]

Let \(b_n(\omega) = \max_{x \in M} |F_n(\omega, x)|\) and \(a_n(\omega) = \max_{x \in M} F_n(\omega, x)\). Then we have for \(\forall n \in \mathbb{N}\), \(a_n(\omega) \leq b_n(\omega)\), \(a^n_+ \in L^1(\mathbb{P})\), \(b_n \in L^1(\mathbb{P})\) and

\[ a_{n+m}(\omega) \leq a_n(\omega) + a_m(\sigma^n(\omega)). \]

That is say \(\{a_n(\omega)\}\) is a sub-additive function sequence on \(\Omega\) with respect to \(\sigma\). By the sub-additive ergodic theorem (see [19]), we have that \(\lim_{n \to \infty} \frac{a_n(\omega)}{n} = A(\omega)\) exists and \(A\) is a constant for \(\mathbb{P}\) a.e. \(\omega\) and \(A = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega} a_n \, d\mathbb{P}\). Let \(B = \sup_{\mu \in \mathcal{M}_{\mathbb{P}}(f)} \Lambda(\mu)\), where \(\Lambda(\mu) = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_n(\omega, x) \, d\mu\).

**Lemma 2.1.** \(\Lambda(\mu)\) is an upper-semi continuous map from \(\mathcal{M}_{\mathbb{P}}(f)\) to \(\mathbb{R}\).
Proof. It is well known. But we cannot find its proof. For the completeness, we give a proof here.

Since \( \Lambda(\mu) = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_n(\omega, x) \, d\mu \), for \( \forall \epsilon > 0 \), there exists \( m \in \mathbb{N} \) such that

\[
\frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\mu < \Lambda(\mu) + \frac{\epsilon}{2}.
\]

The continuity of map \( \mu \mapsto \frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\mu \) implies that there exists \( \delta > 0 \) such that if \( d(\nu, \mu) < \delta \), then

\[
\left| \frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\nu - \frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\mu \right| < \frac{\epsilon}{2}.
\]

Therefore

\[
\frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\nu < \frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\mu + \frac{\epsilon}{2} < \Lambda(\mu) + \epsilon.
\]

Hence

\[
\Lambda(\nu) = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_n(\omega, x) \, d\nu = \inf \left\{ \frac{1}{n} \int_{\Omega \times M} F_n(\omega, x) \, d\nu \right\} < \Lambda(\mu) + \epsilon.
\]

This completes the proof of lemma. \( \square \)

Since \( M_P(f) \) is compact, Lemma 2.1 implies there exists \( \mu^* \in M_P(f) \) such that \( B = \Lambda(\mu^*) \).

Lemma 2.2. \( \lim_{n \to \infty} \frac{b_i(\sigma^n \omega)}{n} = 0 \), for all \( i \) and \( \mathbb{P} \) a.e. \( \omega \).

Proof. Since \( b_i \in L^1(\mathbb{P}) \) for all \( i \), Birkhoff ergodic theorem says that for \( \mathbb{P} \) a.e. \( \omega \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} b_i(\sigma^j \omega) = \int_{\Omega} b_i \, d\mathbb{P}.
\]

Hence for \( \mathbb{P} \) a.e. \( \omega \),

\[
\lim_{n \to \infty} \frac{b_i(\sigma^n \omega)}{n} = \lim_{n \to \infty} \frac{1}{n} \left[ \sum_{j=0}^{n-1} b_i(\sigma^j \omega) - \sum_{j=0}^{n-1} b_i(\sigma^j \omega) \right] = 0. \quad \square
\]

Theorem 2.1. \( B = A \).

Proof. First, we prove that \( B \leq A \). Sub-additive ergodic theorem says that for every \( \mu \in M_P(f) \), there exist a measurable set \( A \) with \( \mu(A) = 1 \) and \( F \in L^1(\mu) \) such that for \( (\omega, x) \in A \)
\[
\lim_{n \to \infty} \frac{F_n(\omega, x)}{n} = \bar{F}(\omega, x) \quad \text{and} \\
\int_{\Omega \times M} \bar{F}(\omega, x) \, d\mu = \lim_{n \to \infty} \int_{\Omega \times M} \frac{F_n(\omega, x)}{n} \, d\mu = \Lambda(\mu).
\]

Since \( F_n(\omega, x) \leq a_n(\omega) \) for all \( x \), \( \bar{F}(\omega, x) \leq A \) for all \((\omega, x) \in A\). Thus
\[
\Lambda(\mu) = \lim_{n \to \infty} \int_{\Omega \times M} \frac{1}{n} F_n(\omega, x) \, d\mu = \int_{\Omega \times M} \bar{F}(\omega, x) \, d\mu \leq A.
\]

Hence
\[
B = \max_{\mu \in \mathcal{M}_2(f)} \Lambda(\mu) = \max_{\mu \in \mathcal{M}_2(f)} \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_n(x, \omega) \, d\mu \leq A.
\]

Next we prove that \( B \geq A \). Suppose to the contrary \( B < A \). Let \( \epsilon = \frac{A - B}{2} \). \( b_1(\omega) \in L^1(\mathbb{P}) \) implies that there exists \( \delta > 0 \) such that
\[
\int_V b_1(\omega) \, d\mathbb{P} < \frac{\epsilon}{2}
\]
for every measurable set \( V \subset \Omega \) with \( \mathbb{P}(V) < \delta \). Since \( \mathbb{P} \) a.e. \( \omega \), \( \lim_{n \to \infty} \frac{1}{n} a_n(\omega) = A \), there exists \( U \subset \Omega \) with \( \mathbb{P}(U) > 1 - \delta \) such that \( \{\frac{1}{n} a_n(\omega)\} \) converges to \( A \) uniformly on \( U \). Thus \( \exists N \), for \( \forall n > N, \forall \omega \in U \),
\[
\frac{a_n(\omega)}{n} - A > -\frac{\epsilon}{2}.
\]
By the definition of \( a_n(\omega) \), \( \exists x_n(\omega) \in M \) such that \( a_n(\omega) = F_n(\omega, x_n(\omega)) \). Hence for \( \omega \in U \), \( \exists x_n(\omega) \in M \) such that
\[
\frac{F_n(\omega, x_n(\omega))}{n} > A - \frac{\epsilon}{2}.
\]
Let \( G = \bigcup_{i=-\infty}^{\infty} \sigma^i(U) \), ergodicity implies that \( \mathbb{P}(G) = 1 \). Then, for each \( \omega \in U \) and each \( n > N \) we define a probability measure
\[
\sigma_n(\omega) = \delta_{x_n(\omega)}
\]
where \( \delta_x \) denotes the Dirac measure at \( x \in M \) and for \( \omega \in G \setminus U \) and \( n > N \) define
\[
\sigma_n(\omega) = \delta_{x_n(\omega)}
for some arbitrary point $x_n(\omega) \in M$ which can be chosen independently of $\omega$ and $n$. We can define random probability measures $\mu_n : \Omega \to Pr(M)$ on $\Omega \times M$ as follows:

$$
\mu_n(\omega) = \frac{1}{n} \sum_{i=0}^{n-1} (f^i \sigma_n)(\omega).
$$

It is easy to prove that the marginal of $\mu_n$ on $\Omega$ coincides with $\mathbb{P}$ and it is well known that $\{\mu_n\}$ has a subsequence converging to an invariant measure $\mu \in \mathcal{M}_P(f)$ (see Arnold [1]). Without loss of generality, we suppose $\lim_{n \to \infty} \mu_n = \mu$.

For the completeness, we will give a proof of $\mu \in \mathcal{M}_P(f)$. First we prove that $\pi^*(\mu) = \mathbb{P}$, where $\pi$ is projection from $\Omega \times M \to \Omega$. The fact $\pi^*(\mu_n) = \mathbb{P}$, for $n > N$ imply that $\pi^*(\mu) = \lim_{n \to \infty} \pi^*(\mu_n) = \mathbb{P}$.

Then we prove that $\mu$ is invariant. In fact, for every random continuous function $h$ on $\Omega \times M$,

$$
\left| \int_{\Omega \times M} h d f^* \mu - \int_{\Omega \times M} h d \mu \right| = \lim_{n \to \infty} \left| \int_{\Omega \times M} h d f^* \mu_n - \int_{\Omega \times M} h d \mu_n \right|
\leq \lim_{n \to \infty} \frac{2}{n} \int_{\Omega} \max_{x \in M} |h(x, \omega)| d \mathbb{P} = 0.
$$

The arbitrariness of $h$ implies that $\mu$ is $f$-invariant.

For a fixed $m$, let $n = ms + l$, $0 \leq l < m$. Sub-additivity implies that for $j = 0, 1, \ldots, m - 1$

$$
F_n(\omega, x_n(\omega)) \leq F_j(\omega, x_n(\omega)) + F_m(f^j(\omega, x_n(\omega))) + \cdots + F_m(f^{m(s-2)}f^j(\omega, x_n(\omega)))
+ F_{m-j+l}(f^{m(s-1)}f^j(\omega, x_n(\omega)))
\leq a_j(\omega) + F_m(f^j(\omega, x_n(\omega))) + \cdots + F_m(f^{m(s-2)}f^j(\omega, x_n(\omega)))
+ a_{m-j+l}(\sigma^{m(s-1)+j}(\omega)).
$$

Summing $j$ from 0 to $m - 1$, we get

$$
F_n(\omega, x_n(\omega)) \leq \sum_{i=0}^{m(s-1)-1} \frac{1}{m} F_m(f^i(\omega, x_n(\omega)))
+ \frac{1}{m} \sum_{j=0}^{m-1} a_{m-j+l}(\sigma^{m(s-1)+j}(\omega))
= \sum_{i=0}^{n-1} \frac{1}{m} F_m(f^i(\omega, x_n(\omega)))
- \sum_{j=(s-1)m}^{n-1} \frac{1}{m} F_m(f^i(\omega, x_n(\omega)))
+ \frac{1}{m} \sum_{j=0}^{m-1} a_{m-j+l}(\sigma^{m(s-1)+j}(\omega))
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\[ \leq \sum_{i=0}^{n-1} \frac{1}{m} F_m(f^i(\omega, x_n(\omega))) + \sum_{j=(s-1)m}^{n-1} \frac{1}{m} b_m(\sigma^j(\omega)) \]

\[ + \frac{1}{m} [b_1(\omega) + \cdots + b_{m-1}(\omega)] + \frac{1}{m} \sum_{j=0}^{m-1} b_{m-j+1}(\sigma^{m(s-1)+j}(\omega)). \]

Since

\[ F_n(\omega, x) \geq -|F_n(\omega, x)| \geq -b_n(\omega) \geq \left[ b_1(\omega) + \cdots + b_1(\sigma^{n-1}(\omega)) \right], \]

we have

\[ \int_{G \setminus U} \frac{F_n(\omega, x_n(\omega))}{n} \, d\mathbb{P} \geq -\frac{1}{n} \sum_{i=0}^{n-1} \int_{G \setminus U} b_1(\sigma^i(\omega)) \, d\mathbb{P} \geq -\frac{1}{n} \sum_{i=0}^{n-1} \int_{\sigma^{-i}(G \setminus U)} b_1(\omega) \, d\mathbb{P}. \]

Since \( \sigma : \Omega \to \Omega \) is an invertible transformation preserving ergodic measure \( \mathbb{P} \), we have \( \mathbb{P}(\sigma^{-i}(G \setminus U)) = \mathbb{P}(G \setminus U) < \delta \). Hence

\[ \int_{\sigma^{-i}(G \setminus U)} b_1(\omega) \, d\mathbb{P} < \frac{\epsilon}{2}. \]

This implies that

\[ \int_{G \setminus U} \frac{F_n(\omega, x_n(\omega))}{n} \, d\mathbb{P} \geq -\frac{\epsilon}{2}. \]

Therefore

\[ \int_{\Omega} \frac{F_n(\omega, x_n(\omega))}{n} \, d\mathbb{P} = \int_{U} \frac{F_n(\omega, x_n(\omega))}{n} \, d\mathbb{P} + \int_{G \setminus U} \frac{F_n(\omega, x_n(\omega))}{n} \, d\mathbb{P} \]

\[ \geq A - \frac{\epsilon}{2} - \frac{\epsilon}{2} = A - \epsilon. \]

Using Lemma 2.2, we have

\[ A - \epsilon \leq \lim_{n \to \infty} \int_{\Omega} \frac{1}{n} F_n(\omega, x_n(\omega)) \, d\mathbb{P} \leq \lim_{n \to \infty} \int_{\Omega} \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{m} F_m(f^i(\omega, x_n(\omega))) \, d\mathbb{P} \]

\[ = \lim_{n \to \infty} \int_{\Omega \times M} \frac{1}{m} F_m(\omega, x) \, d\mu_n = \int_{\Omega \times M} \frac{1}{m} F_m(\omega, x) \, d\mu. \]
From the arbitrariness of $m$, it has
\[ B + \epsilon = A - \epsilon \leq \lim_{m \to \infty} \frac{1}{m} \int_{\Omega \times M} F_m(\omega, x) \, d\mu = \Lambda(\mu), \]
which contradicts to the definition of $B$. Therefore $A = B$. This completes the proof of theorem. \( \square \)

**Theorem 2.2.** There exists \( \tilde{\mu} \in \mathcal{E}_P(f) \) such that for \( P \) a.e. \( \omega \),
\[ \Lambda(\tilde{\mu}) = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_n(\omega, x) \, d\tilde{\mu} = \lim_{n \to \infty} \frac{1}{n} \max_{x \in M} F_n(\omega, x). \]

**Proof.** By Lemma 2.1 and Theorem 2.1, there exists \( \mu^* \in \mathcal{M}_P(f) \) such that
\[ A = B = \Lambda(\mu^*) = \sup_{\mu \in \mathcal{M}_P(f)} \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_n(\omega, x) \, d\mu. \]
For \( \forall \mu \in \mathcal{E}_P(f) \), it has \( \Lambda(\mu) \leq \Lambda(\mu^*) \). Then ergodic decomposition theorem [1] implies that there exists \( \tilde{\mu} \in \mathcal{E}_P(f) \) such that
\[ \Lambda(\tilde{\mu}) = \Lambda(\mu^*) = B = A. \]
Hence for \( P \) a.e. \( \omega \),
\[ \Lambda(\tilde{\mu}) = \lim_{n \to \infty} \frac{1}{n} \max_{x \in M} F_n(\omega, x). \quad \square \]

3. Continuous case for random dynamical systems

Let \( \phi: R_+ \times \Omega \times M \to \Omega \times M \) be a measurable random semi-dynamical systems on the compact metric space \( M \) over \( (\Omega, \mathcal{F}, P, (\sigma(t))_{t \in R_+}) \) with time \( R_+ \), where \( (\Omega, \mathcal{F}, P, \sigma(t)) \) is a \( P \)-complete, countably generated probability space with \( P \) preserving ergodic semi-flow \( \sigma(t) \).

The sets of all \( \phi \) invariant measures on \( \Omega \times M \) and all ergodic \( \phi \) invariant measures whose marginal on \( \Omega \) coincide with \( P \) are denoted by \( \mathcal{M}_P(\phi) \) and \( \mathcal{E}_P(\phi) \), respectively. It is well known invariant measure for the time one map of flow is not flow invariant in general and an ergodic measure for a flow is not necessarily ergodic for the time one map. So the results of time one map cannot be used directly to the corresponding flow.

Suppose that \( \{F_t(\omega, x)\} \) is a random continuous sub-additive function on \( \Omega \times M \) and
\[ \sup_{0 \leq t \leq 1} \sup_{x \in M} |F_t(\omega, x)| \in L^1(P). \]
Let \( a_t(\omega) = \max_{x \in M} F_t(\omega, x) \), \( b_t(\omega) = \max_{x \in M} |F_t(\omega, x)| \) and \( c(\omega) = \sup_{0 \leq t \leq 1} b_t(\omega) \). Then we have \( a_t(\omega) \leq b_t(\omega), \forall t \in R_+, c \in L^1(P) \) and
\[ a_{t+s}(\omega) \leq a_t(\omega) + a_s(\sigma^t \omega). \]
That is, say \( \{a_t(\omega)\} \) is a sub-additive function on \( \Omega \) with respect to the flow \( \sigma \). By the sub-additive ergodic theorem (see [19]), we have that \( \lim_{t \to \infty} \frac{a_t(\omega)}{t} = A(\omega) \) exists and \( A \) is a constant for \( \mathbb{P} \) a.e. \( \omega \in \Omega \) and \( A = \lim_{t \to \infty} \int_\Omega \frac{1}{t} a_t(\omega) \, d\mathbb{P} \).

Define \( A_\phi(\mu) = \lim_{t \to \infty} \int_{\Omega \times M} \frac{1}{t} F_t(\omega, x) \, d\mu \) for \( \forall \mu \in \mathcal{M}_{\mathbb{P}}(\phi) \), and \( B = \sup_{\mu \in \mathcal{M}_{\mathbb{P}}(\phi)} A_\phi(\mu) \). Similarly to discrete random dynamical systems, we have \( A_\phi(\mu) \) is an upper-semi continuous map from \( \mathcal{M}_{\mathbb{P}}(\phi) \) to \( \mathbb{R} \) and the following lemma holds true.

**Lemma 3.1.** For a fixed \( t \in \mathbb{R}^+ \), we have that

\[
\lim_{r \to +\infty} \frac{1}{r} \int_0^t b_s(\omega) \, ds = 0,
\]

\[
\lim_{k \to +\infty} \frac{1}{t(k+1)} \int_{t(k-1)}^{t(k+1)} b_t(\sigma^s(\omega)) \, ds = 0,
\]

and

\[
\lim_{k \to +\infty} \frac{1}{k} \int_0^t b_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \, ds = 0
\]

for \( \mathbb{P} \) a.e. \( \omega \), where \( 0 \leq l < t \).

**Proof.** For a fixed \( t \in \mathbb{R}^+ \), we have

\[
0 \leq \int_0^t b_s(\omega) \, ds \leq c(\omega) + \cdots + c(\sigma^{|l|}(\omega)).
\]

Because for \( \mathbb{P} \) a.e. \( \omega \),

\[
c(\omega) + \cdots + c(\sigma^{|l|}(\omega))
\]

is bounded, we have

\[
\lim_{r \to +\infty} \frac{1}{r} \int_0^t b_s(\omega) \, ds = 0.
\]

Since \( b_t \in L^1(\mathbb{P}) \), Birkhoff ergodic theorem says that for \( \mathbb{P} \) a.e. \( \omega \),

\[
\lim_{k \to +\infty} \frac{1}{t(k+1)} \int_0^{t(k+1)} b_t(\sigma^s(\omega)) \, ds = \int_\Omega b_t(\omega) \, d\mathbb{P}.
\]
Hence for $\mathbb{P}$ a.e. $\omega$, 
\[
\lim_{k \to +\infty} \frac{1}{t(k+1)} \int_{t(k-1)}^{t(k+1)} b_t(\sigma^s(\omega)) \, ds
= \lim_{k \to +\infty} \frac{1}{t(k+1)} \left[ \int_{0}^{t(k+1)} b_t(\sigma^s(\omega)) \, ds - \int_{0}^{t(k-1)} b_t(\sigma^s(\omega)) \, ds \right] = 0.
\]
Since $0 \leq l < t$ and $0 \leq s \leq t$, we have $l + t - s < 2t$. Hence

\[
b_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \leq c(\sigma^{s+(k-1)t}(\omega)) + \cdots + c(\sigma^{s+[2t]-(k-1)t}(\omega)).
\]

This implies that

\[
\int_{0}^{t} b_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \, ds \leq \int_{0}^{t} c(\sigma^{s+(k-1)t}(\omega)) \, ds + \cdots + \int_{0}^{t} c(\sigma^{s+[2t]-(k-1)t}(\omega)) \, ds.
\]

Since $c \in L^1(\mathbb{P})$, Birkhoff ergodic theorem says that for $\mathbb{P}$ a.e. $\omega$,

\[
\lim_{k \to +\infty} \frac{1}{tk} \int_{0}^{tk} c(\sigma^s(\omega)) \, ds = \int_{\Omega} c(\omega) \, d\mathbb{P}.
\]

Thus

\[
\lim_{k \to +\infty} \frac{1}{k} \int_{0}^{t} c(\sigma^{s+(k-1)t}(\omega)) \, ds = \lim_{k \to +\infty} \frac{1}{(k-1)t} \int_{(k-1)t}^{kt} c(\sigma^s(\omega)) \, ds = 0.
\]

Similarly, for $j = 1, \ldots, [2t]$, we have

\[
\lim_{k \to +\infty} \frac{1}{k} \int_{0}^{t} c(\sigma^{s+j+(k-1)t}(\omega)) \, ds = \int_{\Omega} c(\omega) \, d\mathbb{P}.
\]

Therefore

\[
\lim_{k \to +\infty} \frac{1}{k} \int_{0}^{t} b_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \, ds = 0. \quad \Box
\]

**Theorem 3.1.** Suppose $\{F_t(\omega, x)\}$ is a random continuous sub-additive function on $\Omega \times M$ and

\[
\sup_{0 \leq l \leq 1} \sup_{x \in M} |F_t(\omega, x)| \in L^1(\mathbb{P}),
\]
then there exists a $\nu \in E_P(\phi)$ such that for $P$ a.e. $\omega$,

$$A_\phi(v) = \max \{ A_\phi(\mu) : \mu \in \mathcal{M}_P(\phi) \} = \lim_{t \to \infty} \frac{1}{t} a_t(\omega) = \lim_{t \to \infty} \frac{1}{t} \max_{x \in M} F_t(\omega, x).$$

**Proof.** The proof of this theorem is the same as in Theorems 2.1 and 2.2 for discrete random dynamical systems.

Using sub-additive ergodic theorem, we can prove that $B \leq A$. It is the same as in Theorem 2.1 and we omit the detail.

Next we prove that $B \geq A$. Suppose to the contrary $B < A$. Let $\epsilon = \frac{A - B}{2}$. The $b_1(\omega) \in L^1(P)$ implies that there exists $\delta > 0$ such that

$$\int_V b_1(\omega) \, d\mathbb{P} < \frac{\epsilon}{2}$$

for every measurable set $V \subset \Omega$ with $\mathbb{P}(V) < \delta$. Since $\mathbb{P}$ a.e. $\omega$, $\lim_{t \to \infty} \frac{1}{t} a_t(\omega) = A$, there exists $U \subset \Omega$ with $\mathbb{P}(U) > 1 - \delta$ such that $\{\frac{1}{n} a_n(\omega)\}$ converges to $A$ uniformly on $U$. Thus $\exists T$, for $\forall n > T$, $\forall \omega \in U$,

$$\frac{a_n(\omega)}{n} - A > \frac{\epsilon}{2}.$$

By the definition of $a_n(\omega)$, there exists $x_n(\omega) \in M$ such that $a_n(\omega) = F_n(\omega, x_n(\omega))$. Hence for $\omega \in U$, there exists $x_n(\omega) \in M$ such that

$$\frac{F_n(\omega, x_n(\omega))}{n} > A - \frac{\epsilon}{2}.$$

Let $G = \bigcup_{i=-\infty}^{\infty} \sigma^i(U)$, ergodicity implies that $\mathbb{P}(G) = 1$. Then, for each $\omega \in U$ and each $n > T$ we define a probability measure

$$\sigma_n(\omega) = \delta_{x_n(\omega)},$$

where $\delta_x$ denotes the Dirac measure at $x \in M$ and for $\omega \in G \setminus U$ and $n > T$ define

$$\sigma_n(\omega) = \delta_{x_n(\omega)}$$

for some arbitrary point $x_n(\omega) \in M$ which can be chosen independently of $\omega$ and $n$. Using random probability measure $\sigma_n$, we can define new random probability measure $\mu_n : \Omega \to Pr(M)$ on $\Omega \times M$ as follows:

$$\mu_n(\omega) = \frac{1}{n} \int_0^n (\phi_s^\omega \sigma_n)(\omega) \, ds.$$

It is easy to prove that the marginal of $\mu_n$ on $\Omega$ coincides with $\mathbb{P}$ and it is well known that $\{\mu_n\}$ has a subsequence converging to an invariant measure $\mu \in \mathcal{M}_P(\phi)$ (see Arnold [1]). Without loss of generality, we suppose $\lim_{n \to \infty} \mu_n = \mu$. 
For a fixed $t \in \mathbb{R}_+$, let $n = tk + l$, $0 \leq l < t$. Sub-additivity implies for $0 \leq s < t,$

$$F_n(\omega, x_n(\omega)) \leq F_s(\omega, x_n(\omega)) + F_t(\phi^s(\omega, x_n(\omega))) + \cdots + F_t(\phi^{s+(k-2)t}(\omega, x_n(\omega))) + F_{l+t-s}(\phi^{s+(k-1)t}(\omega, x_n(\omega)))$$

$$\leq a_s(\omega) + F_t(\phi^s(\omega, x_n(\omega))) + \cdots + F_t(\phi^{s+(k-2)t}(\omega, x_n(\omega))) + a_{l+t-s}(\sigma^{s+(k-1)t}(\omega)).$$

Then we integrate with respect to $s$ in $[0, t]$, and we have

$$t F_n(\omega, x_n(\omega)) \leq \int_0^t a_s(\omega) \, ds + \int_0^t F_t(\phi^s(\omega, x_n(\omega))) \, ds + \cdots + F_t(\phi^{s+(k-2)t}(\omega, x_n(\omega))) \, ds$$

$$+ \int_0^{t(k-1)} a_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \, ds.$$

$$= \int_0^t a_s(\omega) \, ds + \int_0^{t(k-1)} F_t(\phi^s(\omega, x_n(\omega))) \, ds + \int_{t(k-1)}^t a_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \, ds$$

$$\leq \int_0^t b_s(\omega) \, ds + \int_0^{n} F_t(\phi^s(\omega, x_n(\omega))) \, ds + \int_{t(k-1)}^t b_1(\sigma^s(\omega)) \, ds$$

$$+ \int_0^{t(k-1)} b_{l+t-s}(\sigma^{s+(k-1)t}(\omega)) \, ds.$$

Since

$$F_n(\omega, x) \geq -|F_n(\omega, x)| \geq -b_n(\omega) \geq -b_1(\omega) + \cdots + b_1(\sigma^{n-1}\omega),$$

one has

$$\int_{G \setminus U} F_n(\omega, x_n(\omega)) \, dP \geq -\frac{1}{n} \sum_{i=0}^{n-1} \int_{G \setminus U} b_1(\sigma^i(\omega)) \, dP \geq -\frac{1}{n} \sum_{i=0}^{n-1} \int_{\sigma^{-i}(G \setminus U)} b_1(\omega) \, dP.$$
Since $\sigma : \Omega \to \Omega$ is an invertible transformation preserving ergodic measure $P$, we have $P(\sigma^{-1}(G \setminus U)) = P(G \setminus U) < \delta$. Hence

$$\int_{\sigma^{-1}(G \setminus U)} b_1(\omega) \, dP < \frac{\epsilon}{2}.$$  

This implies that

$$\int_{G \setminus U} \frac{F_n(\omega, x_n(\omega))}{n} \, dP \geq -\frac{\epsilon}{2}.$$  

Therefore

$$\int_{\Omega} \frac{F_n(\omega, x_n(\omega))}{n} \, dP = \int_{U} \frac{F_n(\omega, x_n(\omega))}{n} \, dP + \int_{G \setminus U} \frac{F_n(\omega, x_n(\omega))}{n} \, dP$$

$$\geq -\frac{\epsilon}{2} - \frac{\epsilon}{2} = A - \epsilon.$$  

Hence

$$A - \epsilon \leq \lim_{n \to \infty} \frac{1}{n} \int_{\Omega} F_n(\omega, x_n(\omega)) \, dP \leq \lim_{n \to \infty} \left( \frac{1}{n} \int_{U} F_n(\omega, x_n(\omega)) \, dP + \frac{1}{n} \int_{G \setminus U} F_n(\omega, x_n(\omega)) \, dP \right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times M} F_t(\omega, x) \, d\mu_n = \int_{\Omega \times M} \frac{1}{t} F_t(\omega, x) \, d\mu.$$  

From the arbitrariness of $t$, one has

$$B + \epsilon = A - \epsilon \leq \lim_{t \to \infty} \frac{1}{t} \int_{\Omega \times M} F_t(\omega, x) \, d\mu = A(\mu),$$

which contradicts the definition of $B$. Therefore $A = B$.

Since $A$ is an upper-semi continuous map from $\mathcal{M}(\phi)$ to $R$, there exists $\mu^*$ such that

$$A = B = A(\mu^*) = \sup_{\mu \in \mathcal{M}(\phi)} A(\mu).$$

For $\forall \mu \in \mathcal{E}(\phi)$, $A(\mu) \leq A(\mu^*)$, the ergodic decomposition theorem [1] implies that there exists $v \in \mathcal{E}(\phi)$ such that

$$A(\mu) = A(\mu^*) = B = A.$$  

Hence for $P$ a.e. $\omega$,

$$A(\mu) = \lim_{t \to \infty} \frac{1}{t} \max_{x \in M} F_t(\omega, x).$$

This completes the proof of theorem.  \qed
Remark 1. In [13, 14], the authors consider a skew-product semi-flow \((\Omega \times X, \tau, R_+)\),

\[
\tau : R_+ \times \Omega \times X \to \Omega \times X, \quad (t, \omega, x) \mapsto \left(\sigma^t(\omega), u(t, \omega, x)\right),
\]

where \(\Omega\) is a compact metric space, \(\{\sigma^t(\omega) \mid t \in R\}\) is a continuous minimal flow on \(\Omega\) and \(X\) is a strongly ordered Banach space. They assume \(u\) is \(C^1\) in \(x\), and \(u_x\) is continuous in \(\omega\). They consider the upper Lyapunov exponent on a compact, positively invariant set \(K\) of the skew-product semi-flow. For \((\omega, x) \in K\), the Lyapunov exponent \(\lambda(\omega, x)\) is defined as

\[
\lambda(\omega, x) = \lim \sup_{t \to \infty} \frac{\ln \|u_x(t, \omega, x)\|}{t},
\]

and the upper Lyapunov exponent is defined as \(\lambda_K = \sup_{(\omega, x) \in K} \lambda(\omega, x)\).

In [13, 14], the authors prove that under the condition of compactness of the closed unit sphere, the upper Lyapunov exponent can be achieved by an ergodic measure. This result can be deduced directly from our Theorem 3.1, even if the closed unit sphere is not compact and \(\Omega\) is a complete, countably generated probability space.

4. Determined dynamical systems

4.1. Discrete case

For discrete random dynamical systems

\[
f : \Omega \times M \to \Omega \times M, \quad f(\omega, x) = \left(\sigma(\omega), \phi_{\omega}(x)\right),
\]

over an abstract dynamical system \((\Omega, \mathcal{F}, P, \sigma)\). When the \(\sigma\)-invariant measure \(P\) is a Dirac-\(\delta\) measure supported on a single fixed point \(\{p\}\), the setting stated above then reduces to the case in which \(f : M \to M\) is a standard deterministic dynamical system. We consider all Borel probability measures supported on \(M\) and denote it by \(\mathcal{P}(M)\). The sets of all \(f\)-invariant measures and all ergodic \(f\)-invariant measures are denoted by \(\mathcal{M}(f)\) and \(\mathcal{E}(f)\), respectively. Suppose that \(\{F_n(x)\}\) is continuous functions sequence and satisfies sub-additivity with respect to \(f\):

\[
F_{n+m}(x) \leq F_n(x) + F_m(f^n(x)).
\]

Let \(A(\mu) = \lim_{n \to \infty} \int_M \frac{1}{n} F_n(x) \, d\mu\) and \(B = \sup_{\mu \in \mathcal{M}(f)} A(\mu)\). Then we have the following theorems which are direct consequence of Theorems 2.1, 2.2.

Theorem 4.1. There exists \(\nu \in \mathcal{E}(f)\) such that

\[
B = A(\nu) = \sup_{\mu \in \mathcal{M}(f)} \{A(\mu)\} = \max_{\mu \in \mathcal{M}(f)} \{A(\mu)\}.
\]

Next let \(A_n = \max_{x \in M} F_n(x)\). Then \(\{A_n\}\) is a sub-additive sequence. Therefore the limit \(\lim_{n \to \infty} \frac{A_n}{n}\) exists and \(\lim_{n \to \infty} \frac{A_n}{n} = \inf_{n \geq 0} \{\frac{A_n}{n}\}\).
Theorem 4.2.

\[
\lim_{n \to \infty} \frac{A_n}{n} = \sup_{x \in M} \limsup_{n \to \infty} \frac{1}{n} F_n(x) = \max_{x \in M} \limsup_{n \to \infty} \frac{1}{n} F_n(x) = B.
\]

An analogous problem has been considered in [17] and they prove that

\[
\sup_{x \in M} \limsup_{n \to \infty} \frac{1}{n} F_n(x) = \sup_{\Lambda(\mu)} \{ \Lambda(\mu) : \mu \in \mathcal{E}(f) \}.
\]

Therefore Theorems 4.1 and 4.2 are refinement of the results in [17].

4.2. Continuous case

Similarly, for continuous random dynamical systems

\[ \phi: R_+ \times \Omega \times M \to \Omega \times M \]

over \((\Omega, \mathcal{F}, \mathbb{P}, (\sigma(t))_{t \in R_+})\) with time \(R_+\). When the \(\sigma\)-invariant measure \(\mathbb{P}\) is a Dirac-\(\delta\) measure supported on a single fixed point \(\{p\}\), the setting stated above then reduces to the case in which \(\phi: R_+ \times M \to M\) is a standard deterministic dynamical system.

A Borel measure \(\mu\) is called \(\phi^t\) invariant if for any Borel set \(B \subset M\) it holds that \(\mu(\phi^{-t}(B)) = \mu(B)\). It is called \(\phi\) invariant if it is \(\phi^t\) invariant for all \(t\). A \(\phi\) invariant measure is called ergodic with respect to \(\phi\) if any Borel set \(\phi^t\) invariant for any \(t\) has measure 0 or 1. The sets of all \(\phi\) invariant measures and ergodic \(\phi\) invariant measures are denoted by \(\mathcal{M}(\phi)\) and \(\mathcal{E}(\phi)\), respectively.

Suppose that \(\{F_t(x)\}\) is a continuous sub-additive function on \(M\) with respect to \(\phi\). In order to study the growth rate of \(\{F_t(x)\}\), we first consider the time-one map \(\phi^1\) of the flow \(\phi\) and \(\{F_n(x)\} \mid n \in \mathbb{N}\) which is a continuous sub-additive function on \(M\) with respect to \(\phi^1\). By Theorem 4.1, there exists \(v \in \mathcal{E}(\phi^1)\) such that

\[
B = \sup_{\mu \in \mathcal{M}(\phi^1)} \{ \Lambda(\mu) \} = \max_{\mu \in \mathcal{M}(\phi^1)} \{ \Lambda(\mu) \} = \Lambda(v).
\]

Then we can define a measure

\[
\hat{v} = \int_0^1 v_t \, dt
\]

where \(v_t(A) = v(\phi^{-t}A)\). It is easy to prove that \(\hat{v}\) is \(\phi\) invariant and ergodic.

Sub-additivity of \(\{F_t(x)\}\) implies that for \(\forall \mu \in \mathcal{M}(\phi)\), \(\int_M F_t(x) \, d\mu\) is a sub-additive sequence. Therefore limit \(\lim_{t \to \infty} \int_M \frac{F_t(x)}{t} \, d\mu\) exists and we denote it by \(\Lambda_\phi(\mu)\).

Theorem 4.3. Let \(F: R_+ \times M \to R\) be a continuous sub-additive function with respect to the semi-flows \(\phi\). Then
\[
\sup_{x \in M} \limsup_{t \to \infty} \frac{1}{t} F_t(x) = \max_{x \in M} \limsup_{t \to \infty} \frac{1}{t} F_t(x) = \lim_{t \to \infty} \frac{1}{t} \max_{x \in M} F_t(x) \\
= \max\{ \Lambda_\phi(\mu) : \mu \in \mathcal{M}(\phi) \} = \Lambda_\phi(\hat{\nu}).
\]

**Proof.** Theorem 4.3 can be deduced directly from Theorem 3.1. \(\square\)

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