Asymptotic Behavior of the Nonautonomous Two-Species Lotka-Volterra Competition Models*

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Abstract—In the present paper, the nonautonomous two-species Lotka-Volterra competition models are considered, where all the parameters are time-dependent and asymptotically approach periodic functions, respectively. Under some conditions, it is shown that any positive solutions of the models asymptotically approach the unique strictly positive periodic solution of the corresponding periodic system.

Keywords—Competition model, Asymptotically approach, Positive periodic solution, Global asymptotic stability, Comparison theorem.

1. INTRODUCTION

We consider the nonautonomous system of differential equations

\[ \begin{align*}
\dot{x} &= x[b_1(t) - a_{11}(t)x - a_{12}(t)y], \\
\dot{y} &= y[b_2(t) - a_{21}(t)x - a_{22}(t)y],
\end{align*} \tag{1.1} \]

where the functions \( b_i(t), a_{ij}(t) \) \((i, j = 1, 2)\) are continuous and bounded above and below by positive constants on the half infinite interval \( 0 \leq t < +\infty \). This system models the competition between two species in a time-dependent environment. In the case when (1.1) is autonomous, i.e., when the functions \( b_i(t), a_{ij}(t) \) are positive constants \( b_i, a_{ij} \) \((i, j = 1, 2)\), respectively, it has long been known that the conditions

\[ b_1 > \frac{a_{12}b_2}{a_{22}} \quad \text{and} \quad b_2 > \frac{a_{21}b_1}{a_{11}} \]

are necessary and sufficient for the existence of a stable equilibrium point \( \text{col}(x_0, y_0) \) of the system (1.1) such that both components are positive and it attracts all solutions with initial values in the open first quadrant of the \( x, y \)-plane. Moreover,

\[ x_0 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad y_0 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}. \]

(See, for example, [1]).

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In [2], K. Gopalsamy gave a partial extension of this result to the nonautonomous periodic case. To state Gopalsamy’s result, we introduce the following notations: Given a function \( f(t) \), which is bounded above and below by positive constants for \( 0 \leq t < +\infty \), we set

\[
f^M = \sup_t f(t), \quad f^L = \inf_t f(t).
\]

In [2], Gopalsamy let the growth rates \( b_i(t) \) and \( b_j(t) \) be continuous, positive and \( T \)-periodic functions and \( a_{ij}(t) \) be positive constants \( a_{ij} \) \((i, j = 1, 2)\), respectively. As an application of a theorem of Krasnoselskii concerning strictly monotone and strictly convex operators in cones, he showed that the conditions

\[
b^L_1 > \frac{a_{12}b^M_2}{a^L_{22}}, \quad b^L_2 > \frac{a_{21}b^M_1}{a^L_{11}}
\]

imply the existence of a \( T \)-periodic solution of (1.1), which remains in a certain open rectangle in the first quadrant of the \( x, y \)-plane and attracts all solutions of (1.1) that start in this rectangle. In [3] and [4], it was assumed that all parameters \( b_i(t), a_{ij}(t) \) \((i, j = 1, 2)\) were continuous, positive and \( T \)-periodic functions. Using differential inequalities and topological degree, respectively, S. Ahmad in [3] and C. Alvarez and A.C. Lazer in [4] showed that if the conditions

\[
(1.2)
\]

hold, then the system (1.1) has a unique \( T \)-periodic solution \( \text{col}(x_0(t), y_0(t)) \) with all components positive, and this solution is asymptotically stable and attracts all solutions whose components have positive initial values. Moreover, they established upper and lower bounds for the components of the unique \( T \)-periodic solution \( \text{col}(x_0(t), y_0(t)) \) with positive components as following:

\[
\begin{align*}
\frac{b^L_1 a_{22}^L - a^M_{12} b^L_2}{a^L_{11} a^L_{22} - a^M_{12} a^L_{21}} & \leq x_0(t) \leq \frac{b^M_1 a_{22}^M - a^M_{12} b^L_2}{a^M_{11} a^M_{22} - a^M_{12} a^L_{21}}, \\
\frac{a^L_{11} b^L_2 - a^M_{21} b^L_1}{a^L_{11} a^L_{22} - a^M_{12} a^L_{21}} & \leq y_0(t) \leq \frac{a^L_{11} b^M_2 - a^M_{21} b^L_1}{a^M_{11} a^M_{22} - a^M_{12} a^L_{21}}.
\end{align*}
\]

Incidentally, we point out that by Massera’s theorem (see [5]), we can show that the conditions (1.2) imply the existence of \( T \)-periodic solution of (1.1).

In [6], Freedman et al. studied the asymptotic behavior of single-species model

\[
\dot{x} = xg(x, k(t))
\]

where \( k(t) \) is asymptotic to a periodic function \( \tilde{k}(t) \) and \( g(x, k(t)) \) satisfies some hypotheses under which there is a globally attracting periodic solution in (1.3).

In this paper, we use comparison theorem and standard theorems concerning continuity of solutions of differential equations with respect to initial conditions and parameters to prove that if the parameters \( b_i(t), a_{ij}(t) \) are asymptotic to periodic functions \( \tilde{b}_i(t), \tilde{a}_{ij}(t) \) \((i, j = 1, 2)\), respectively, and if in addition

\[
(1.4)
\]

for a certain sufficiently small \( \varepsilon_0 > 0 \), there exists a globally attracting periodic solution in (1.1). Furthermore, using the method of the present paper, it is easy to show that the similar conclusion holds for the time-dependent \( n \)-species Lotka-Volterra cooperative systems.
2. RESULT

We begin with some definitions.

**DEFINITION 1.** Let \( \phi, \psi : [0, +\infty) \rightarrow R \). \( \phi \) is said to approach \( \psi \) asymptotically, in notation \( \phi \sim \psi \), if

\[
\lim_{t \to \infty} |\phi(t) - \psi(t)| = 0.
\]

**DEFINITION 2.** Let \( \text{col}(x, y), \text{col}(\tilde{x}, \tilde{y}) : [0, +\infty) \rightarrow R^2 \). Then, \( \text{col}(x, y) \) is said to approach \( \text{col}(\tilde{x}, \tilde{y}) \) asymptotically, in notation \( \text{col}(x, y) \sim \text{col}(\tilde{x}, \tilde{y}) \), if \( x \sim \tilde{x} \) and \( y \sim \tilde{y} \).

It is easy to verify that "\( \sim \)" is an equivalent relation. In the next section, we prove the main result of this paper, namely the following theorem.

**THEOREM.** For all \( i, j = 1, 2 \), let \( b_i, \tilde{b}_i, a_{ij}, \tilde{a}_{ij} : [0, +\infty) \rightarrow R^+ \) be \( b_i \sim \tilde{b}_i, a_{ij} \sim \tilde{a}_{ij} \) and let \( \tilde{b}_i(t), \tilde{a}_{ij}(t) \) be continuous, positive and \( T \)-periodic functions. If, in addition, the inequalities (1.4) hold, then for any positive solution \( \text{col}(x(t), y(t)) \) of (1.1), we have \( \text{col}(x, y) \sim \text{col}(\tilde{x}, \tilde{y}) \), where \( \text{col}(\tilde{x}, \tilde{y}) \) is the unique positive \( T \)-periodic solution of

\[
\begin{align*}
x'(t) &= \tilde{b}_1(t) - \tilde{a}_{11}(t)x - \tilde{a}_{12}(t)y, \\
y'(t) &= \tilde{b}_2(t) - \tilde{a}_{21}(t)x - \tilde{a}_{22}(t)y.
\end{align*}
\]

A corollary of this theorem is that there exists a globally attracting positive \( T \)-periodic solution in (1.1).

3. PROOF

To prove the theorem above stated, we need a series of lemmas.

**LEMMA 1.** Both the open first quadrant and the first closed quadrant in the \( x, y \)-plane are invariant with respect to (1.1).

**PROOF.** Since

\[
\begin{align*}
x(t) &= x(0) \exp \left( \int_0^t \left[ b_1(s) - a_{11}(s)x(s) - a_{12}(s)y(s) \right] ds \right), \\
y(t) &= y(0) \exp \left( \int_0^t \left[ b_2(s) - a_{21}(s)x(s) - a_{22}(s)y(s) \right] ds \right),
\end{align*}
\]

the assertion of the Lemma immediately follows for all \( t \in [0, +\infty) \) and the proof is complete.

Let us consider the following two systems:

\[
\begin{align*}
\dot{u}_i &= u_i[r_i(t) - r_{ij}(t)u_i - r_{ij}(t)u_j], \quad i = 1, 2; \\
\dot{v}_i &= v_i[s_i(t) - s_{ij}(t)v_i - s_{ij}(t)v_j], \quad i = 1, 2;
\end{align*}
\]

where \( r_i(t), s_i(t), r_{ij}(t), s_{ij}(t) \) (\( i, j = 1, 2 \)) are continuous and bounded above and below by positive constants on the half infinite interval \([0, +\infty)\).

**LEMMA 2.** (Comparison Theorem). Suppose that \( \text{col}(u_1(t), u_2(t)) \) and \( \text{col}(v_1(t), v_2(t)) \) are solutions of (3.1) and (3.2), respectively, satisfying \( u_i(t_0) = v_i(t_0) > 0 \) (\( i = 1, 2 \)). If, for all \( 0 < t_0 \leq t < +\infty \), there exist inequalities

\[
\begin{align*}
r_1(t) > s_1(t), \quad r_{1j}(t) < s_{1j}(t), \quad \text{and} \\
r_2(t) < s_2(t), \quad r_{2j}(t) > s_{2j}(t), \quad j = 1, 2;
\end{align*}
\]

then, \( u_1(t) > v_1(t) \) and \( u_2(t) < v_2(t) \) for all \( t \in (t_0, +\infty) \).
Proof. Since 
\[ \dot{u}_1(t_0) - \dot{v}_1(t_0) > 0 \quad \text{and} \quad \dot{u}_2(t_0) - \dot{v}_2(t_0) < 0, \]
the inequalities
\[ u_1(t) > v_1(t), \quad u_2(t) < v_2(t) \tag{3.3} \]
will hold for \( t - t_0 \) sufficiently small and positive. If (3.3) do not hold for all \( t > t_0 \), there exists \( t_1 > t_0 \) such that (3.3) hold for \( t_0 < t < t_1 \) and either: (a) \( u_1(t_1) = v_1(t_1) \), or (b) \( u_2(t_1) = v_2(t_1) \).
Suppose that (a) holds. By continuity we must have \( u_2(t_1) = v_2(t_1) \), and by invariance of the open first quadrant, \( u_1(t_1) = v_1(t_1) > 0 \). Now, since \( u_1(t) - v_1(t) > 0 \) on \( (t_0, t_1) \) and \( u_1(t_1) - v_1(t_1) = 0 \), we must have \( \dot{u}_1(t_1) = \dot{v}_1(t_1) \leq 0 \). On the other hand, from (3.1) and (3.2), we have that
\[ \dot{u}_1(t_1) - \dot{v}_1(t_1) = u_1(t_1)[(r_1(t_1) - s_1(t_1)) + (s_{11}(t_1) - r_{11}(t_1)) u_1(t_1) + s_{12}(t_1) v_2(t_1) - r_{12}(t_1) u_2(t_1)] > 0. \]
This contradiction shows that case (a) is impossible. In the same way, (b) also leads to a contradiction. This shows that
\[ u_1(t) > v_1(t) \quad \text{and} \quad u_2(t) < v_2(t), \quad \text{for all} \quad t \in (t_0, +\infty). \]
The proof is complete.

Assume that \( b_i \sim \bar{b}_i \) and \( a_{ij} \sim \bar{a}_{ij} \) \((i, j = 1, 2)\). Then, for any \( \varepsilon \in (0, \varepsilon_0) \), there exists \( t_\varepsilon > 0 \) such that
\[ \left| b_i(t) - \bar{b}_i(t) \right| < \varepsilon \quad \text{and} \quad \left| a_{ij}(t) - \bar{a}_{ij}(t) \right| < \varepsilon \tag{3.4} \]
for all \( t \geq t_\varepsilon \). We construct the following two systems:
\[ \begin{align*}
\dot{x} &= x[(\bar{b}_1(t) - s_1(t) + \varepsilon)x - (\bar{a}_{12}(t) + \varepsilon)y], \\
\dot{y} &= y[(\bar{b}_2(t) + \varepsilon)x - (\bar{a}_{22}(t) - \varepsilon)y];
\end{align*} \tag{3.5} \]
and
\[ \begin{align*}
\dot{x} &= x[(\bar{b}_1(t) + \varepsilon)x - (\bar{a}_{11}(t) + \varepsilon)x - (\bar{a}_{12}(t) + \varepsilon)x - (\bar{a}_{22}(t) + \varepsilon)y], \\
\dot{y} &= y[(\bar{b}_2(t) - \varepsilon)x - (\bar{a}_{21}(t) + \varepsilon)x - (\bar{a}_{22}(t) - \varepsilon)y].
\end{align*} \tag{3.6} \]
From Lemma 2 and inequalities (3.4), we have the following lemma.

**Lemma 3.** Suppose that \( \text{col}(x^\varepsilon(t), y^\varepsilon(t)) \) and \( \text{col}(x^{-\varepsilon}(t), y^{-\varepsilon}(t)) \) are solutions of (3.5) and (3.6), respectively, satisfying
\[ x^\varepsilon(t_\varepsilon) = x^\varepsilon(t_\varepsilon) > 0 \quad \text{and} \quad y^\varepsilon(t_\varepsilon) = y^\varepsilon(t_\varepsilon) > 0. \]
Then
\[ x^{-\varepsilon}(t) < x(t) < x^\varepsilon(t), \quad y^{-\varepsilon}(t) < y(t) < y^\varepsilon(t) \tag{3.7} \]
for all \( t > t_\varepsilon \), where \( \text{col}(x(t), y(t)) \) is the solution of (1.1) satisfying \( x(t_\varepsilon) = x^{-\varepsilon}(t_\varepsilon) = x^\varepsilon(t_\varepsilon) > 0 \) and \( y(t_\varepsilon) = y^\varepsilon(t_\varepsilon) = y^{-\varepsilon}(t_\varepsilon) > 0. \)

Proof. Inequalities (3.4) lead to
\[ \begin{align*}
\bar{b}_1(t) - \varepsilon &< b_i(t) < \bar{b}_1(t) + \varepsilon, \\
\bar{a}_{ij}(t) - \varepsilon &< a_{ij}(t) < \bar{a}_{ij}(t) + \varepsilon
\end{align*} \]
for all \( t \geq t_\varepsilon \), which imply that compared with (1.1), respectively, (3.5) and (3.6) satisfy the conditions described in Lemma 2. So from Lemma 2, inequalities (3.7) hold immediately. The proof is complete.
If \( \tilde{b}_i(t), \tilde{a}_{ij}(t) \) \((i, j = 1, 2)\) are continuous, positive and \(T\)-periodic functions, we wish that there exist positive \(T\)-periodic solutions of all (2.1), (3.5) and (3.6), respectively, which are globally asymptotically stable. In fact, we have the following two lemmas which assure the existences and stabilities of periodic solutions.

**Lemma 4.** Conditions (1.4) imply

\[
\frac{b^L_1}{b^M_1} > \frac{\tilde{a}^L_{12}}{\tilde{a}^L_{22}}, \quad (3.8)
\]

and

\[
\frac{\tilde{b}^L_1 - \varepsilon}{\tilde{b}^M_1 + \varepsilon} > \frac{\tilde{a}^L_{12} + \varepsilon}{\tilde{a}^L_{22} - \varepsilon} \quad (3.9)
\]

for all \(0 < \varepsilon \leq c_1 - c_0/2\).

**Proof.** Since (3.10) \(\implies\) (3.8), (3.13) and (3.12) \(\implies\) (3.9), (3.11), we only need to prove (3.10) and (3.12). From inequalities (3.4) and the periodicity of \(b_i\) and \(a_{ij} \) \((i, j = 1, 2)\), for all \(t > t_0\), there exist

\[
b_1(t) - \varepsilon < \tilde{b}_1(t) < b_1(t) + \varepsilon,
\]

\[
b_2(t) - \varepsilon < \tilde{b}_2(t) < b_2(t) + \varepsilon,
\]

\[
b_{12}(t) - \varepsilon < \tilde{a}_{12}(t) < a_{12}(t) + \varepsilon,
\]

\[
a_{22}(t) - \varepsilon < \tilde{a}_{22}(t) < a_{22}(t) + \varepsilon,
\]

as results of which, we have

\[
b^L_1 - \varepsilon < \tilde{b}^L_1, \quad \tilde{b}^M_1 < b^M_1 + \varepsilon, \quad \tilde{a}^L_{12} < a^L_{12} + \varepsilon, \quad \tilde{a}^L_{22} > a^L_{22} - \varepsilon. \quad (3.14)
\]

Now, for all \(0 < \varepsilon \leq \varepsilon_1 = c_0/2\), (3.14) and (1.4) result in

\[
\frac{\tilde{b}^L_1 - \varepsilon}{\tilde{b}^M_1 + \varepsilon} > \frac{\tilde{b}^L_1 - 2\varepsilon}{\tilde{b}^M_1 + 2\varepsilon} > \frac{\tilde{a}^L_{12} + 2\varepsilon}{\tilde{a}^L_{22} - 2\varepsilon} > \frac{\tilde{a}^L_{12} + \varepsilon}{\tilde{a}^L_{22} - \varepsilon}.
\]

Consequently, (3.10) holds for all \(0 < \varepsilon \leq \varepsilon_1\). Similarly, (3.12) also holds for all \(0 < \varepsilon \leq \varepsilon_1\). By previous remarks, the Lemma is proved.

Therefore, as an application of the results obtained in [3] and [4], we have the following lemma.

**Lemma 5.** Under the conditions (1.4), there exist positive \(T\)-periodic solutions \(\text{col}(\tilde{x}(t), \tilde{y}(t)), \text{col}(u^{-}\varepsilon(t), v^{+}(t))\) and \(\text{col}(u^{+}(t), v^{-}\varepsilon(t))\) of (2.1), (3.5) and (3.6), respectively, which are all globally asymptotically stable. Moreover,

\[
0 < \frac{\tilde{a}^L_{12} - \tilde{a}^M_{12}}{\tilde{a}^L_{12} - \tilde{a}^M_{12}} \equiv r_1 \leq \tilde{x}(t) \leq r_2 = \frac{\tilde{b}^M_{12} - \tilde{a}^M_{12}}{\tilde{b}^L_{12} - \tilde{a}^L_{12}} \quad (3.15)
\]

and

\[
0 < \frac{\tilde{a}^L_{12} - \tilde{a}^M_{12}}{\tilde{a}^L_{12} - \tilde{a}^M_{12}} \equiv S_1 \leq \tilde{y}(t) \leq S_2 = \frac{\tilde{b}^M_{12} - \tilde{a}^M_{12}}{\tilde{b}^L_{12} - \tilde{a}^L_{12}}. \quad (3.16)
\]
where

\[ r_1^{\pm \varepsilon} = \frac{(\bar{b}_1^+ - \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon)}{\bar{a}_1^+ + \varepsilon}(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ r_2^{\pm \varepsilon} = \frac{(\bar{b}_1^+ - \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon)}{\bar{a}_1^+ + \varepsilon}(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ s_1^\varepsilon = \frac{(\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon) - (\bar{a}_1^+ - \varepsilon)(\bar{b}_2^+ - \varepsilon)}{(\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ s_2^\varepsilon = \frac{(\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon) - (\bar{a}_1^+ - \varepsilon)(\bar{b}_2^+ - \varepsilon)}{(\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ r_1^{\pm \varepsilon} = \frac{(\bar{b}_1^+ - \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon)}{(\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ r_2^{\pm \varepsilon} = \frac{(\bar{b}_1^+ - \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon)}{(\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ s_1^\varepsilon = \frac{(\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon) - (\bar{a}_1^+ - \varepsilon)(\bar{b}_2^+ - \varepsilon)}{(\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

\[ s_2^\varepsilon = \frac{(\bar{a}_1^+ + \varepsilon)(\bar{b}_2^+ + \varepsilon) - (\bar{a}_1^+ - \varepsilon)(\bar{b}_2^+ - \varepsilon)}{(\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon) - (\bar{a}_1^+ + \varepsilon)(\bar{a}_2^+ - \varepsilon)} \]

From the expressions of \( r_i^{\pm \varepsilon} \) and \( s_i^{\pm \varepsilon} \) \((i = 1, 2)\), we have the following lemma.

**Lemma 6.** There exists \( \varepsilon_2 : 0 < \varepsilon_2 \leq \varepsilon_1 \) such that

\[ |r_1^{\pm \varepsilon}, r_2^{\pm \varepsilon}| \subset \left[ \frac{r_1^{-}}{2}, r_2^{-} + 1 \right] \]

and

\[ |s_1^{\pm \varepsilon}, s_2^{\pm \varepsilon}| \subset \left[ \frac{s_1^{-}}{2}, s_2^{-} + 1 \right], \]

for \( 0 < \varepsilon \leq \varepsilon_2 \), where \( \delta = 1 \) or \( \delta = -1 \).

Before proving the main result stated in Section 2, we give the last lemma.

**Lemma 7.** Suppose that \( \text{col}(\bar{x}(t), \bar{y}(t)) \), \( \text{col}(u^{-\varepsilon}(t), v^{-\varepsilon}(t)) \) and \( \text{col}(u^{\varepsilon}(t), v^{\varepsilon}(t)) \) are the positive \( T \)-periodic solutions described in Lemma 5. If we denote them by \( \Gamma, \Gamma_{\varepsilon} \) and \( \Gamma_{\varepsilon} \), respectively, then for all \( \eta > 0 \), there exists \( \alpha > 0 \) such that \( \Gamma_{-\varepsilon} \subset O(\Gamma, \eta) \) and \( \Gamma_{\varepsilon} \subset O(\Gamma, \eta) \) (the \( \eta \)-neighborhood of \( \Gamma \)) for all \( 0 < \varepsilon < \alpha \), in notations \( \text{col}(u^{-\varepsilon}, v^{-\varepsilon}) \nsim \text{col}(\bar{x}, \bar{y}) \) and \( \text{col}(u^{\varepsilon}, v^{\varepsilon}) \nsim \text{col}(\bar{x}, \bar{y}) \).

**Remark.** Because of the continuity of solutions of differential equations with respect to initial conditions and parameters (see, for example [7]), it follows that \( \text{col}(u^{-\varepsilon}, v^{-\varepsilon}) \nsim \text{col}(\bar{x}, \bar{y}) \) and \( \text{col}(u^{\varepsilon}, v^{\varepsilon}) \nsim \text{col}(\bar{x}, \bar{y}) \iff \) for all \( \eta > 0 \), there exists \( \alpha > 0 \), such that

\[ \inf_{0 \leq t \leq T} |u^{-\varepsilon}(t) - \bar{x}(0)| + |v^{-\varepsilon}(t) - \bar{y}(0)| < \eta \]

and

\[ \inf_{0 \leq t \leq T} |u^{\varepsilon}(t) - \bar{x}(0)| + |v^{\varepsilon}(t) - \bar{y}(0)| < \eta. \]
for all $0 < \varepsilon < \alpha$. The method to establish the two inequalities is similar, so to prove Lemma 7, we only need to show that, for all $\eta > 0$, there exists $\varepsilon_3 : 0 < \varepsilon_3 \leq \varepsilon_2$ such that

$$\inf_{0 \leq t < T} \left[ |u^{\varepsilon}(t) - \tilde{x}(0)| + |v^{\varepsilon}(t) - \tilde{y}(0)| \right] < \eta, \quad \text{for all } 0 < \varepsilon < \varepsilon_3.$$  \hspace{1cm} (3.15)

**Proof of Lemma 7.** Suppose (3.15) does not hold. Then

$$\limsup_{\varepsilon \to 0} \inf_{0 \leq t < T} \left[ |u^{\varepsilon}(t) - \tilde{x}(0)| + |v^{\varepsilon}(t) - \tilde{y}(0)| \right] = \limsup_{\varepsilon \to 0} d(\varepsilon) - d > 0.$$  

This is because $u^{\varepsilon}(t)$ and $v^{\varepsilon}(t)$ are continuous and $T$ periodic functions. There exists $t_\varepsilon \in [0, T)$ such that

$$d(\varepsilon) = \inf_{0 \leq t < T} \left[ |u^{\varepsilon}(t) - \tilde{x}(0)| + |v^{\varepsilon}(t) - \tilde{y}(0)| \right] = |u^{\varepsilon}(t_\varepsilon) - \tilde{x}(0)| + |v^{\varepsilon}(t_\varepsilon) - \tilde{y}(0)|.$$  

Without loss of generality, we can assume that $t_\varepsilon = 0$, namely that

$$d(\varepsilon) = |u^{\varepsilon}(0) - \tilde{x}(0)| + |v^{\varepsilon}(0) - \tilde{y}(0)|.$$  

In fact, if we denote $\text{col}(\tilde{u}^{\varepsilon}(t), \tilde{v}^{\varepsilon}(t)) = \text{col}(u^{\varepsilon}(t - t_\varepsilon), v^{\varepsilon}(t - t_\varepsilon))$, then $\text{col}(\tilde{u}^{\varepsilon}(t), \tilde{v}^{\varepsilon}(t))$ is also the unique $T$-periodic solution of (3.5) and

$$\inf_{0 \leq t < T} \left[ |\tilde{u}^{\varepsilon}(t) - \tilde{x}(0)| + |\tilde{v}^{\varepsilon}(t) - \tilde{y}(0)| \right] = |\tilde{u}^{\varepsilon}(0) - \tilde{x}(0)| + |\tilde{v}^{\varepsilon}(0) - \tilde{y}(0)|.$$  

From $\limsup_{\varepsilon \to 0} d(\varepsilon) = d > 0$, there exist $\{\varepsilon_j\}_{j} \subset (0, \varepsilon_2)$ such that $\lim_{\varepsilon_j \to 0} d(\varepsilon_j) = d > 0$. By Lemma 6 we know that $\{\text{col}(u^{\varepsilon}(0), v^{\varepsilon}(0)) : 0 < \varepsilon < \varepsilon_2\}$ is a bounded set. Because the Euclid space $\mathbb{R}^2$ is complete, there exist $\{\varepsilon_{j_k}\}_{k} \subset \{\varepsilon_j\}_{j}$, without loss of generality we can assume $\{\varepsilon_{j_k}\}_{k} = \{\varepsilon_j\}_{j}$, such that

$$\lim_{\varepsilon_j \to 0} \text{col}(u^{\varepsilon_j}(0), v^{\varepsilon_j}(0)) = \text{col}(u(0), v(0)).$$  

Let $\text{col}(u(t), v(t))$ denote the solution of (2.1) having the initial value $\text{col}(u(0), v(0))$. Continuous dependence on initial conditions and parameters for (3.5) leads to

$$\lim_{\varepsilon_j \to 0} \text{col}(u^{\varepsilon_j}(t), v^{\varepsilon_j}(t)) = \text{col}(u(t), v(t)).$$  

$\text{col}(u^{\varepsilon_j}(t), v^{\varepsilon_j}(t))$ is $T$-periodic; so is $\text{col}(u(t), v(t))$. But under (1.4), there exists a unique $T$-periodic solution of (2.1). Consequently,

$$\text{col}(u(t), v(t)) = \text{col}(\tilde{x}(t), \tilde{y}(t)),$$  

as a consequence of which, we have

$$0 = |u(0) - \tilde{x}(0)| + |v(0) - \tilde{y}(0)| = d > 0.$$  

This is a contradiction. Therefore, (3.15) holds. By previous remark, the proof of Lemma 7 is complete.

Now, let us show the main result of this paper.

**Proof of Theorem.** Let $\text{col}(x(t), y(t))$ be a “fixed” positive solution of (1.1). After constructing systems (3.5) and (3.6), from Lemma 7, for all $\eta > 0$, $\exists \varepsilon_3 : 0 < \varepsilon_3 \leq \varepsilon_2 \leq \varepsilon_1 = \delta_0/2$ such that

$$|u^{\varepsilon}(t) - \tilde{x}(t)| < \eta \quad \text{and} \quad |v^{\varepsilon}(t) - \tilde{y}(t)| < \eta, \quad \text{for } 0 < \varepsilon < \varepsilon_3,$$  \hspace{1cm} (3.15)
where $\delta = 1$ or $\delta = -1$. Pick any $0 < \varepsilon < \varepsilon_3$. Since $b_i \sim \tilde{b}_i, a_{ij} \sim \tilde{a}_{ij}$ ($i, j = 1, 2$), there exists $t_2 > 0$ such that

$$|b_i(t) - \tilde{b}_i(t)| < \varepsilon, \quad |a_{ij}(t) - \tilde{a}_{ij}(t)| < \varepsilon,$$

for all $t \geq t_2$.

Let col$(x^-\varepsilon(t), y^-\varepsilon(t))$ and col$(x^\varepsilon(t), y^\varepsilon(t))$ be the solutions of (3.5) and (3.6), respectively, satisfying

$$x^-\varepsilon(t_2) - x^\varepsilon(t_2) = x(t_2) \quad \text{and} \quad y^\varepsilon(t_2) - y^-\varepsilon(t_2) = y(t_2).$$

From Lemma 2 or Lemma 3, we have

$$x^-\varepsilon(t) < x(t) < x^\varepsilon(t) \quad \text{and} \quad y^-\varepsilon(t) < y(t) < y^\varepsilon(t), \quad (3.16)$$

for all $t > t_2$. From Lemma 5 and Lemma 7, we have

$$\text{col}(x^-\varepsilon, y^-\varepsilon) \sim \text{col}(u^-\varepsilon, v^-\varepsilon) \sim \text{col}(\bar{x}, \bar{y}) \quad (3.17)$$

and

$$\text{col}(x^\varepsilon, y^-\varepsilon) \sim \text{col}(u^\varepsilon, v^-\varepsilon) \sim \text{col}(\bar{x}, \bar{y}). \quad (3.18)$$

So (3.16), (3.17) and (3.18) lead to

$$\text{col}(x, y) \sim \text{col}(\bar{x}, \bar{y})$$

and the proof is complete.

Hence, although there do not exist positive periodic solutions of such a model (1.1) satisfying (1.4), any positive solutions of (1.1) asymptotically approach a strictly positive periodic solution of the corresponding periodic system.

REFERENCES