On a Problem in Monotone Approximation

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Let \( f \in C[-1, 1] \). A sufficient condition is given which ensures that the \( n \)th polynomial of best approximation to \( f \) is increasing for \( n \) sufficiently large. Using this condition, we are able to give a counterexample to a theorem announced by Tzimbalario [6].

1. INTRODUCTION

Let \( n, k \) be nonnegative integers and let \( \Pi_n \) denote the set of algebraic polynomials of degree \( n \) or less. Let \( C^k[-1, 1] \) denote the class of functions which have a continuous \( k \)th derivative on \([-1, 1](C[-1, 1] \) will mean \( C^1[-1, 1] \)).

For \( f \in C[-1, 1] \), define

\[
E_n(f) = \min_{p \in \Pi_n} \| p - f \|
\]

where \( \| \| \) denotes the uniform norm on \([-1, 1]\). It is well known that for each \( n \) the above minimum is attained by a unique element in \( \Pi_n \). We call this element the \( n \)th algebraic polynomial of best approximation to \( f \). Later, if no confusion is likely to occur, we will always denote it by \( p_n \) for any given \( f \).

The problem we study in this paper is the following: Let \( f \in C[-1, 1] \), and assume that there exists a \( \delta > 0 \), such that

\[
(f(x_1) - f(x_2))/(x_1 - x_2) \geq \delta \tag{1.1}
\]

for all \( x_1, x_2 \in [-1, 1] \) with \( x_1 \neq x_2 \). What extra condition on \( f \) is needed to ensure that \( p_n \) is increasing for all \( n \) sufficiently large?

Roulier [4] showed that \( f \in C^2[-1, 1] \) is such a condition. Also in [4], Roulier asked: if \( f \in C^1[-1, 1] \) and satisfies (1.1) (or equivalently \( f''(x) \geq \delta \))...
for $x \in [-1, 1]$, is $p_n$ increasing for $n$ sufficiently large? In [5], Roulier conjectured that the answer is negative.

In answering this question, Tzimbalario [6] announced the following theorem:

**Theorem 1.1.** Let $f$ be a continuous function on $[-1, 1]$ with $f'$ not in some $\text{Lip} \, \alpha$, $\alpha < 1$, and $f' \geq \delta$ for some strictly positive $\delta$. Then there are infinitely many $n$ for which $p_n$ is not increasing.

Fletcher and Roulier discussed this problem in [2]. Their main results are the following two theorems.

**Theorem 1.2.** Let $\alpha$ be given in the interval $0 < \alpha < 1$. There exists $f \in C^1[-1, 1]$ for which

$$f'(x) \geq \delta > 0, \quad x \in [-1, 1]$$

(1.2)

and

$$f' \in \text{Lip} \, \alpha, \quad \text{but} \quad f' \notin \text{Lip}(\alpha + \varepsilon)$$

(1.3)

for any $\varepsilon > 0$, such that there are infinitely many $n$ for which $p_n$ is not increasing on $[-1, 1]$.

**Theorem 1.3.** Let $0 < \alpha < 1$ be given. There exists a function $f \in C^1[-1, 1]$ for which (1.2) and (1.3) hold and such that $p_n$ is increasing for all $n$ sufficiently large.

Also in [2], Fletcher and Roulier drew the conclusion that Theorem 1.3 provides counterexamples to Tzimbalario's Theorem (Theorem 1.1).

We have noted that there might be the following different interpretations of Tzimbalario's Theorem:

**Theorem 1.1a.** Let $f$ be a function in $C^1[-1, 1]$ for which (1.2) holds. If there exists $\alpha$ in the interval $(0, 1)$, such that $f' \notin \text{Lip} \, \alpha$, then there are infinitely many $n$ for which $p_n$ is not increasing.

**Theorem 1.1b.** Let $f$ be a function in $C^1[-1, 1]$ for which (1.2) holds. If $f' \notin \bigcup_{0 < \alpha < 1} \text{Lip} \, \alpha$, then there are infinitely $n$ for which $p_n$ is not increasing.

Theorem 1.3 only provides counterexamples to Theorem 1.1a, and it is obvious that a counterexample to Theorem 1.1b will automatically be one to Theorem 1.1a.

The purpose of this paper is to prove a stronger result on the positive aspect of the problem, and to provide a counterexample to Theorem 1.1b.
2. Main Results

**Theorem 2.1.** Let $f$ be a function in $C[-1, 1]$, satisfying (1.1). If $E_n(f) = o(n^{-2})$, then $p_n$ is increasing for all $n$ sufficiently large.

Since $f \in C^2[-1, 1]$ necessarily means $E_n(f) = o(n^{-2})$, by Jackson's Theorem [1, pp. 147], Theorem 2.1 is stronger than Roulier's result [3].

**Theorem 2.2.** There exists $f \in C^1[-1, 1]$ for which (1.2) holds, but $f' \notin \bigcup_{0 < \alpha < 1} \text{Lip } \alpha$, such that $p_n$ is increasing on $[-1, 1]$ when $n$ is sufficiently large.

The contradiction between Theorems 2.2 and 1.1 is apparent.

**Lemma.** Let $f \in C^1[-1, 1]$ and $q_n \in \Pi_n$. If \( \|q_n - f\| = o(n^{-2}) \) then \( \|q_n''\| = o(n^2) \).

**Proof.** Let $n$ be fixed, and choose $k$ so that $2^k < n \leq 2^{k+1}$. Write

\[
q_n = (q_n - q_{2^k+1}) + \sum_{i=0}^{k} (q_{2^{i+1}} - q_{2^i}) + q_1.
\]

Since $q_1^n = 0$, \( \|q_n''\| \leq \|q_n'' - q_{2^k+1}''\| + \sum_{i=0}^{k} \|q_{2^{i+1}}'' - q_{2^i}''\| \). Since \( \|q_n - f\| \leq \|q_n - f\| + \|f - q_{2^k+1}\| = o(n^{-2}) \), by Markov's Inequality, \( \|q_n'' - q_{2^k+1}''\| = o(n^2) \).

Let $A(n) = \sum_{i=0}^{k} \|q_{2^{i+1}}'' - q_{2^i}''\||$. It remains to show that $A(n) = o(n^2)$, i.e., that for any given $\epsilon > 0$, $A(n) < \epsilon n^2$ for $n$ sufficiently large.

We introduce some new notations by letting $v_i = q_{2^i}$ and $\beta_i = \sup_{j > i} \|v_j - f\|$. We have \( \|v_{i+1} - v_i\| \leq \beta_{i+1} + \beta_i \leq 2\beta_i \).

By Markov's Inequality, \( \|v_{i+1} - v_i\| \leq 2\beta_i(2^{i+1})^4 \). As \( \|q_n - f\| = o(n^{-2}) \), we may assume that $\beta_i \leq \alpha_i(2^i)^{-2}$ where $\alpha_i > 0$.

Now we have

\[
A(n) \leq \sum_{i=0}^{k} 2\beta_i(2^{i+1})^4 = \sum_{i=0}^{k} 2\alpha_i(2^i)^{-2}(2^{i+1})^4 = 32 \sum_{i=0}^{k} \alpha_i 4^i.
\]

Given $\epsilon > 0$, select $m$ so that $\alpha_i < \epsilon$ when $i \geq m$. Select $N \geq m$ so that $n^{-2} \sum_{i=0}^{m-1} \alpha_i 4^i < \epsilon$ when $n \geq N$. Then for any $n \geq N$ we will have

\[
(1/32)n^{-2} A(n) \leq n^{-2} \sum_{i=0}^{m-1} \alpha_i 4^i + n^{-2} \sum_{i=m}^{k} \alpha_i 4^i
\]

\[
\leq \epsilon + n^{-2} \alpha_i 4^i \leq \epsilon + n^{-2} \alpha_i 4^i \leq 5\epsilon.
\]
Proof of Theorem 2.1. Let $\delta$ be as in (1.1); we will show that $p'(x) \geq \delta/4$ for $n$ sufficiently large. Suppose not; then there is an infinite subset of natural numbers $N^*$ such that the following is true for $n \in N^*$,

$$p'_n(x_n) < \delta/4,$$  \hspace{1cm} (2.1)

where $x_n, n \in N^*$, is a sequence of points in the interval $[-1, 1]$. By the Mean-Value Theorem and the lemma we have just proved, we have, for $n$ sufficiently large, that

$$|p'_n(x_n) - p'_n(x_n \pm h)| = |p''_n(\xi)| h \leq \|p''_n\| n^{-2} < \delta/4,$$ \hspace{1cm} (2.2)

where $0 \leq h \leq n^{-2}$ and the sign $+$ or $-$ is chosen so that $x_n + h$ or $x_n - h$ is in the interval $[-1, 1]$. In the following, for the convenience of writing, we assume that $+$ has always been chosen.

By (2.1) and (2.2)

$$p'_n(x_n + h) = p'_n(x_n + h) - p'_n(x_n) + p'_n(x_n) < \delta/4 + \delta/4 = \delta/2.$$

Using the Mean-Value Theorem again, we have

$$p_n(x_n + n^{-2}) - p_n(x_n) < \delta/(2n^2).$$ \hspace{1cm} (2.3)

As $\|p_n - f\| = o(n^{-2})$ we may assume that $\|f - p_n\| < \delta/(4n^2)$. Using this last inequality and (2.3), we get

$$f(x_n + n^{-2}) - f(x_n)$$

$$= \left[ f(x_n + n^{-2}) - p_n(x_n + n^{-2}) \right]$$

$$+ \left[ p_n(x_n + n^{-2}) - p_n(x_n) \right] + \left[ p_n(x_n) - f(x_n) \right]$$

$$< \delta/(4n^2) + \delta/(2n^2) + \delta/(4n^2) = \delta/n^2.$$

This contradicts the assumption (1.1), and completes the proof.

Proof of Theorem 2.2. We choose the basic interval here to be $[0, 1]$ instead of $[-1, 1]$; there is no loss of generality in doing this.

Let

$$g(x) = \begin{cases} 
    x/\ln(2/x), & x \in (0, 1] \\
    0, & x = 0.
\end{cases}$$

Then

$$g'(x) = \begin{cases} 
    (\ln(2/x) + 1)/\ln(2/x)^2, & x \in (0, 1] \\
    0, & x = 0.
\end{cases}$$
Let \( f(x) = g(x) + \delta x \), where \( \delta \) is as in (1.1). It is obvious that \( f \) satisfies (1.2). Since
\[
\lim_{x \to 0} \frac{g'(x)}{x^\alpha} = \infty
\]
for any \( 0 < \alpha < 1 \), we infer
\[
g'(x) \notin \text{Lip } \alpha \quad \text{for every } \alpha \text{ satisfying } 0 < \alpha < 1.
\]
By Theorem 2.1, the proof will be completed if we can show that \( E_n(f) = o(n^{-2}) \). As \( E_n(f) = E_n(g) \) for \( n \geq 1 \), it suffices to show that \( E_n(g) = o(n^{-2}) \). Consider
\[
G(x) = \begin{cases} 
\frac{x^2}{\ln(2/x^2)}, & x \in [-1, 0) \cup (0, 1] \\
0, & x = 0.
\end{cases}
\]
Differentiating \( G(x) \) twice, we observe that \( G \in C^2[-1, 1] \). By Jackson's Theorem \( E_n(G) = o(n^{-2}) \). Let \( Q_{2n} \) be the 2nth best approximation polynomial of \( G \). As \( G \) is even, so is \( Q_{2n} \) [3, Chapt. 2, Problem 3], and therefore \( Q_{2n}(x) = q_n(x^2) \) where \( q_n \) is a polynomial of degree \( n \) or less. We have now
\[
\| g(x) - q_n(x) \|_{[0,1]} = \left\| \frac{x}{\ln \frac{2}{x^2}} - q_n(x) \right\|_{[0,1]} \\
= \left\| \frac{x^2}{\ln \frac{2}{x^2}} - q_n(x^2) \right\|_{[-1,1]} \\
= \|G(x) - Q_{2n}(x)\|_{[-1,1]} = o(n^{-2}).
\]
So we have \( E_n(g) = o(n^{-2}) \) and Theorem 2.2 is proved.

3. Comment and Conjecture

If we read carefully the proofs of Theorems 1.2 and 1.3 by Fletcher and Roulier [2], we find that the example in the proof of Theorem 1.3 satisfies the condition of Theorem 2.1, i.e., \( E_n(f) = o(n^{-2}) \), while the one in that of Theorem 1.2 does not. The result of Theorem 2.2 also exhibits the power of Theorem 2.1.

We make the following conjecture:

**Conjecture.** Theorem 2.1 cannot be improved, in the sense that there exists \( f \in C[-1, 1] \), satisfying (1.1), and \( E_n(f) = O(n^{-2}) \), such that \( p_n(x) \) is not increasing for infinitely many \( n \).
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