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CONVERGENCE IN DISTRIBUTION OF QUOTIENTS OF ORDER STATISTICS

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Let $X_1, X_2, \ldots$ be i.i.d. random variables with continuous distribution function $F < 1$. It is known that if $1 - F(x)$ varies regularly of order $-\rho$, the successive quotients of the order statistics in decreasing order of $X_1, \ldots, X_n$ are asymptotically independent, as $n \to \infty$, with distribution functions $x^{k\rho}$, $k = 1, 2, \ldots$. A strong converse is proved, viz. convergence in distribution of this type of one of the quotients implies regular variation of $1 - F(x)$.

order statistics	partial maxima
limit theorem	regular variation
Wiener–Tauber theorem	

1. Introduction and results

Let $X_1, X_2, \ldots$ be independent random variables with common continuous distribution function F such that $F(x) < 1$ for all x. By $M_{n0}, M_{n1}, \ldots, M_{n,n-1}$ we denote the order statistics in decreasing order of $X_1, \ldots, X_n$, so that $X_{n0} = \max\{X_1, \ldots, X_n\}$. The index n will be suppressed occasionally. The following result is contained in the work of Dwass [2, 3], Lamperti [6] and Polfeldt [8]:

Theorem 1. *If* $1 - F(x)$ *varies regularly of order* $-\rho \leqslant 0$, *as* $x \to \infty$, *then for* $k = 1, 2, \ldots$ *and* $0 < \xi_i < 1$, $i = 1, \ldots, k$,

$$\lim_{n \to \infty} \mathbf{P}[M_{nj} M_{n,j-1}^{-1} < \xi_j,\ j = 1, \ldots, k] = \xi_1^{\rho} \xi_2^{2\rho} \ldots \xi_k^{k\rho} . \qquad (1)$$

We give a proof in Section 2, but our main goal is proving the following strong converse:

Theorem 2. *If for some* $j \geq 1$, $\xi \in (0, 1)$ *and* $\rho \geq 0$,

$$\lim_{n\to\infty} \mathbf{P}[M_{nj} M_{n,j-1}^{-1} < \xi] = \xi^{j\rho}, \tag{2}$$

then

$$\lim_{y\to\infty} (1 - F(\xi^{-1} y))/(1 - F(y)) = \xi^{\rho}. \tag{3}$$

Corollary. *If* (2) *holds for all* $\xi \in (0, 1)$ (*or even for* ξ_1 *and* ξ_2 *with* $(\log \xi_1)/(\log \xi_2)$ *irrational, see* [5, *Theorem* 1.1.2]), *then* $1 - F(x)$ *varies regularly of order* $-\rho$ *as* $x \to \infty$.

Let $Y_1, Y_2, \ldots$ be independent random variables with common continuous distribution function $G < 1$ and let $N_{n0}, \ldots, N_{n,n-1}$ be the order statistics in decreasing order of $Y_1, \ldots, Y_n$. By considering $X_i = \exp(Y_i)$ with distribution function $F(x) = G(\log x)$ and order statistics $M_{nj} = \exp(N_{nj})$, one immediately derives from Theorems 1 and 2:

Theorem 3. *If* $(1 - G(y + \eta))/(1 - G(y)) \to \exp(-\rho\eta)$ *as* $y \to \infty$, *for all* $\eta > 0$, *then for* $k = 1, 2, \ldots$ *and* $\eta_i > 0$, $i = 1, \ldots, k$,

$$\lim_{n\to\infty} \mathbf{P}[N_{n,j-1} - N_{n,j} > \eta_j,\ j = 1, \ldots, k] = \exp(-\rho\eta_1 - 2\rho\eta_2 - \ldots - k\rho\eta_k).$$

Theorem 4. *If for some* $j \geq 1$, $\eta > 0$ *and* $\rho \geq 0$,

$$\lim_{n\to\infty} \mathbf{P}[N_{n,j-1} - N_{n,j} > \eta] = \exp(-\rho j \eta),$$

then

$$(1 - G(y + \eta))/(1 - G(y)) \to \exp(-\rho\eta) \quad \textit{as } y \to \infty.$$

Remarks. Theorems 1 and 2 with $\rho = 0$ mean that $M_{nj} M_{n,j-1}^{-1} \to 0$ in probability if and only if $1 - F(x)$ varies slowly at infinity. We may extend Theorems 1 and 2 to $\rho = \infty$, with $\xi^{\rho} = 0$, $0 < \xi < 1$, by stating that $M_{nj} M_{n,j-1}^{-1} \to 1$ in probability if and only if $1 - F(x)$ varies rapidly at infinity, i.e. $(1 - F(ax))/(1 - F(x)) \to 0$, $a > 1$. No change in the proofs is required. An analogous remark applies to Theorems 3 and 4.

2. Proof of Theorem 1

Since $F(x) < 1$, we have $M_{nj} \to \infty$ a.s. as $n \to \infty$. So it is no restriction to assume $F(0^+) = 0$. By symmetry we have, for $1 \leqslant r < n$,

$$\mathbf{P}[M_{r-1} > a \mid M_{n-1}, \ldots, M_r] = \left\{\frac{1 - F(a)}{1 - F(M_r)}\right\}^r, \quad a > M_r .$$

So, for $1 \leqslant r \leqslant k$,

$$\mathbf{P}[M_k M_{k-1}^{-1} < \xi_k, \ldots, M_r M_{r-1}^{-1} < \xi_r] = \int_A (1 - F(\xi_r^{-1} M_r))^r (1 - F(M_r))^{-r} \, \mathrm{d}P ,$$

where $A = \Omega$ if $r = k$ and $A = \{M_k M_{k-1}^{-1} < \xi_k, \ldots, M_{r+1} M_r^{-1} < \xi_{r+1}\}$ if $r < k$. Since by the regular variation of $1 - F$,

$$\left|\xi_r^{+\rho r} \mathbf{P}[A] - \int_A (1 - F(\xi_r^{-1} M_r))^r (1 - F(M_r))^{-r} \, \mathrm{d}P\right| \leqslant$$

$$\leqslant \int_\Omega |\xi_r^{+\rho r} - (1 - F(\xi_r^{-1} M_r))^r (1 - F(M_r))^{-r}| \, \mathrm{d}P \to 0 .$$

we have

$$\lim_{n \to \infty} |\xi_r^{+\rho r} \mathbf{P}[A] - \mathbf{P}[M_{nk}/M_{n,k-1} < \xi_k, \ldots, M_{nr}/M_{n,r-1} < \xi_r]| = 0 ,$$

from which the theorem follows by induction with respect to r.

3. Proof of Theorem 2

Since $F(x) < 1$, we have $M_{nk} \to \infty$ a.s. as $n \to \infty$, so that it is no restriction to assume $F(0^+) = 0$. Put

$$\Theta(x) = 1/(1 - F(x)), \quad x \geqslant 0 , \tag{4}$$

$$\beta(y) = \inf\{x: x \geqslant 0, \Theta(x) \geqslant y\}, \quad y \geqslant 1 . \tag{5}$$

Then by the continuity of F, we have

$$\beta(\Theta(x)) = x \quad \text{a.e. } [F] \text{ on } [0, \infty) , \tag{6}$$

$$\Theta(\beta(y)) = y, \quad y \geq 1 . \tag{7}$$

For $n \geq j+1 \geq 2$ we obtain

$$\mathbf{P}[M_{j-1} > \xi^{-1} M_j] =$$

$$= n(n-1)\binom{n-2}{j-1} \int_0^\infty (F(u))^{n-j-1}\, \mathrm{d}F(u) \int_{u\xi^{-1}}^\infty (1-F(v))^{j-1}\, \mathrm{d}F(v)$$

$$= \frac{n(n-1)\dots(n-j)}{j!} \int_0^\infty (F(u))^{n-j-1}\, (1-F(\xi^{-1}u))^j\, \mathrm{d}F(u) .$$

Taking $\Theta(u)$ as new integration variable, we find, using (6),

$$\mathbf{P}[M_{j-1} > \xi^{-1} M_j] = \frac{n(n-1)\dots(n-j)}{j!} \int_1^\infty (1-x^{-1})^{n-j-1}\, x^{-j-2}\, \gamma(x)\, \mathrm{d}x, \tag{8}$$

with $\gamma(x)$ for $x \geq 1$ defined by

$$\gamma(x) = x^j\, \{1 - F(\xi^{-1}\beta(x))\}^j = \{\Theta(\beta(x))/\Theta(\xi^{-1}\beta(x))\}^j . \tag{9}$$

Defining the probability density q_j on $(0, \infty)$ by

$$q_j(x) = \frac{1}{j!}\, x^{-j-2} \exp(-x^{-1}), \quad x > 0 , \tag{10}$$

we have

$$\lim_{n\to\infty} \int_1^\infty \left| n^{-1} q_j(n^{-1}x) - \frac{n(n-1)\dots(n-j)}{j!} (1-x^{-1})^{n-j-1}\, x^{-j-2} \right| \mathrm{d}x = 0, \tag{11}$$

$$\lim_{\substack{\lambda\to\infty \\ |\lambda-\mu|\leq 1}} \int_0^\infty |\lambda^{-1} q_j(\lambda^{-1}x) - \mu^{-1} q_j(\mu^{-1}x)|\, \mathrm{d}x = 0 . \tag{12}$$

We thank the referee for the following short proof of (11) and (12). By putting $x = ny$ in (11) and $x = \lambda y$ in (12), the integrals are reduced to

$$\int |p_n(y) - p(y)|\, \mathrm{d}y ,$$

where $p_n(y) \to p(y)$ as $n \to \infty$, and p_n, p are probability densities. Now apply Scheffé's lemma ([9], [1, App. II]).

So from (2), (8) and (11), since γ is bounded,

$$\lim_{n\to\infty} \int_1^\infty n^{-1} q_j(n^{-1}x)\, \gamma(x)\, \mathrm{d}x = \xi^{j\rho} . \tag{13}$$

Defining $\gamma(x) = 1$, say, for $0 \leqslant x < 1$, we find from (13) and (12),

$$\lim_{\lambda\to\infty} \int_0^\infty \lambda^{-1} q_j(\lambda^{-1}x)\, \gamma(x)\, \mathrm{d}x = \xi^{j\rho} . \tag{14}$$

We now use the Wiener–Tauber Theorem 3.2 given in [11, Section (8.3)]. Since γ is bounded and $q_j \in U$, i.e.

$$\int_0^\infty \mu^{-1}\, q_j(\mu^{-1}x)\, a(x)\, \mathrm{d}x = 0, \qquad \mu > 0 ,$$

implies $a(x) \equiv 0$ for bounded continuous a, we have

$$\lim_{\lambda\to\infty} \int_0^\infty \lambda^{-1} f(\lambda^{-1}x)\, \gamma(x)\, \mathrm{d}x = \xi^{j\rho} , \tag{15}$$

for any probability density f. We now prove

$$\lim_{\lambda\to\infty} \lambda\{1 - F(\xi^{-1}\beta(\lambda y))\} = y^{-1}\xi^{\rho}, \qquad y > 0 . \tag{16}$$

Since $\lambda\{1 - F(\xi^{-1}\beta(\lambda y))\}$ decreases with y and is bounded by y^{-1}, it is sufficient to show that $\lambda_k \to \infty$ and

$$\lim_{k\to\infty} \lambda_k\{1 - F(\xi^{-1}\beta(\lambda_k y))\} = \psi(y), \qquad y \notin D(\psi) , \tag{17}$$

where $D(\psi)$ is the set of discontinuity points of ψ on $(0, \infty)$, imply $\psi(y) = y^{-1}\xi^{\rho}$. Now (17) implies, by (9) and (15), since γ is bounded and $D(\psi)$ is countable, that

$$\int_0^\infty f(x)\, x^j\, \psi^j(x)\, \mathrm{d}x = \xi^{j\rho}$$

for every probability density f, so that

$$x^j\, \psi^j(x) = \xi^{j\rho} ,$$

and (16) follows. From (16) we easily derive (3), since $\beta(y) \to \infty$ as $y \to \infty$. See the proofs of [4, Lemma 2, Section VIII.8] and [5, Theorem 1.1.3].

Remark. Shorrock [10, Theorem 3] proved a result analogous to our Theorem 1 for quotients of upper record values. Attempts to prove a converse failed, even when using the strengthened version of Wiener's Tauberian theorem given by Moh [7].

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