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ACADEMIC PRESS

Historia Mathematica 31 (2004) 62–75

HISTORIA MATHEMATICA

www.elsevier.com/locate/hm

The origins of quasi-concavity: a development between mathematics and economics

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Received 28 January 2002; revised 6 February 2003; accepted 28 July 2003

Abstract

The origins of the notion of quasi-concave function are considered, with special interest in some work by John von Neumann, Bruno de Finetti, and W. Fenchel. The development of such pioneering studies subsequently led to a whole field of research, known as "generalized convexity." The different styles of the three authors and the various motivations for introducing quasi-concavity are compared, without losing sight of economic applications characteristic of the whole field of generalized convexity.

Sommario

Il lavoro considera le origini della nozione di funzione quasi-concava, con particolare riguardo ad alcuni scritti di John von Neumann, di Bruno de Finetti, e di W. Fenchel. Lo sviluppo di tali studi pionieristici ha successivamente consentito lo sviluppo di un intero campo di ricerca detto "convessità generalizzata." I diversi stili dei tre autori e le differenti motivazioni che hanno portato all'introduzione della quasi-concavità vengono confrontati, senza perdere di vista il riferimento alle applicazioni economiche che costituiscono una caratteristica in tema di convessità generalizzata.

MSC: 01A60; 26B25; 52A01; 91B16; 91A10

Keywords: Convexity; Quasi-convexity; Utility theory; Von Neumann; De Finetti; Fenchel

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0315-0860/$ – see front matter
doi:10.1016/j.hm.2003.07.001

1. Introduction

The first modern formalization of the concept of *convex function* appears in Jensen [1905]. Since then, at first referring to "Jensen's convex functions," then more openly, without needing any explicit reference, the definition of convex function becomes a standard element in calculus handbooks.

A function $f : C \subseteq R^n \to R$, where C is a convex set, is said to be *convex* when the following inequality holds, for any $x, y \in C$ and for any $t \in [0, 1]$: $f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$. If the reverse inequality holds, i.e., $-f$ is convex, the function f is said to be *concave*. Such a definition (which is based upon the three points x, y, and $tx + (1-t)y$) becomes simpler when we consider a smaller class of functions. A differentiable function is convex whenever, $\forall x, y \in C$, the following inequality, whose geometric meaning is apparent, holds: $f(x) - f(y) \geqslant \nabla f(x)(y - x)$. Moreover, a twice differentiable function f is convex whenever, $\forall x \in C$, $\mathrm{d}^2 f(x) \geqslant 0$; the last inequality extends, in a natural way, the elementary rule that studies the convexity of a real function of a real variable through the sign of its second derivative.

We shall not mention the importance and the applications of a basic instrument such as convexity. We just give some hints on the topics that will be brought up later. Convexity is one of the most frequently used hypotheses in optimization theory. It is usually introduced to give global validity to propositions otherwise only locally true (for convex functions, for instance, a local minimum is also a global minimum) and to obtain sufficiency for conditions that are generally only necessary, as with the classical Fermat theorem or with Kuhn–Tucker conditions in nonlinear programming. For the history of linear and nonlinear programming one can turn to Grattan-Guinness [1989], Giorgi and Guerraggio [1998], Kjeldsen [2000]. In microeconomics, convexity plays a fundamental role in general equilibrium theory and in duality results. In particular, in consumer theory, the so-called convexity of preferences ensures the existence of a demand function; moreover, in production theory, the convexity of the production sets (which eliminates increasing marginal returns) ensures the existence of an equilibrium production. In decision theory, the concavity of the utility function corresponds to risk aversion for the agent. In game theory, convexity ensures the existence of an equilibrium solution: the introduction of the fundamental notion of a mixed strategy corresponds to a convexification of the strategy set. For a historical reference, see, e.g., Weintraub [1992].

In the past century, the notion of a convex function has been generalized in various ways, either by an extension to abstract spaces, or by a change in the inequalities presented above. One of the more recent generalizations, for instance, is due to M.A. Hanson, who introduced *invex functions* in the 1980s: $f : R^n \to R$ is invex whenever it is differentiable and there exists a function $\eta : R^n \times R^n \to R^n$ such that $f(x) - f(y) \geqslant \nabla f(x)\eta(x, y)$. A more classical extension of convexity, only partially comparable to invex functions, is the class of *quasi-convex* functions. At an intermediate level of generality (at least in the continuous case) we mention *pseudo-convex functions*. These classes can be characterized through the generalized monotonicity of gradient maps; see Hadjisavvas and Schaible [2001a, 2001b].

Many important properties of convex functions are preserved within a wider functional environment. If f is convex, its lower level sets $A_k = \{x \in C : f(x) \leqslant k,\ k \in R\}$ are convex; the converse implication does not hold true, as it is apparent in the case of a monotone function of a real variable. The same holds true in the concave case, referring to the convexity of the upper level sets $B_k = \{x \in C : f(x) \geqslant k,\ k \in R\}$. Quasi-convex (quasi-concave) functions are characterized by the last property: they are those functions whose level sets A_k (B_k) are convex or, equivalently, those functions that satisfy the following (three points) condition: $\forall x, y \in C$, $\forall t \in [0, 1]$, $f(tx + (1-t)y) \leqslant \max\{f(x), f(y)\}$, $(f(tx + (1-t)y) \geqslant$

$\min\{f(x), f(y)\})$. It is easily seen that, in the case of real functions of one real variable, every monotone function is quasi-convex (and quasi-concave). The description of the class of quasi-convex functions and of their properties is developed following the same pattern as in the convex case; for differentiable functions, instead of considering the inequality $f(y) - f(x) \geqslant \nabla f(x)(y - x)$, it is enough to require that $\nabla f(x)(y - x) \geqslant 0$ imply $f(y) - f(x) \geqslant 0$.

We do not mention other variations on the topic of generalized convexity here. Thus we refer to Schaible and Ziemba [1981] and Avriel et al. [1988]. All the definitions follow a pattern that reproduces the relationships among convex, strictly convex, and strongly convex functions where some special properties are preserved.

In this paper we wish to analyze the first contributions to quasi-concavity, either to pinpoint the reasons such a notion was introduced or to describe how the same idea of a quasi-concave function has been developed following different routes. Its history develops along the second half of the last century, but we find some puzzle, even on priority.

2. Bruno de Finetti

Bruno de Finetti (1906–1985) is one of the mathematicians whose name is usually linked to the introduction of the class of quasi-concave functions. In 1949—the year of publication of his work “Sulle stratificazioni convesse” [de Finetti, 1949]—de Finetti was a full professor of financial mathematics at Trieste. He obtained his degree in “applied mathematics” in 1927 in Milan, where he attended the courses in economics held by Ulisse Gobbi at the Politecnico. Then he worked first in Rome at the mathematical department of the “Istituto Centrale di Statistica” and subsequently in Trieste, as actuary in a major insurance company. In the years between the two wars de Finetti conceived the main ideas that would make him a protagonist in modern probability theory: first the subjective approach, then the study of exchangeable processes and a formulation of the general theory of processes with independent increments. We limit ourselves to giving just some chronological references in order to emphasize the preeminent role played by this young researcher in the field of probability theory. For a detailed bibliography on the work of de Finetti, see also Daboni [1987]. Between 1928 and 1930 the journal *Rendiconti dell’Istituto Lombardo* published some articles stimulating controversy between de Finetti and Fréchet about the hypothesis of σ-additivity. In 1937, the *Annales de l’Institut M. Poincaré* published the text of a cycle of five lessons held at the Institute two years before, where de Finetti stated in an almost final version his subjectivist point of view on probability. Moreover, in 1937, de Finetti is among the protagonists of the *Colloque consacré à la theorie des probabilités*, organized by the University of Genève in the frame of a series of meetings held yearly in order to examine a given subject thoroughly. In the same years de Finetti was an outstanding figure in the Italian mathematical world. He was one of the first scholars to study the applications of mathematics to economic and social sciences. Together with masters such as Guido Castelnuovo and Francesco Paolo Cantelli, he may be considered one of the founders of the Italian school of probabilistic studies. He devoted special attention to the methodological aspects in his studies: he declared the importance of abstract and general research, in order to obtain a deeper understanding of social behavior. Contemporary Italian mathematicians, influenced by a tradition going back to the 17th century, hesitated to follow the modern algebraic formal methods. Besides, the cultural atmosphere urged the development of a pragmatic science, devoted to solve the real problems of the country, and de Finetti himself had never forsaken a social conscience.

In the 1930s, when the fascist dictatorship took hold in Italy, de Finetti saw in the new government the revolutionary possibility of creating a "third way," superior both to liberalism and to socialism. The refusal of every compromise with the new liberal world allowed him to conceive *ex novo* a globally fascist idea of life (and of economy): "in the free-trade economy the individual is subject to the system that presupposes egoism and compels to egoism even those who understand that the result of all egoism is chaos and ruin; only the discipline of an outer command may rescue one from the slavery of freedom, from the slavery of anarchy" [de Finetti, 1943, p. 48]. The 1929 depression was experienced as an unambiguous sign of the crisis of the system and of the defeat of economic theory, not fit to account for the new phenomena: "it is already too evident that the doctrine and the system urgently need a deep revision for anybody to deny it in good faith, unless he lives in the clouds or on a university chair" [de Finetti, 1935a, p. 364]. The criticism of the Paretian system, developed on the basis of strong ethical expectations in a group of articles that introduce the themes of *welfare economics*, led him to a position that was absolutely original in the framework of Italian mathematics in the 1930s: one should "not give up, but increasingly refine the subtle mathematical sharpness that clearly distinguishes Pareto from other economists, one should let go of the contacts with historical reality instead of holding them fast, one should make a cleaner and stricter distinction between science and the assessment of the aim for which one can exploit it" [de Finetti, 1935b, p. 230]. In short, to reach the truth, one should not "give up the too abstract character of Pareto's ideas to watch reality closer, but, on the contrary, one should make abstractness more perfectly coherent" [de Finetti, 1935b, p. 230].

In 1954, de Finetti (who was in Trieste when "Sulle stratificazioni convesse" was published) moved to the University of Rome. The institutions had changed. De Finetti changed his political options, but kept his attention fixed upon political themes. He assumed a critical tone: "the necessity to design and to accomplish a different economic system is urgent not only in answer to the needs of the community: it must be achieved so as to save humanity from self-destruction as well" [de Finetti, 1973, p. 77].

Hereafter we study the work "Sulle stratificazioni convesse," a technical article issued in 1949 in "Annali di Matematica Pura e Applicata," where the notion of quasi-concavity is introduced. It is the first time that de Finetti devoted his attention to this topic, even if the main issues of his paper can find a natural setting in the ideas of contemporary mathematicians, always paying attention to social and economic applications.

The work begins with three problems that we formulate in modern terms:

(a) Is a quasi-concave function concave as well? Or, in the words of de Finetti, "given a family of convex regions, one inside the other or, as we say for the sake of brevity, a *convex stratification*, is it possible to associate a convex function $f(P)$ with it, as stated before? or, briefly, is it a *stratification of a convex function?*"
(b) Assuming a negative answer to the first question, does there always exist, for a given quasi-concave function φ, an increasing transformation F such that the composition $f = F[\varphi]$ is concave?
(c) Assuming a positive answer to the second question (considering as equivalent all the functions $z = h + kf$, with $h, k \in R$), does a function $F[\varphi]$ that is the *least concave* exist, i.e., one satisfying the inequality $F[\varphi] \leqslant G[\varphi]$ for any increasing transformation G such that $G[\varphi]$ is concave?

In his work, de Finetti uses the term "convex function" to denote what is usually called a *concave* function. In the sequel we will use the modern terminology. Now, we will study the details of de Finetti's work and the answers he gives to his own problems, focusing on some inexactnesses and on what may be considered an extreme reliance on geometric intuition (due to his special mathematical style). Nevertheless, the originality of his intuitions and the consciousness of the deep connections between the analytical problems, subjects of the preceding questions, and the economic models should not be undervalued.

The first paragraph—"Generalities"—containing the enunciation of the three problems mentioned above, also shows the style and the methods of the whole article (it is de Finetti's typical mathematical style). In de Finetti's own words there is reference—immediate and explicit—to the problems of economic analysis. The analysis of what will be called quasi-concavity was suggested to de Finetti from utility theory, as the author acknowledges. The discussion of the first problem is justified by the statement that, in mathematical economics, "it seems to be deemed" that from the convexity of the regions delimited from the indifference varieties the concavity of the utility index necessarily follows. The second problem makes an implicit reference to the dispute between *ordinalists* and *cardinalists*. The basic assumption of the *utilitarian* revolution was the rational behavior of the consumer, who was considered to be able to rank his needs; it seemed that the existence of a function (of consumable goods) that measured utility was unavoidable. This fundamental concept—utility—was then defined as a measurable quantity in the cardinal sense, unique but for *linear* increasing transformations. But, at the end of the 19th century, Pareto's work appears with a new awareness that will lead to the ordinalist paradigm proposed by Hicks and Allen (1934). They say that it is not at all necessary to assume the cardinal measurability of utility; utility is just the expression of the preferences of the consumer, represented by the indifference curves of the agent (the level sets of the function), unique but for *monotone* increasing transformations. In the second problem, the author wonders if in the class of functions that represent the utility of the consumer, there is a concave function.

So the idea of a quasi-concave function f, which represent a "convex stratification," for which "the regions defined by the inequality $f(p) \geqslant c$ obviously make up (as the constant c changes) a family of convex regions" is not based merely on formal reasons, with a subsequent application and exemplification in economic analysis—almost to prove the significance of the mathematical procedure. On the contrary, in de Finetti the introduction of quasiconcavity is justified by economic theory. We can say, with some emphasis, according to the spirit of de Finetti's work, that generalized convexity has been intimately connected with utility theory since it began.

Afterwards the topic is developed in strictly mathematical terms. The formal level is characterized by the strong presence of intuition and of geometric methods and language. Geometric terms such as "bottlenecks," "infinitely thin layers," "contours," and "indentations" are present. For the sake of clarity, the author makes reference to pictures of special cases, in a period when the mathematical style is already sufficiently aligned to modern abstract standards. Other propositions are explicitly proved in a geometric way. On the whole, the paper is not segmented into lemmas, theorems, proofs, remarks, examples, but is the development of a unique subject thought of and "seen" in geometric terms and then written in an analytic language that maintains some hints of the underlying structure, where, e.g., some continuity assumption is left out or some proof that is deemed evident and immediate in a geometric three-dimensional intuition is omitted.

In this way de Finetti is able to give an answer to the second question. Generally his answer is negative, but it becomes positive whenever the function φ "is assumed to have bounded first and second derivatives." The example, considered in a footnote, of a function φ (of one real variable) and of its transformation $f(x) = e^{-\lambda\varphi(x)}$, for which one has $f''(x) = -\lambda e^{-\lambda\varphi(x)}[\lambda(\varphi'(x))^2 - \varphi''(x)]$, shows that concavity of f is ensured if one takes λ "large enough" ($\lambda > \varphi''(x)/[\varphi'(x)]^2$), but this is true if φ'' is bounded and φ' is far from zero. Another doubtful point is the sentence concerning the domain of the functions considered. De Finetti says that every remark will refer to finite-dimensional spaces, even if its extension to the infinite-dimensional case is not a problem. What you have to do is "to give up—it is not a clear remark—those conclusions arising from the possibility of considering maxima instead of suprema."

If we go back to the three main problems of the paper, we find that a generally negative answer to the first two problems is given immediately, through some geometrical examples: there are some convex stratifications that are not stratifications of concave functions; moreover, the concavification of a quasi-concave function "would not be true if some restriction were not imposed." Now, taking for granted that such restrictions are implicitly imposed to be certain that the class of concavifying transformations is nonempty, de Finetti devotes himself to the third question. Here there is no direct reference to economic problems, but during the same years de Finetti worked on the utility theory, where he emphasized the relevance of the notion of relative concavity (see, e.g., de Finetti [1952]). These ideas would be developed later, when the notion of risk aversion was introduced by Pratt and Arrow [Pratt, 1964; Arrow, 1970], and their relation to concavity would be developed. An economic agent is risk-averse if and only if his utility function is concave and its "concavity degree" represents a measure of risk-aversion.

The first step in the construction of the least concave function is a lemma concerning the construction of the least concave function greater than a given function ψ defined on a set C: "let $f(P) = \sup . \sum_h \lambda_h \psi(P_h)$ where the supremum is calculated considering all the possible expressions of $P = \sum_k \lambda_k P_k$ as a linear combination with coefficients $\lambda_k > 0$ of any (finite) number of points P_k of C; it results that $f(P)$ is convex and that, on the whole C, $f(P) \geqslant \psi(P)$, while $f(P) \leqslant \varphi(P)$ whenever φ is any other convex function $\geqslant \varphi(P)$." In de Finetti's opinion "the proof is obvious," so it is omitted. From a modern point it is not so trivial. As a matter of fact, in a recent monograph on *convex analysis* [Hiriart-Urruty and Lemaréchal, 1993] we can find the following theorem (Proposition 2.5.1, I, p. 169): "Let $g : R^n \to R \cup \{+\infty\}$, not identically $+\infty$, be minorized by an affine function: for some $(s, b) \in R^n \times R$, $g(x) \geqslant (s, x) - b$, for all $x \in R^n$. Then, the following three functions f_1, f_2, f_3 are convex and coincide on R^n:

$$f_1(x) = \inf\big\{r\colon\ (x, r) \in \mathit{co\,epi}\, g\big\}$$

$$f_2(x) = \sup\big\{h(x)\colon\ h \in \mathit{Conv}\, R^n,\ h \leqslant g\big\}$$

$$f_3(x) = \inf\left\{\sum_{j=1}^{k} \alpha_j g(x_j)\colon\ k = 1, 2, \ldots, \alpha \in \Delta_k,\ x_j \in \mathit{dom}\, g,\ \sum_{j=1}^{k} \alpha_j x_j = x\right\}."$$

The meaning of the symbols is apparent: $\mathit{co\,epi}\, g$, $\mathit{Conv}\, R^n$, $\mathit{dom}\, g$ are, respectively, the smallest convex set containing the epigraph of a function g, the set of convex functions defined on R^n, and the effective domain of the function g. If such modern language is translated into de Finetti's terms, it can be immediately seen that the preceding theorem is nothing but de Finetti's lemma (which does not consider the function f_1), with one, or better yet, two differences: the proof given by Hiriart-Urruty and Lemaréchal is longer than one page and, moreover, there is an assumption that is ignored by de Finetti (and not replaced by anything else). And it is an essential hypothesis, as can easily be seen by considering the function $g(x) = x^3$; there exists no affine minorizing function for g, and the three functions f_1, f_2, and f_3 are identically $-\infty$.

The monograph by Hiriart-Urruty and Lemaréchal was published only recently, but de Finetti had the possibility of realizing that he needed a hypothesis or, in someway, doubting of the generality of his result. In [Ascoli, 1935] a preceding work was quoted on the minimal nondecreasing function greater than a given real function defined on an interval $[a, b]$. Ascoli was a known mathematician in Italy, even if not first rank; moreover, he published his note in a prestigious journal, whose fame was well-established. The work is based on a theorem—well emphasized—where the least concave (continuous) function is constructed greater than any *bounded* function on an interval $[a, b]$. The boundedness assumption just allows Ascoli to assert the existence of majorizing linear functions.

De Finetti's lemma is followed by "two different procedures to construct a least convex function." The former is an iterative procedure, which uses the preceding lemma and may be resumed as follows. Given the (convex) indifference varieties of a function φ defined on an affine space S, with $\min\varphi = a$, $\max\varphi = b$, take the function $\varphi_0(P)$ as follows,

$$\varphi_0(P) = \begin{cases} a, & \varphi(P) \neq b, \\ b, & \varphi(P) = b, \end{cases}$$

where $P \in S$.

The function φ_1 is given by the preceding lemma, so it is the least concave function greater than φ_0; we can build φ_2 by considering the least function greater than φ_1 and constant where φ was constant. We can now iterate this procedure and we build two sequences of functions: $\{\varphi_{2k+1}\}$ is a sequence of concave functions, while $\{\varphi_{2k}\}$ is constant wherever φ is constant. The existence of the limit of $\{\varphi_k\}$, for $k \to +\infty$, is ensured by monotonicity. The limit function f is what we were looking for, since it is concave (it is the limit of the concave functions φ_{2k+1}) and it is "constant on the prescribed level varieties" (it is the limit of the functions φ_{2k}). De Finetti just fears the circumstances that can render the geometric procedure described above, or the situations that "prevent the existence of the convex function one was looking for (or better: that make it degenerate into a constant)." But the only degenerate solution is $\varphi(x) \equiv b$; as a matter of fact, if it were the case that $\varphi(x) = c$, even only on a set containing an interior point Q, from $\varphi(P) = b > c$ and $\varphi(R) = c$ (where R is a point of the segment PQ beyond Q) it would follow that $\varphi(Q) = t\varphi(P) + (1-t)\varphi(R) > tc + (1-t)c = c$.

The latter is a more analytical procedure for the construction of a least concave function. First de Finetti proves a necessary and sufficient condition, which implicitly confirms the negative answer given to the first problem: a quasi-concave function φ is concave if and only if all its profiles φ_ξ are concave, where the function φ_ξ is defined through the linear transformations ξ: $\varphi_\xi(x) = \sup_P \varphi(P)$, $\xi(P) = x$. In a hint contained in a footnote, the proof brings forward the notion of the subgradient of a convex function, introduced by R.T. Rockafellar in the 1970s. We can find some inexactness (for instance, the maxima of φ are treated without having ensured their existence and, in particular, the boundedness of a given closed set). Later the concavity condition on the profiles is rewritten equivalently, through a functional inequality, by the use of Rademacher's theorem (1919) that de Finetti exploits in a special form and without any explicit quotation: a concave function is "derivable except for at most a numerable infinity of cusps." The least concave function is the function that satisfies this inequality in the limit case of equality, i.e., solves the differential equation $\varphi'_\xi(x) = kW_x(x_1, x)$, with $k = \varphi'_\xi(x_1)$ and W_x built through elements of φ, and is therefore characterized by the following profiles:

$$\varphi_\xi(x) = h + k \int_{x_1}^{x_2} W(x_1, u)\, \mathrm{d}u.$$

The second procedure (the more analytical one) for constructing the least concave function is exposed in the fifth paragraph, where de Finetti remarks, without proof, that “it is useful to remark, as a corollary, the following characteristic of least convex functions: if and only if φ is the least convex function, $f = F[\varphi]$ is convex only if F is increasing and convex.”

This property leads us to treat a recent contribution [Debreu, 1976]. Debreu studies the third problem of de Finetti, whom he quotes explicitly, with a formally different definition of the least concave function. He defines a preorder on the set U of continuous, concave, real-valued functions on X representing the same stratification: v is more concave than u if there is a real-valued, concave function f on $u(X)$ such that $v = f(u)$. Actually the two notions are equivalent, as Debreu himself proves by constructing the minimal element according to his own definition. This coincidence was already stated by de Finetti without any proof. In Debreu’s work it is emphasized in a long and complex proof. Another author, who studied the second problem of de Finetti, i.e., the concavificability of a convex stratification, is Y. Kannai. The concavificability conditions which are the main result of his works—see, e.g., Kannai [1977]—follow some results by W. Fenchel, who first carried on some original ideas by de Finetti; moreover, the concave function, which results from the concavification procedure proposed by Kannai in his proofs, is the least concave function representing the convex stratification. Once again, we have to focus on the heavy assumptions that Kannai makes in order to prove his results.

3. Werner Fenchel

The works where the notion of quasi-concavity (or quasi-convexity) is quoted and where some hint to its origins is made usually make reference to the name of Werner Fenchel. Actually his monograph—*Convex Cones, Sets and Functions* [Fenchel, 1953]—is the second published work where quasi-convex functions are studied but it is the first one where this word is explicitly used.

Werner Fenchel (1905–1988), born in Berlin, after obtaining a degree in mathematics, took the first steps of his academic carrier in Göttingen, as an assistant of E. Landau. After some months spent, thanks to a scholarship, in Rome with T. Levi-Civita and in Copenhagen with H. Bohr and T. Bonnesen, he went to Denmark—with a pause in Sweden—to escape the rise of Nazism. In Denmark he taught—at the Technical School and at the University of Copenhagen until 1974. The monograph we will consider was written in the brief American period (from 1949 to 1951), when he visited the University of Southern California, and then Stanford and Princeton. In the last university, Fenchel was invited by A.W. Tucker to hold a series of seminars on convexity, which offered him hints and materials for the book he would publish in 1953. It is not by chance that *Convex Cones, Sets and Functions* begins with acknowledgments to “Professor A.W. Tucker” (and to H.W. Kuhn, for his “critical remarks”) for drawing his attention to the work of de Finetti and to the concavifiability problem. Fenchel published about 50 works on real and complex analysis, on convex analysis, on geometry, and on differential geometry; the monograph on the *Theorie der Körper* [Bonnesen and Fenchel, 1934]—translated into English, published in various editions, and quoted by de Finetti—may be regarded, together with *Convex Cones, Sets and Functions*, as his most significant publication.

Convex Cones, Sets and Functions is divided into three chapters, whose topics appear in the title. Every subject is developed in finite-dimensional spaces and a special attention is devoted to “results having applications in the theory of games and in programming problems.” But, in spite of this setting, which might lead to a mainly “applied” approach, the structure and the style of Fenchel’s work are completely different from what we found in de Finetti's. Here, the distinction between the original problem and the theory, which is developed with the aim to solve it, is just a fact. The possible applications are just mentioned, in fact, in the *Introduction*; afterwards the theme is developed in total autonomy, just based on the principle of its own organic unity, and it is much easier to follow than in de Finetti, with a result closer to the modern standard.

We are directly interested in the seventh and the eighth paragraph of the third chapter. They start from the same issue as the second problem of de Finetti in terms of convex functions. Fenchel, who is not interested in the existence and construction of most convex functions, considers the lower level sets of a lower semicontinuous real-valued function φ (defined on a convex set $D \subseteq \Re^n$) and immediately obtains some of their properties. Now he wonders whether, "conversely," such a family of sets L_τ (indexed by a real number and satisfying the same properties) may be considered *in some sense* generated by a convex function: "under what conditions is a family of sets L_τ satisfying I–IV transformable into the family of level sets of a convex function." The conditions I–IV are

I $\quad \bigcup_{\tau} L_\tau = D,$

II $\quad \tau_1 < \tau_2$ implies $L_{\tau_1} \subset L_{\tau_2}$,

III $\quad \bigcap_{\tau > \tau_0} L_\tau = L_{\tau_0},$

IV $\quad L_\tau$ is a closed set,

while the previously quoted transformation is made through a continuous and strictly increasing function. This is nothing but the second problem issued by de Finetti, with some continuity requirements that were not explicitly made in his work but that in fact were geometrically considered. Conditions I–IV are not sufficient, nor do they become sufficient even if we add the obvious requirement that the sets L_τ are convex. Here Fenchel introduces the term "quasi-convexity," enunciating and proving for the first time the equivalence of the two definitions that we recalled in our Introduction.

Paragraph 7 is developed through a group of necessary and sufficient conditions to ensure that a family of convex sets (satisfying the above-mentioned properties) is the family of the lower level sets of a convex function. The first of these characterizations states that a quasi-convex function is convex if and only if the inclusion

$$\lambda L_{\tau_0} + (1-\lambda) L_{\tau_1} \subset L_{\tau_\lambda},$$

holds, where $\lambda \in [0, 1]$ and $\tau_\lambda = \lambda\tau_0 + (1-\lambda)\tau_1$. In force of this necessary and sufficient condition (and of other propositions, which make use of "modern" instruments of convex analysis, such as the *support function* and the *asymptotic cone*), Fenchel proves the same necessary and sufficient condition we found in de Finetti on the convexity of the profiles. In de Finetti's proof we pointed out a lack of exactness in the implicit use of a boundedness assumption on some sets. It is the same remark that, in the final notes and in general terms, leads Fenchel to claim the originality of his own procedure and to underline the greater generality of his results (without any direct criticism on the Italian mathematician): "[...] is a generalization to the case considered here of a result of de Finetti" who always used "the assumption that the domain D and, thus, all level sets are compact and convex."

Paragraph 8 is devoted to the same problem, but under differentiability assumptions. Fenchel begins to prove three necessary conditions that a quasi-convex function must satisfy in order to obtain the existence of a function F strictly increasing such that $f = F[\varphi]$ is a convex function: φ cannot have critical points, except for those where it reaches its global minimum; for any $x \in D$, the quadratic form

$$\sum_{i,j} \varphi''_{ij}(x) y_i y_j,$$

restricted to the hyperplane

$$\sum_i \varphi'_i(x) y_i = 0$$

must be positive semidefinite and the rank of the same unconstrained quadratic form is at most r (where $r - 1$ denotes the rank of the constrained quadratic form); the rate $F''(\tau)/F'(\tau)$ must satisfy a boundedness condition from below. The same conditions, as a whole, are also sufficient. Here the comparison with de Finetti can be referred only to a note, where the Italian mathematician made a hint toward the differentiable case. We have already remarked that his argument—a quasi-concave function is concavifiable if it has its first and second derivatives bounded—is not convincing, besides the old terminology used there. Also, Fenchel expressed the same doubts. De Finetti's result, if it were correct, would be in contraposition with his own necessary and sufficient condition, because it is based just on the regularity of φ. A more careful analysis reveals the weakness of the remark of the Italian mathematician; the words by Fenchel are, as usual, really moderate: "apparently de Finetti had overlooked the fact that the smoothness of φ does not imply the smoothness of the support function $h(\xi, \tau)$."

4. John von Neumann

In the introductory section we mentioned some puzzles on priority. The traditional attribution of the notion of quasi-concavity to de Finetti and Fenchel is substantially correct; nevertheless, a first formalization appeared 20 years before in a famous work by a mathematician who played an important role in the history of 20th century mathematics.

We are referring to von Neumann [1928], based on a conference held two years earlier on December at the Mathematical Society of Göttingen. These were the "German years" of von Neumann. Born in Budapest in 1903, he received a degree in chemical engineering in Zürich in 1926 and in the same year he obtained a doctorate in mathematics at the University of Budapest. In the academic year 1926–1927 he moved to Germany, where he became Privatdozent at the University of Berlin and obtained then a grant from the *Rockefellar Foundation* that allowed him to study with Hilbert at Göttingen. Von Neumann moved to the United States just at the beginning of the 1930s. In 1931 he was a professor at Princeton University; in 1933 he became a member of the new *Institute for Advanced Study* in Princeton.

The work of 1928 has a relevant place in the unusually wide range of works by von Neumann, who had already offered fundamental contributions in fields very different one from another: from quantum mechanics to algebra, from measure theory to ergodic theory, from economics to game theory, from hydrodynamics to meteorology. "Zur Theorie der Gesellschaftspiele" may be considered a work in "applied" mathematics, where (in an economic environment) modern topological instruments are used, such as fixed point theorems, far from the traditional techniques which made use of geometric intuition and calculus. Here we find the intentional development of a new way to treat mathematical economics that can reduce the gap with other disciplines: "Economics is simply still a million miles away from the state in which an advanced science is, such as physics" [Morgenstern, 1976, p. 810]. Hilbert's program comes to life, with the principle that to establish new foundations for a discipline, a system of coherent axioms should be characterized.

The work begins with the definition of a "game of strategy," expressed through a series of "rules" that are just a primary elaboration of the modern concept of extended form of a game. The subject of the theory of games is clearly explicated: "We shall try to investigate the effects which the players have on each other, the consequence of the fact (so typical of all social happenings) that each player influences the results of all other players, even though he is only interested in his own." Von Neumann uses the monetary value as pay-off function, even if, in a footnote, he makes a hint as to the objections that might be raised against such a choice; his position on the debate over utility is actually measured: "the difficulties that form the subject of our consideration are of a different nature."

In the first paragraph the definitions of strategy and of normal form of a game immediately introduce not only a simplification in the structure of the game, but also a change in perspective. The dynamic element and the possibility to follow the evolution of the game disappear, and time is compressed to the single instant when the game is solved.

There is a normative aim: von Neumann studies the optimal behavior of every player, starting from the case (considered in Paragraph 2) of a zero-sum game between two players. Here the formulation of the theorem of minimax is almost immediate, if you consider the matrix of the game A. The elements of its ith row denote the amount won by the first player, using its ith strategy, corresponding to any possible strategy chosen by the opponent; analogously, the elements of the jth column represent the loss obtained by the second player when he uses the jth strategy, in correspondence with every possible choice of his opponent. Following a maximum prudence criterion, for any possible strategy, each player considers the worst situation that can happen to him; so the first player calculates the minimum on every row, while the second player calculates the maximum on every column. The quantity $\max_i \min_j a_{ij}$ is the lower bound to the value that the fist player can obtain, while $\min_j \max_i a_{ij}$ represents a roof over the losses possibly inflicted to the second player. Obviously, for any rational solution of the game, the value v of the amount won by the first player (and therefore the loss of the second one) will satisfy the following inequalities:

$$\max_i \min_j a_{ij} \leqslant v \leqslant \min_j \max_i a_{ij}.$$

Whenever the inequalities hold as equalities, the game is solved; otherwise, an equilibrium can never be reached. In the last case, von Neumann suggests a "trick": he introduces the concept of a mixed strategy as a probability distribution on the set of pure strategies and proposes to value the whole game through an expected value; if we denote by x and y the mixed strategies chosen by the two players, the value of the game is xAy, where A is the pay-off matrix. So the solution of the game is equivalent to a saddle point of a bilinear form.

Now the formalization of the problem is complete and von Neumann proves (in Paragraph 3) the minimax theorem, which is the main result of this work. In fact, a previous formalization of this result (together with the notion of mixed strategy) had already been published in Borel [1924]. This issue is well known, and we refer to Dell'Aglio [1995] for a complete analysis of the relationships between the two different approaches. While Borel starts from the particular case of two strategies and by the unsuccessful effort to extend his result he is led to conjecture a lack of general validity of the theorem, von Neumann studies the problem in a very general environment. Moreover he makes assumptions much less restrictive than those that usually are verified in the setting of the theory of zero-sum games. First he studies the properties of a function $h(x, y)$ that are useful in the proof. Indeed, he proves that, if h is continuous and satisfies the property

$$\begin{aligned} &\text{if } h(x',y) \geqslant \alpha \text{ and } h(x'',y) \geqslant \alpha,\\ &\text{then } h(x,y) \geqslant \alpha \text{ for any } 0 \leqslant t \leqslant 1,\ x = tx' + (1-t)x'';\\ &\text{if } h(x,y') \leqslant \alpha \text{ and } h(x,y'') \leqslant \alpha,\\ &\text{then } h(x,y) \leqslant \alpha \text{ for any } 0 \leqslant t \leqslant 1,\ y = ty' + (1-t)y'', \end{aligned} \tag{K}$$

then it holds that

$$\max_{\substack{x_i \geqslant 0\\ \sum x_i = 1}} \min_{\substack{y_j \geqslant 0\\ \sum y_j = 1}} h(x,y) = \min_{\substack{y_j \geqslant 0\\ \sum y_j = 1}} \max_{\substack{x_i \geqslant 0\\ \sum x_i = 1}} h(x,y).$$

The reason the paper by von Neumann is so important for the history of generalized convexity is now apparent. Property (K), which is always satisfied in the particular case where h is a bilinear form, is nothing but the requirement on the function h of quasi-concavity in the variable x and of quasi-convexity in the variable y. Such definitions are introduced by von Neumann as technical conditions. There is no hint to the relationships with the convex case, nor the class of functions defined by condition (K) is studied. This property simply allows von Neumann to reduce the problem, through successive projections, to the bi-dimensional case and afterward to assess the convexity of the solution sets of the problems $\min_y h(x,y)$ and $\max_x h(x,y)$.

The proof of the theorem of minimax contains an extension of Brouwer's fixed-point theorem to the case of multivalued mappings: let H be a closed point-to-set mapping (i.e., its graph is a closed set), defined on the real interval $[0,1]$ and with value on closed intervals of $[0,1]$, then a point x^0 exists such that $x^0 \in H(x^0)$. The link of the minimax theorem with the fixed points of point-to-set maps is immediate. Let $F(x) = \arg\min_y h(x,y)$ and $G(y) = \arg\max_x h(x,y)$, then the equality

$$\max_x \min_y h(x,y) = \min_y \max_x h(x,y)$$

is equivalent to the existence of a point $(x^0, y^0) \in G(y^0) \times F(x^0)$.

This approach abounds in seminal ideas that anticipate subsequent results. One can already foresee the notion of "best reply" that will lead to the idea of Nash equilibrium (1951), a starting point for the whole theory of non-cooperative games. On the other hand, working in a more general setting than the one strictly necessary to reach the aim, von Neumann opens a way to the results of Ky Fan and Nikaido on the fixed-point theorem and on the minimax theorem. In these subsequent developments, the link with the theory of games will be somehow preserved, as it is evidenced in the use of the bibliographic references in Nikaido [1954]. Moreover, the theorems of minimax will lead to develop or to deepen various notions of generalized convexity—see, for instance, Ky Fan [1954] for the definition of convexlikeness and Nikaido [1954] for quasiconvexity—and the first efforts to compare the various classes of generalized convex functions will be developed.

The work by von Neumann ends with the study of the three-player case and, more generally, of the n-player case ($n > 3$). By means of the concepts of coalition and of characteristic function, a basis for the theory of cooperative games is set and the problem is simplified to resume the conclusions of the case $n = 2$ already solved.

The impact of von Neumann with economics and game theory will not be isolated. In [von Neumann, 1937] he introduces a new model of general equilibrium. It is formulated as a system of linear equalities and inequalities that may be considered, in some sense, a precursor of linear programming and activity analysis, as Kuhn and Tucker acknowledge in Kuhn and Tucker [1958]. The relationship between this model and the theorem of minimax is close: the existence of an equilibrium point corresponds to the solution of a suitable zero-sum two-players game. This work will stimulate O. Morgenstern to collaborate with von Neumann on a fundamental book on game theory that will be published in 1944: "Theory of Games and Economic Behavior." In von Neumann and Morgenstern [1944], the proof of the minimax theorem is changed, following the ideas introduced in Ville [1938]. The authors use linear algebra methods instead of a fixed point technique and the quasi-concavity assumption disappears.

5. Conclusions

Generalized convexity is by now a common instrument in many mathematical areas and, in particular, in optimization. It is a typical mathematical concept, especially appreciated by those mathematicians whose work overlaps with economics and social sciences. Our historical analysis proves that in the rising of the notion of quasi-concavity these features are present since the beginning. The works by von Neumann and by de Finetti are works of applied mathematics (with reference to mathematical economics and, chiefly, to decision theory), not in the sense that a mathematical theory is subsequently applied to another discipline, but that in a unitary framework the main ideas of utility theory are strictly interwoven with their mathematical development.

The study of von Neumann's "Zür Theorie der Gesellschaftspiele" has led us to verify that the first formulation of quasi-concavity is actually due to the Hungarian mathematician, in 1928, with an important feature: the notion is not introduced as a definition, almost a premise to the study of a specific functional class, but as a simple technical hypothesis (condition (K)) useful to prove the famous minimax theorem. From this point of view, the comparison with the more "modern" (not only in a chronological sense, but also in terms of expository standards) Fenchel is enlightening. Therefore, the first formulation of the concept of quasi-concavity is not due to de Finetti, but there must be credited to him the consciousness of its relevance for economic applications and an organic analysis that is developed with the intuitive style characteristic of Italian mathematics in the first half of the previous century.

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