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On Jacquet–Langlands isogeny over function fields

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ABSTRACT

We propose a conjectural explicit isogeny from the Jacobians of hyperelliptic Drinfeld modular curves to the Jacobians of hyperelliptic modular curves of $\mathcal{D}$-elliptic sheaves. The kernel of the isogeny is a subgroup of the cuspidal divisor group constructed by examining the canonical maps from the cuspidal divisor group into the component groups.

1. Introduction

Let N be a square-free integer, divisible by an even number of primes. It is well known that the new part of the modular Jacobian $J_0(N)$ is isogenous to the Jacobian of a Shimura curve; see [33]. The existence of this isogeny can be interpreted as a geometric incarnation of the global Jacquet–Langlands correspondence over $\mathbb{Q}$ between the cusp forms on GL(2) and the multiplicative group of a quaternion algebra [24]. Jacquet–Langlands isogeny has important arithmetic applications, for example, to level lowering [35]. In this paper we are interested in the function field analogue of the Jacquet–Langlands isogeny.

Let $\mathbb{F}_q$ be the finite field with q elements, and let $F = \mathbb{F}_q(T)$ be the field of rational functions on $\mathbb{P}^1_{\mathbb{F}_q}$. The set of places of F will be denoted by $|F|$. Let $A := \mathbb{F}_q[T]$. This is the subring of F consisting of functions which are regular away from the place generated by $1/T$ in $\mathbb{F}_q[1/T]$. The place

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generated by $1/T$ will be denoted by ∞ and called the *place at infinity*; it will play a role similar to the archimedean place for $\mathbb{Q}$. The places in $|F| - \infty$ are the *finite places*.

Let $v \in |F|$. We denote by F_v, $\mathcal{O}_v$ and $\mathbb{F}_v$ the completion of F at v, the ring of integers in F_v, and the residue field of F_v, respectively. We assume that the valuation $\mathrm{ord}_v : F_v \to \mathbb{Z}$ is normalized by $\mathrm{ord}_v(\pi_v) = 1$, where π_v is a uniformizer of $\mathcal{O}_v$. The *degree of* v is $\deg(v) = [\mathbb{F}_v : \mathbb{F}_q]$. Let $q_v := q^{\deg(v)} = \#\mathbb{F}_v$. If v is a finite place, then with an abuse of notation we denote the prime ideal of A corresponding to v by the same letter.

Given a field K, we denote by $\bar{K}$ an algebraic closure of K.

Let $R \subset |F| - \infty$ be a nonempty finite set of places of even cardinality. Let D be the quaternion algebra over F ramified exactly at the places in R. Let X_F^R be the modular curve of $\mathcal{D}$-elliptic sheaves (see Section 2.2). This curve is the function field analogue of a Shimura curve parametrizing abelian surfaces with multiplication by a maximal order in an indefinite division quaternion algebra over $\mathbb{Q}$. Denote the Jacobian of X_F^R by J^R. The role of classical modular curves in this context is played by Drinfeld modular curves. With an abuse of notation, let R also denote the square-free ideal of A whose support consists of the places in R. Let $X_0(R)_F$ be the Drinfeld modular curve defined in Section 2.1. Let $J_0(R)$ be the Jacobian of $X_0(R)_F$. The same strategy as over $\mathbb{Q}$ shows that J^R is isogenous to the new part of $J_0(R)$ (see Theorem 7.1 and Remark 7.4). The proof relies on Tate's conjecture, so it provides no information about the isogenies $J^R \to J_0(R)^{\mathrm{new}}$ beyond their existence. In this paper we carefully examine the simplest non-trivial case, namely $R = \{x, y\}$ with $\deg(x) = 1$ and $\deg(y) = 2$. (When $R = \{x, y\}$ and $\deg(x) = \deg(y) = 1$, both X_F^R and $X_0(R)_F$ have genus 0.)

Notation 1.1. Unless indicated otherwise, throughout the paper x and y will be two fixed finite places of degree 1 and 2, respectively. When $R = \{x, y\}$, we write X_F^{xy} for X_F^R, J^{xy} for J^R, $X_0(xy)_F$ for $X_0(R)_F$, and $J_0(xy)$ for $J_0(R)$.

The genus of X_F^{xy} is q, which is also the genus of $X_0(xy)_F$. Hence $J_0(xy)$ and J^{xy} are q-dimensional Jacobian varieties, which are isogenous over F. We would like to construct an explicit isogeny $J_0(xy) \to J^{xy}$. A natural place to look for the kernel of an isogeny defined over F is in the cuspidal divisor group $\mathcal{C}$ of $J_0(xy)$. To see which subgroup of $\mathcal{C}$ could be the kernel, one needs to compute, besides $\mathcal{C}$ itself, the component groups of $J_0(xy)$ and J^{xy}, and the canonical specialization maps of $\mathcal{C}$ into the component groups of $J_0(xy)$. These calculations constitute the bulk of the paper. Based on these calculations, in Section 7 we propose a conjectural explicit isogeny $J_0(xy) \to J^{xy}$, and prove that the conjecture is true for $q = 2$. We note that X_F^{xy} is hyperelliptic, and in fact for odd q these are the only X_F^R which are hyperelliptic [31]. The curve $X_0(xy)_F$ is also hyperelliptic, and for levels which decompose into a product of two prime factors these are the only hyperelliptic Drinfeld modular curves [36]. Hence this paper can also be considered as a study of hyperelliptic modular Jacobians over F which interrelates [31] and [36].

The approach to explicating the Jacquet–Langlands isogeny through the study of component groups and cuspidal divisor groups was initiated in the classical context by Ogg. In [27], Ogg proposed in several cases conjectural explicit isogenies between the modular Jacobians and the Jacobians of Shimura curves (as far as I know, these conjectures are still mostly open, but see [19] and [23] for some advances).

We summarize the main results of the paper.

- The cuspidal divisor group $\mathcal{C} \subset J_0(xy)(F)$ is isomorphic to

$$\mathcal{C} \cong \mathbb{Z}/(q+1)\mathbb{Z} \oplus \mathbb{Z}/\big(q^2+1\big)\mathbb{Z}.$$

- The component groups of $J_0(xy)$ and J^{xy} at x, y, and ∞ are listed in Table 1. ($J_0(xy)$ and J^{xy} have good reduction away from x, y and ∞, so the component groups are trivial away from these three places.)
- If we denote the component group of $J_0(xy)$ at $*$ by Φ_*, and the canonical map $\mathcal{C} \to \Phi_*$ by ϕ_*, then there are exact sequences

Table 1

	x	y	∞
$J_0(xy)$	$\mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$	$\mathbb{Z}/(q+1)\mathbb{Z}$	$\mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$
J^{xy}	$\mathbb{Z}/(q+1)\mathbb{Z}$	$\mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$	$\mathbb{Z}/(q+1)\mathbb{Z}$

$$0 \to \mathbb{Z}/(q+1)\mathbb{Z} \to \mathcal{C} \xrightarrow{\phi_x} \Phi_x \to \mathbb{Z}/(q+1)\mathbb{Z} \to 0,$$

$$0 \to \mathbb{Z}/\big(q^2+1\big)\mathbb{Z} \to \mathcal{C} \xrightarrow{\phi_y} \Phi_y \to 0,$$

$$\phi_\infty : \mathcal{C} \xrightarrow{\sim} \Phi_\infty \quad \text{if } q \text{ is even},$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{C} \xrightarrow{\phi_\infty} \Phi_\infty \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{if } q \text{ is odd}.$$

- The kernel $\mathcal{C}_0 \cong \mathbb{Z}/(q^2+1)\mathbb{Z}$ of ϕ_y maps injectively into Φ_x and Φ_∞.

Conjecture 7.3 then states that there is an isogeny $J_0(xy) \to J^{xy}$ whose kernel is $\mathcal{C}_0$. As an evidence for the conjecture, we prove that the quotient abelian variety $J_0(xy)/\mathcal{C}_0$ has component groups of the same order as J^{xy}. This is a consequence of a general result (Theorem 4.3), which describes how the component groups of abelian varieties with toric reduction change under isogenies. Finally, we prove Conjecture 7.3 for $q=2$ (Theorem 7.12); the proof relies on the fact that $J_0(xy)$ in this case is isogenous to a product of two elliptic curves. Two other interesting consequences of our results are the following. First, we deduce the genus formula for X_F^R proven in [30] by a different argument (Corollary 6.3). Second, assuming q is even and Conjecture 7.3 is true, we are able to tell how the optimal elliptic curve with conductor $xy\infty$ changes in a given F-isogeny class when we change the modular parametrization from $X_0(xy)_F$ to X_F^{xy} (Proposition 7.10).

2. Preliminaries

2.1. Drinfeld modular curves

Let K be an A-field, i.e., K is a field equipped with a homomorphism $\gamma : A \to K$. In particular, K contains $\mathbb{F}_q$ as a subfield. The *A-characteristic* of K is the ideal $\ker(\gamma) \lhd A$. Let $K\{\tau\}$ be the twisted polynomial ring with commutation rule $\tau s = s^q \tau$, $s \in K$. A *rank-2 Drinfeld A-module over K* is a ring homomorphism $\phi : A \to K\{\tau\}$, $a \mapsto \phi_a$ such that $\deg_\tau \phi_a = -2\operatorname{ord}_\infty(a)$ and the constant term of ϕ_a is $\gamma(a)$. A *homomorphism* of two Drinfeld modules $u : \phi \to \psi$ is $u \in K\{\tau\}$ such that $\phi_a u = u\psi_a$ for all a in A; u is an *isomorphism* if $u \in K^\times$. Note that ϕ is uniquely determined by the image of T:

$$\phi_T = \gamma(T) + g\tau + \Delta\tau^2,$$

where $g \in K$ and $\Delta \in K^\times$. The *j-invariant* of ϕ is $j(\phi) = g^{q+1}/\Delta$. It is easy to check that if K is algebraically closed, then $\phi \cong \psi$ if and only if $j(\phi) = j(\psi)$.

Treating τ as the automorphism of K given by $k \mapsto k^q$, the field K acquires a new A-module structure via ϕ. Let $\mathfrak{a} \lhd A$ be an ideal. Since A is a principal ideal domain, we can choose a generator $a \in A$ of $\mathfrak{a}$. The A-module $\phi[\mathfrak{a}] = \ker \phi_a(\bar{K})$ does not depend on the choice of a and is called the $\mathfrak{a}$-*torsion* of ϕ. If $\mathfrak{a}$ is coprime to the A-characteristic of K, then $\phi[\mathfrak{a}] \cong (A/\mathfrak{a})^2$. On the other hand, if $\mathfrak{p} = \ker(\gamma) \neq 0$, then $\phi[\mathfrak{p}] \cong (A/\mathfrak{p})$ or 0; when $\phi[\mathfrak{p}] = 0$, ϕ is called *supersingular*.

Lemma 2.1. *Up to isomorphism, there is a unique supersingular rank-2 Drinfeld A-module over $\overline{\mathbb{F}}_x$: it is the Drinfeld module with j-invariant equal to 0. Up to isomorphism, there is a unique supersingular rank-2 Drinfeld A-module over $\overline{\mathbb{F}}_y$, and its j-invariant is non-zero.*

Proof. This follows from [9, (5.9)] since $\deg(x) = 1$ and $\deg(y) = 2$. □

Let $\mathrm{End}(\phi)$ denote the centralizer of $\phi(A)$ in $\bar{K}\{\tau\}$, i.e., the ring of all homomorphisms $\phi \to \phi$ over $\bar{K}$. The automorphism group $\mathrm{Aut}(\phi)$ is the group of units $\mathrm{End}(\phi)^\times$.

Lemma 2.2. *If $j(\phi) \neq 0$, then $\mathrm{Aut}(\phi) \cong \mathbb{F}_q^\times$. If $j(\phi) = 0$, then $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^2}^\times$.*

Proof. If $u \in \bar{K}^\times$ commutes with $\phi_T = \gamma(T) + g\tau + \Delta\tau^2$, then $u^{q^2-1} = 1$ and $u^{q-1} = 1$ if $g \neq 0$. This implies that $u \in \mathbb{F}_q^\times$ if $j(\phi) \neq 0$, and $u \in \mathbb{F}_{q^2}^\times$ if $j(\phi) = 0$. On the other hand, we clearly have the inclusions $\mathbb{F}_q^\times \subset \mathrm{Aut}(\phi)$ and, if $j(\phi) = 0$, $\mathbb{F}_{q^2}^\times \subset \mathrm{Aut}(\phi)$. This finishes the proof. □

Lemma 2.3. *Let $\mathfrak{p} \lhd A$ be a prime ideal and $\mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$. Let ϕ be a rank-2 Drinfeld A-module over $\overline{\mathbb{F}}_\mathfrak{p}$. Let $\mathfrak{n} \lhd A$ be an ideal coprime to $\mathfrak{p}$. Let $C_\mathfrak{n}$ be an A-submodule of $\phi[\mathfrak{n}]$ isomorphic to $A/\mathfrak{n}$. Denote by $\mathrm{Aut}(\phi, C_\mathfrak{n})$ the subgroup of automorphisms of ϕ which map $C_\mathfrak{n}$ to itself. Then $\mathrm{Aut}(\phi, C_\mathfrak{n}) \cong \mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$. The second case is possible only if $j(\phi) = 0$.*

Proof. The action of $\mathbb{F}_q^\times$ obviously stabilizes $C_\mathfrak{n}$, hence, using Lemma 2.2, it is enough to show that if $\mathrm{Aut}(\phi, C_\mathfrak{n}) \neq \mathbb{F}_q^\times$, then $\mathrm{Aut}(\phi, C_\mathfrak{n}) \cong \mathbb{F}_{q^2}^\times$. Let $u \in \mathrm{Aut}(\phi, C_\mathfrak{n})$ be an element which is not in $\mathbb{F}_q$. Then $\mathrm{Aut}(\phi) = \mathbb{F}_q[u]^\times \cong \mathbb{F}_{q^2}^\times$, where $\mathbb{F}_q[u]$ is considered as a finite subring of $\mathrm{End}(\phi)$. It remains to show that $\alpha + u\beta$ stabilizes $C_\mathfrak{n}$ for any $\alpha, \beta \in \mathbb{F}_q$ not both equal to zero. But this is obvious since α and $u\beta$ stabilize $C_\mathfrak{n}$ and $C_\mathfrak{n} \cong A/\mathfrak{n}$ is cyclic. □

One can generalize the notion of Drinfeld modules over an A-field to the notion of Drinfeld modules over an arbitrary A-scheme S [8]. The functor which associates to an A-scheme S the set of isomorphism classes of pairs $(\phi, C_\mathfrak{n})$, where ϕ is a Drinfeld A-module of rank 2 over S and $C_\mathfrak{n} \cong A/\mathfrak{n}$ is an A-submodule of $\phi[\mathfrak{n}]$, possesses a coarse moduli scheme $Y_0(\mathfrak{n})$ that is affine, flat and of finite type over A of pure relative dimension 1. There is a canonical compactification $X_0(\mathfrak{n})$ of $Y_0(\mathfrak{n})$ over $\mathrm{Spec}(A)$; see [8, §9] or [41]. The finitely many points $X_0(\mathfrak{n})(\bar{F}) - Y_0(\mathfrak{n})(\bar{F})$ are called the *cusps* of $X_0(\mathfrak{n})_F$.

Denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of F_∞. Let $\Omega = \mathbb{C}_\infty - F_\infty$ be the *Drinfeld upper half-plane*; Ω has a natural structure of a smooth connected rigid-analytic space over F_∞. Denote by $\Gamma_0(\mathfrak{n})$ the *Hecke congruence subgroup* of level $\mathfrak{n}$:

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \in \mathfrak{n} \right\}.$$

The group $\Gamma_0(\mathfrak{n})$ naturally acts on Ω via linear fractional transformations, and the action is *discrete* in the sense of [8, p. 582]. Hence we may construct the quotient $\Gamma_0(\mathfrak{n}) \setminus \Omega$ as a 1-dimensional connected smooth analytic space over F_∞.

The following theorem can be deduced from the results in [8]:

Theorem 2.4. *$X_0(\mathfrak{n})$ is a proper flat scheme of pure relative dimension 1 over $\mathrm{Spec}(A)$, which is smooth away from the support of $\mathfrak{n}$. There is an isomorphism of rigid-analytic spaces $\Gamma_0(\mathfrak{n}) \setminus \Omega \cong Y_0(\mathfrak{n})^{\mathrm{an}}_{F_\infty}$.*

There is a genus formula for $X_0(\mathfrak{n})_F$ which depends on the prime decomposition of $\mathfrak{n}$; see [16, Thm. 2.17]. By this formula, the genera of $X_0(x)_F$, $X_0(y)_F$ and $X_0(xy)_F$ are 0, 0 and q, respectively.

2.2. Modular curves of $\mathcal{D}$-elliptic sheaves

Let D be a quaternion algebra over F. Let $R \subset |F|$ be the set of places which ramify in D, i.e., $D \otimes F_v$ is a division algebra for $v \in R$. It is known that R is finite of even cardinality, and, up to isomorphism, this set uniquely determines D; see [42]. Assume $R \neq \emptyset$ and $\infty \notin R$. In particular, D is a division algebra. Let $C := \mathbb{P}^1_{\mathbb{F}_q}$. Fix a locally free sheaf $\mathcal{D}$ of $\mathcal{O}_C$-algebras with stalk at the generic point equal to D and such that $\mathcal{D}_v := \mathcal{D} \otimes_{\mathcal{O}_C} \mathcal{O}_v$ is a maximal order in $D_v := D \otimes_F F_v$.

Let S be an $\mathbb{F}_q$-scheme. Denote by Frob_S its Frobenius endomorphism, which is the identity on the points and the qth power map on the functions. Denote by $C \times S$ the fibered product $C \times_{\mathrm{Spec}(\mathbb{F}_q)} S$. Let $z : S \to C$ be a morphism of $\mathbb{F}_q$-schemes. A *$\mathcal{D}$-elliptic sheaf over S*, with pole ∞ and zero z, is a sequence $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, where each $\mathcal{E}_i$ is a locally free sheaf of $\mathcal{O}_{C\times S}$-modules of rank 4 equipped with a right action of $\mathcal{D}$ compatible with the $\mathcal{O}_C$-action, and where

$$j_i : \mathcal{E}_i \to \mathcal{E}_{i+1},$$
$$t_i : {}^\tau\mathcal{E}_i := (\mathrm{Id}_C \times \mathrm{Frob}_S)^*\mathcal{E}_i \to \mathcal{E}_{i+1}$$

are injective $\mathcal{O}_{C\times S}$-linear homomorphisms compatible with the $\mathcal{D}$-action. The maps j_i and t_i are sheaf modifications at ∞ and z, respectively, which satisfy certain conditions, and it is assumed that for each closed point w of S, the Euler–Poincaré characteristic $\chi(\mathcal{E}_0|_{C\times w})$ is in the interval $[0, 2)$; we refer to [26, §2] and [22, §1] for the precise definition. Moreover, to obtain moduli schemes with good properties at the closed points w of S such that $z(w) \in R$ one imposes an extra condition on $\mathbb{E}$ to be "special" [22, p. 1305]. Note that, unlike the original definition in [26], ∞ is allowed to be in the image of S; here we refer to [1, §4.4] for the details. Denote by $\mathcal{E}\ell\ell^{\mathcal{D}}(S)$ the set of isomorphism classes of $\mathcal{D}$-elliptic sheaves over S. The following theorem can be deduced from some of the main results in [26] and [22]:

Theorem 2.5. *The functor $S \mapsto \mathcal{E}\ell\ell^{\mathcal{D}}(S)$ has a coarse moduli scheme X^R, which is proper and flat of pure relative dimension* 1 *over C and is smooth over $C - R - \infty$.*

Remark 2.6. Theorems 2.4 and 2.5 imply that $J_0(R)$ and J^R have good reduction at any place $v \in |F| - R - \infty$; cf. [2, Ch. 9].

3. Cuspidal divisor group

For a field K, we represent the elements of $\mathbb{P}^1(K)$ as column vectors $\binom{u}{v}$ where $u, v \in K$ are not both zero and $\binom{u}{v}$ is identified with $\binom{\alpha u}{\alpha v}$ if $\alpha \in K^\times$. We assume that $\mathrm{GL}_2(K)$ acts on $\mathbb{P}^1(K)$ on the left by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}.$$

Let $\mathfrak{n} \lhd A$ be an ideal. The cusps of $X_0(\mathfrak{n})_F$ are in natural bijection with the orbits of $\Gamma_0(\mathfrak{n})$ acting from the left on $\mathbb{P}^1(F)$.

Lemma 3.1. *If $\mathfrak{n}$ is square-free, then there are 2^s cusps on $X_0(\mathfrak{n})_F$, where s is the number of prime divisors of $\mathfrak{n}$. All the cusps are F-rational.*

Proof. See Proposition 3.3 and Corollary 3.4 in [11]. □

For every $\mathfrak{m}|\mathfrak{n}$ with $(\mathfrak{m}, \mathfrak{n}/\mathfrak{m}) = 1$ there is an *Atkin–Lehner involution* $W_\mathfrak{m}$ on $X_0(\mathfrak{n})_F$, cf. [36]. Its action is given by multiplication from the left with any matrix $\begin{pmatrix} ma & b \\ n & m \end{pmatrix}$ whose determinant generates $\mathfrak{m}$, and where $a, b, m, n \in A$, $(n) = \mathfrak{n}$, $(m) = \mathfrak{m}$.

From now on assume $\mathfrak{n} = xy$. Recall that we denote by x and y the prime ideals of A corresponding to the places x and y, respectively. With an abuse of notation, we will denote by x also the monic irreducible polynomial in A generating the ideal x, and similarly for y. It should be clear from the context in which capacity x and y are being used. With this notation, $X_0(xy)_F$ has 4 cusps, which can be represented by

$$[\infty] := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad [0] := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad [x] := \begin{pmatrix} 1 \\ x \end{pmatrix}, \qquad [y] := \begin{pmatrix} 1 \\ y \end{pmatrix},$$

cf. [36, p. 333] and [15, p. 196].

There are 3 non-trivial Atkin–Lehner involutions W_x, W_y, W_{xy} which generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$: these involutions commute with each other and satisfy

$$W_x W_y = W_{xy}, \qquad W_x^2 = W_y^2 = W_{xy}^2 = 1.$$

By [36, Prop. 9], none of these involutions fixes a cusp. In fact, a simple direct calculation shows that

$$\begin{aligned} W_{xy}\big([\infty]\big) &= [0], \qquad W_{xy}\big([x]\big) = [y]; \\ W_x\big([\infty]\big) &= [y], \qquad W_x\big([0]\big) = [x]; \\ W_y\big([\infty]\big) &= [x], \qquad W_y\big([0]\big) = [y]. \end{aligned} \tag{3.1}$$

Let $\Delta(z)$, $z \in \Omega$, denote the Drinfeld discriminant function; see [11] or [15] for the definition. This is a holomorphic and nowhere vanishing function on Ω. In fact, $\Delta(z)$ is a type-0 and weight-(q^2-1) cusp form for $\mathrm{GL}_2(A)$. Its order of vanishing at the cusps of $X_0(\mathfrak{n})_F$ can be calculated using [15]. When $\mathfrak{n} = xy$, [15, (3.10)] implies

$$\operatorname{ord}_{[\infty]} \Delta = 1, \qquad \operatorname{ord}_{[0]} \Delta = q_x q_y, \qquad \operatorname{ord}_{[x]} \Delta = q_y, \qquad \operatorname{ord}_{[y]} \Delta = q_x. \tag{3.2}$$

The functions

$$\Delta_x(z) := \Delta(xz), \qquad \Delta_y(z) := \Delta(yz), \qquad \Delta_{xy}(z) := \Delta(xyz)$$

are type-0 and weight-(q^2-1) cusp forms for $\Gamma_0(xy)$. Hence the fractions Δ/Δ_x, Δ/Δ_y, Δ/Δ_{xy} define rational functions on $X_0(xy)_{\mathbb{C}_\infty}$. We compute the divisors of these functions.

The matrix $W_{xy} = \left(\begin{smallmatrix} 0 & 1 \\ xy & 0 \end{smallmatrix}\right)$ normalizes $\Gamma_0(xy)$ and interchanges $\Delta(z)$ and $\Delta_{xy}(z)$. Thus by (3.1) and (3.2)

$$\operatorname{ord}_{[\infty]} \Delta_{xy} = q_x q_y, \qquad \operatorname{ord}_{[0]} \Delta_{xy} = 1, \qquad \operatorname{ord}_{[x]} \Delta_{xy} = q_x, \qquad \operatorname{ord}_{[y]} \Delta_{xy} = q_y.$$

A similar argument involving the actions of W_x and W_y gives

$$\begin{aligned} &\operatorname{ord}_{[\infty]} \Delta_x = q_x, \qquad \operatorname{ord}_{[0]} \Delta_x = q_y, \qquad \operatorname{ord}_{[x]} \Delta_x = q_x q_y, \qquad \operatorname{ord}_{[y]} \Delta_x = 1; \\ &\operatorname{ord}_{[\infty]} \Delta_y = q_y, \qquad \operatorname{ord}_{[0]} \Delta_y = q_x, \qquad \operatorname{ord}_{[x]} \Delta_y = 1, \qquad \operatorname{ord}_{[y]} \Delta_y = q_x q_y. \end{aligned}$$

From these calculations we obtain

$$\begin{aligned} \operatorname{div}(\Delta/\Delta_{xy}) &= (1 - q_x q_y)[\infty] + (q_x q_y - 1)[0] + (q_y - q_x)[x] + (q_x - q_y)[y] \\ &= \big(q^3 - 1\big)\big([0] - [\infty]\big) + \big(q^2 - q\big)\big([x] - [y]\big), \end{aligned}$$

and similarly,

$$\mathrm{div}(\Delta/\Delta_x) = (q-1)\big([y]-[\infty]\big) + \big(q^3-q^2\big)\big([0]-[x]\big),$$
$$\mathrm{div}(\Delta/\Delta_y) = \big(q^2-1\big)\big([x]-[\infty]\big) + \big(q^3-q\big)\big([0]-[y]\big).$$

Next, by [15, p. 200], the largest positive integer k such that Δ/Δ_{xy} has a kth root in the field of modular functions for $\Gamma_0(xy)$ is $(q-1)^2/(q-1) = (q-1)$. We can apply the same argument to Δ/Δ_x as a modular function for $\Gamma_0(x)$ to deduce that Δ/Δ_x has $(q-1)^2/(q-1)$th root. Similarly, Δ/Δ_y has $(q-1)(q^2-1)/(q-1)$th root. Therefore, the following relations hold in $\mathrm{Pic}^0(X_0(xy)_F)$:

$$\begin{aligned} \big(q^2+q+1\big)\big([0]-[\infty]\big) + q\big([x]-[y]\big) &= 0,\\ \big([y]-[\infty]\big) + q^2\big([0]-[x]\big) &= 0,\\ \big([x]-[\infty]\big) + q\big([0]-[y]\big) &= 0. \end{aligned} \tag{3.3}$$

There is one more relation between the cuspidal divisors which comes from the fact that $X_0(xy)_F$ is hyperelliptic. By a theorem of Schweizer [36, Thm. 20], $X_0(xy)_F$ is hyperelliptic, and W_{xy} is the hyperelliptic involution. Consider the degree-2 covering

$$\pi : X_0(xy)_F \to X_0(xy)_F/W_{xy} \cong \mathbb{P}^1_F.$$

Denote $P := \pi([\infty])$, $Q := \pi([x])$. Since $W_{xy}([\infty]) \neq [x]$, $P \neq Q$. There is a function f on $\mathbb{P}^1_F$ with divisor $P - Q$. Now

$$\begin{aligned} \mathrm{div}\big(\pi^* f\big) &= \pi^*\big(\mathrm{div}(f)\big) = \pi^*(P-Q)\\ &= \big([\infty] + W_{xy}\big([\infty]\big)\big) - \big([x] + W_{xy}\big([x]\big)\big) = [\infty] + [0] - [x] - [y]. \end{aligned}$$

This gives the relation in $\mathrm{Pic}^0(X_0(xy)_F)$

$$[\infty] + [0] - [x] - [y] = 0. \tag{3.4}$$

Fixing $[\infty] \in X_0(xy)(F)$ as an F-rational point, we have the Abel–Jacobi map $X_0(xy)_F \to J_0(xy)$ which sends a point $P \in X_0(xy)_F$ to the linear equivalence class of the degree-0 divisor $P - [\infty]$.

Definition 3.2. Let $c_0, c_x, c_y \in J_0(xy)(F)$ be the classes of $[0]-[\infty]$, $[x]-[\infty]$, and $[y]-[\infty]$, respectively. These give F-rational points on the Jacobian since the cusps are F-rational. The *cuspidal divisor group* is the subgroup $\mathcal{C} \subset J_0(xy)$ generated by c_0, c_x, and c_y.

From (3.3) and (3.4) we obtain the following relations:

$$\begin{aligned} \big(q^2+q+1\big)c_0 + qc_x - qc_y &= 0,\\ q^2c_0 - q^2c_x + c_y &= 0,\\ qc_0 + c_x - qc_y &= 0,\\ c_0 - c_x - c_y &= 0. \end{aligned}$$

Lemma 3.3. *The cuspidal divisor group $\mathcal{C}$ is generated by c_x and c_y, which have orders dividing $q+1$ and q^2+1, respectively.*

Proof. Substituting $c_0 = c_x + c_y$ into the first three equations above, we see that $\mathcal{C}$ is generated by c_x and c_y subject to relations:

$$(q+1)c_x = 0,$$
$$\left(q^2+1\right)c_y = 0. \qquad \square$$

The following simple lemma, which will be used later on, shows that the factors (q^2+1) and $(q+1)$ appearing in Lemma 3.3 are almost coprime.

Lemma 3.4. *Let n be a positive integer. Then*

$$\gcd\left(n^2+1, n+1\right) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $d = \gcd(n^2+1, n+1)$. Then d divides $(n^2+1) - (n+1) = n(n-1)$. Since n is coprime to $n+1$, d must divide $n-1$, hence also must divide $(n+1) - (n-1) = 2$. For n even, d is obviously odd, so $d = 1$. For n odd, $n+1$ and n^2+1 are both even, so $d = 2$. $\square$

4. Néron models and component groups

4.1. Terminology and notation

The notation in this section will be somewhat different from the rest of the paper. Let R be a complete discrete valuation ring, with fraction field K and algebraically closed residue field k.

Let A_K be an abelian variety over K. Denote by A its Néron model over R and denote by A_k^0 the connected component of the identity of the special fiber A_k of A. There is an exact sequence

$$0 \to A_k^0 \to A_k \to \Phi_A \to 0,$$

where Φ_A is a finite (abelian) group called the *component group of* A_K. We say that A_K has *semi-abelian reduction* if A_k^0 is an extension of an abelian variety A_k' by an affine algebraic torus T_A over k (cf. [2, p. 181]):

$$0 \to T_A \to A_k^0 \to A_k' \to 0.$$

We say that A_K has *toric reduction* if $A_k^0 = T_A$. The *character group*

$$M_A := \operatorname{Hom}(T_A, \mathbb{G}_{m,k})$$

is a free abelian group contravariantly associated to A.

Let X_K be a smooth, proper, geometrically connected curve over K. We say that X is a *semi-stable model* of X_K over R if (cf. [2, p. 245]):

(i) X is a proper flat R-scheme.
(ii) The generic fiber of X is X_K.
(iii) The special fiber X_k is reduced, connected, and has only ordinary double points as singularities.

We will denote the set of irreducible components of X_k by $C(X)$ and the set of singular points of X_k by $S(X)$. Let $G(X)$ be the *dual graph of* X: The set of vertices of $G(X)$ is the set $C(X)$, the set of edges is the set $S(X)$, the end points of an edge x are the two components containing x. Locally at $x \in S(X)$ for the étale topology, X is given by the equation $uv = \pi^{m(x)}$, where π is a uniformizer of R. The integer $m(x) \geqslant 1$ is well defined, and will be called the *thickness* of x. One obtains from $G(X)$ a graph with length by assigning to each edge $x \in S(X)$ the length $m(x)$.

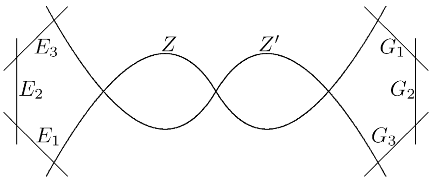

Fig. 1. $\widetilde{X}_k$ for $n=5$ and $m=4$.

4.2. Raynaud's theorem

Let X_K be a curve over K with semi-stable model X over R. Let J_K be the Jacobian of X_K, let J be the Néron model of J_K over R, and $\Phi := J_k/J_k^0$. Let $\widetilde{X} \to X$ be the minimal resolution of X. Let $B(\widetilde{X})$ be the free abelian group generated by the elements of $C(\widetilde{X})$. Let $B^0(\widetilde{X})$ be the kernel of the homomorphism

$$B(\widetilde{X}) \to \mathbb{Z}, \qquad \sum_{C_i \in C(\widetilde{X})} n_i C_i \mapsto \sum n_i.$$

The elements of $C(\widetilde{X})$ are Cartier divisors on $\widetilde{X}$, hence for any two of them, say C and C', we have an intersection number $(C \cdot C')$. The image of the homomorphism

$$\alpha : B(\widetilde{X}) \to B(\widetilde{X}), \quad C \mapsto \sum_{C' \in C(\widetilde{X})} \big(C \cdot C'\big) C'$$

lies in $B^0(\widetilde{X})$. A theorem of Raynaud [2, Thm. 9.6/1] says that Φ is canonically isomorphic to $B^0(\widetilde{X})/\alpha(B(\widetilde{X}))$.

The homomorphism $\phi : J_K(K) \to \Phi$ obtained from the composition

$$J_K(K) = J(R) \to J_k(k) \to \Phi$$

will be called the *canonical specialization map*. Let $D = \sum_Q n_Q Q$ be a degree-0 divisor on X_K whose support is in the set of K-rational points. Let $P \in J_K(K)$ be the linear equivalence class of D. The image $\phi(P)$ can be explicitly described as follows. Since X and $\widetilde{X}$ are proper, $X(K) = X(R) = \widetilde{X}(R)$. Since $\widetilde{X}$ is regular, each point $Q \in X(K)$ specializes to a unique element $c(Q)$ of $C(\widetilde{X})$. With this notation, $\phi(P)$ is the image of $\sum_Q n_Q c(Q) \in B^0(\widetilde{X})$ in Φ.

We apply Raynaud's theorem to compute Φ explicitly for a special type of X. Assume that X_k consists of two components Z and Z' crossing transversally at $n \geqslant 2$ points $x_1, \dots, x_n$. Denote $m_i := m(x_i)$. Let $r : \widetilde{X} \to X$ denote the resolution morphism; it is a composition of blow-ups at the singular points. It is well known that $r^{-1}(x_i)$ is a chain of $m_i - 1$ projective lines. More precisely, the special fiber $\widetilde{X}_k$ consists of Z and Z' but now, instead of intersecting at x_i, these components are joined by a chain $E_1, \dots, E_{m_i-1}$ of projective lines, where E_i intersect E_{i+1}, E_1 intersects Z at x_i and E_{m_i-1} intersects Z' at x_i. All the singularities are ordinary double points.

Assume $m_1 = m_n = m \geqslant 1$ and $m_2 = \cdots = m_{n-1} = 1$ if $n \geqslant 3$.

If $m = 1$, then $X = \widetilde{X}$, so $B^0(\widetilde{X})$ is freely generated by $z := Z - Z'$. In this case Raynaud's theorem implies that Φ is isomorphic to $B^0(\widetilde{X})$ modulo the relation $nz = 0$.

If $m \geqslant 2$, let $E_1, \dots, E_{m-1}$ be the chain of projective lines at x_1 and $G_1, \dots, G_{m-1}$ be the chain of projective lines at x_n, with the convention that Z in $\widetilde{X}_k$ intersects E_1 and G_1, cf. Fig. 1. The elements $z := Z - Z'$, $e_i := E_i - Z'$, $g_i := G_i - Z'$, $1 \leqslant i \leqslant m-1$ form a $\mathbb{Z}$-basis of $B^0(\widetilde{X})$. By Raynaud's theorem, Φ is isomorphic to $B^0(\widetilde{X})$ modulo the following relations:

if $m = 2$,

$$-nz + e_1 + g_1 = 0, \qquad z - 2e_1 = 0, \qquad z - 2g_1 = 0;$$

if $m = 3$,

$$-nz + e_1 + g_1 = 0, \qquad z - 2e_1 + e_2 = 0, \qquad z - 2g_1 + g_2 = 0,$$
$$e_1 - 2e_2 = 0, \qquad g_1 - 2g_2 = 0;$$

if $m \geqslant 4$

$$-nz + e_1 + g_1 = 0, \qquad z - 2e_1 + e_2 = 0, \qquad z - 2g_1 + g_2 = 0,$$
$$e_i - 2e_{i+1} + e_{i+2} = 0, \qquad g_i - 2g_{i+1} + g_{i+2} = 0, \quad 1 \leqslant i \leqslant m-3,$$
$$e_{m-2} - 2e_{m-1} = 0, \qquad g_{m-2} - 2g_{m-1} = 0.$$

Theorem 4.1. *Denote the images of z, e_i, g_i in Φ by the same letters, and let $\langle z\rangle$ be the cyclic subgroup generated by z in Φ. Then for any $n \geqslant 2$ and $m \geqslant 1$:*

(i) *$\Phi \cong \mathbb{Z}/m(m(n-2)+2)\mathbb{Z}$.*
(ii) *If $m \geqslant 2$, then Φ is generated by e_{m-1}. Explicitly, for $1 \leqslant i \leqslant m-1$,*

$$e_i = (m-i)e_{m-1},$$
$$g_i = \big(i(nm+1) - (2i-1)m\big)e_{m-1},$$
$$z = me_{m-1}.$$

(iii) *$\Phi/\langle z\rangle \cong \mathbb{Z}/m\mathbb{Z}$.*

Proof. When $m = 1$ the claim is obvious, so assume $m \geqslant 2$. By [2, Prop. 9.6/10], Φ has order

$$\sum_{i=1}^{n} \prod_{j \neq i} m_j = m^2(n-2) + 2m.$$

From the relations

$$e_{m-2} - 2e_{m-1} = 0,$$
$$e_i - 2e_{i+1} + e_{i+2} = 0, \quad 1 \leqslant i \leqslant m-3,$$
$$z - 2e_1 + e_2 = 0$$

it follows inductively that $e_i = (m-i)e_{m-1}$ for $1 \leqslant i \leqslant m-1$, and $z = me_{m-1}$. Next, from the relations

$$-nz + e_1 + g_1 = 0 \quad \text{and} \quad z - 2g_1 + g_2 = 0$$

we get $g_1 = (nm - m + 1)e_{m-1}$ and $g_2 = (2nm - 3m + 2)e_{m-1}$. Finally, if $m \geqslant 4$, the relations $g_i - 2g_{i+1} + g_{i+2} = 0$, $1 \leqslant i \leqslant m-3$, show inductively that

$$g_i = \big(i(nm+1) - (2i-1)m\big)e_{m-1}, \quad 1 \leqslant i \leqslant m-1.$$

This proves (i) and (ii), and (iii) is an immediate consequence of (ii). $\square$

Remark 4.2. Note that by the formula in Theorem 4.1

$$g_{m-1} = \big(m^2(n-2) + 2m - \big(m(n-2)+1\big)\big)e_{m-1} = -\big(m(n-2)+1\big)e_{m-1}.$$

It is easy to see that $m(n-2)+1$ is coprime to the order $m(m(n-2)+2)$ of Φ. Hence g_{m-1} is also a generator. This is of course not surprising since the relations defining Φ remain the same if we interchange e_i's and g_i's.

4.3. *Grothendieck's theorem*

Grothendieck gave another description of Φ in [20]. This description will be useful for us when studying maps between the component groups induced by isogenies of abelian varieties.

Let A_K be an abelian variety over K with semi-abelian reduction. Denote by $\hat{A}_K$ the dual abelian variety of A_K. As discussed in [20], there is a non-degenerate pairing $u_A : M_A \times M_{\hat{A}} \to \mathbb{Z}$ (called *monodromy pairing*) having nice functorial properties, which induces an exact sequence

$$0 \to M_{\hat{A}} \xrightarrow{u_A} \mathrm{Hom}(M_A, \mathbb{Z}) \to \Phi_A \to 0. \tag{4.1}$$

Let $H \subset A_K(K)$ be a finite subgroup of order coprime to the characteristic of k. Since $A(R) = A_K(K)$, H extends to a constant étale subgroup-scheme $\mathcal{H}$ of A. The restriction to the special fiber gives a natural injection $\mathcal{H}_k \cong H \hookrightarrow A_k(k)$, cf. [2, Prop. 7.3/3]. Composing this injection with $A_k \to \Phi_A$, we get the canonical homomorphism $\phi : H \to \Phi_A$. Denote $H_0 := \ker(\phi)$ and $H_1 := \mathrm{im}(\phi)$, so that there is a tautological exact sequence

$$0 \to H_0 \to H \xrightarrow{\phi} H_1 \to 0.$$

Let B_K be the abelian variety obtained as the quotient of A_K by H. Let $\varphi_K : A_K \to B_K$ denote the isogeny whose kernel is H. By the Néron mapping property, φ_K extends to a morphism $\varphi : A \to B$ of the Néron models. On the special fibers we get a homomorphism $\varphi_k : A_k \to B_k$, which induces an isogeny $\varphi_k^0 : A_k^0 \to B_k^0$ and a homomorphism $\varphi_\Phi : \Phi_A \to \Phi_B$. The isogeny φ_k^0 restricts to an isogeny $\varphi_t : T_A \to T_B$, which corresponds to an injective homomorphisms of character groups $\varphi^* : M_B \to M_A$ with finite cokernel.

Theorem 4.3. *Assume A_K has toric reduction. There is an exact sequence*

$$0 \to H_1 \to \Phi_A \xrightarrow{\varphi_\Phi} \Phi_B \to H_0 \to 0.$$

Proof. The kernel of φ_k is $\mathcal{H}_k \cong H$. It is clear that $\ker(\varphi_\Phi) = H_1$. Let $\hat{\varphi}_K : \hat{B}_K \to \hat{A}_K$ be the isogeny dual to φ_K. Using (4.1), one obtains a commutative diagram with exact rows (cf. [34, p. 8]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\hat{A}} & \longrightarrow & \mathrm{Hom}(M_A, \mathbb{Z}) & \longrightarrow & \Phi_A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\varphi}^*} & & \downarrow{\scriptstyle \mathrm{Hom}(\varphi^*, \mathbb{Z})} & & \downarrow{\scriptstyle \varphi_\Phi} & & \\
0 & \longrightarrow & M_{\hat{B}} & \longrightarrow & \mathrm{Hom}(M_B, \mathbb{Z}) & \longrightarrow & \Phi_B & \longrightarrow & 0.
\end{array}$$

From this diagram we get the exact sequence

$$0 \to \ker(\varphi_\Phi) \to M_{\hat{B}}/\hat{\varphi}^*(M_{\hat{A}}) \to \mathrm{Ext}^1_{\mathbb{Z}}\big(M_A/\varphi^*(M_B), \mathbb{Z}\big) \to \mathrm{coker}(\varphi_\Phi) \to 0.$$

Using the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, it is easy to show that

$$\mathrm{Ext}^1_{\mathbb{Z}}\big(M_A/\varphi^*(M_B), \mathbb{Z}\big) \cong \mathrm{Hom}\big(M_A/\varphi^*(M_B), \mathbb{Q}/\mathbb{Z}\big) =: \big(M_A/\varphi^*(M_B)\big)^\vee,$$

so there is an exact sequence of abelian groups

$$0 \to \ker(\varphi_\Phi) \to M_{\hat{B}}/\hat{\varphi}^*(M_{\hat{A}}) \to \big(M_A/\varphi^*(M_B)\big)^\vee \to \mathrm{coker}(\varphi_\Phi) \to 0. \tag{4.2}$$

So far we have not used the assumption that A_K has toric reduction. Under this assumption, B_K also has toric reduction, and H_0 is the kernel of $\varphi_t : T_A \to T_B$. Hence $(M_A/\varphi^*(M_B))^\vee \cong H_0$. Next, [5, Thm. 8.6] implies that $M_{\hat{B}}/\hat{\varphi}^*(M_{\hat{A}}) \cong H_1$. Thus, we can rewrite (4.2) as

$$0 \to \ker(\varphi_\Phi) \to H_1 \to H_0 \to \mathrm{coker}(\varphi_\Phi) \to 0.$$

Since $\ker(\varphi_\Phi) = H_1$, this implies that $\mathrm{coker}(\varphi_\Phi) \cong H_0$. □

5. Component groups of $J_0(xy)$

5.1. Component groups at x and y

We return to the notation in Section 3. As we mentioned in Section 2.1, $X_0(xy)$ is smooth over $A[1/xy]$.

Proposition 5.1.

(i) *$X_0(xy)_{F_x}$ has a semi-stable model over $\mathcal{O}_x$ such that $X_0(xy)_{\mathbb{F}_x}$ consists of two irreducible components both isomorphic to $X_0(y)_{\mathbb{F}_x} \cong \mathbb{P}^1_{\mathbb{F}_q}$ intersecting transversally in $q+1$ points. Two of these singular points have thickness $q+1$, and the other $q-1$ points have thickness 1.*

(ii) *$X_0(xy)_{F_y}$ has a semi-stable model over $\mathcal{O}_y$ such that $X_0(xy)_{\mathbb{F}_y}$ consists of two irreducible components both isomorphic to $X_0(x)_{\mathbb{F}_y} \cong \mathbb{P}^1_{\mathbb{F}_{q^2}}$ intersecting transversally in $q+1$ points. All these singular points have thickness 1.*

Proof. The fact that $X_0(xy)_F$ has a model over $\mathcal{O}_x$ and $\mathcal{O}_y$ with special fibers of the stated form follows from the same argument as in the case of $X_0(v)_F$ over $\mathcal{O}_v$ ($v \in |F| - \infty$) discussed in [11, §5]. We only clarify why the number of singular points and their thickness are as stated.

(i) The special fiber $X_0(xy)_{\mathbb{F}_x}$ consists of two copies of $X_0(y)_{\mathbb{F}_x}$. The set of points $Y_0(y)(\bar{\mathbb{F}}_x)$ is in bijection with the isomorphism classes of pairs (ϕ, C_y), where ϕ is a rank-2 Drinfeld A-module over $\bar{\mathbb{F}}_x$ and $C_y \cong A/y$ is a cyclic subgroup of ϕ. The two copies of $X_0(y)_{\mathbb{F}_x}$ intersect exactly at the points corresponding to (ϕ, C_y) with ϕ supersingular; more precisely, (ϕ, C_y) on the first copy is identified with $(\phi^{(x)}, C_y^{(x)})$ on the second copy where $\phi^{(x)}$ is the image of ϕ under the Frobenius isogeny and $C_y^{(x)}$ is subgroup of $\phi^{(x)}$ which is the image of C_y, cf. [11].

Now, by Lemma 2.1, up to an isomorphism over $\bar{\mathbb{F}}_x$, there is a unique supersingular Drinfeld module ϕ in characteristic x and $j(\phi) = 0$. It is easy to see that ϕ has $q_y + 1 = q^2 + 1$ cyclic subgroups isomorphic to A/y, so the set $S = \{(\phi, C_y) \mid C_y \subset \phi[y]\}$ has cardinality $q^2 + 1$. By Lemma 2.2, $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^2}^\times$. This group naturally acts S, and the orbits are in bijection with the singular points of $X_0(xy)_{\mathbb{F}_x}$. Since the genus of $X_0(xy)_F$ is q, the arithmetic genus of $X_0(xy)_{\mathbb{F}_x}$ is also q due to the flatness of $X_0(xy) \to \mathrm{Spec}(A)$; see [21, Cor. III.9.10]. Using the fact that the genus of $X_0(y)_F$ is zero, a simple calculation shows that the number of singular points of $X_0(xy)_{\mathbb{F}_x}$ is $q+1$, cf. [21, p. 298]. Next, by Lemma 2.3, the stabilizer in $\mathrm{Aut}(\phi)$ of (ϕ, C_y) is either $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$. Let s be the number of pairs (ϕ, C_y) with stabilizer $\mathbb{F}_{q^2}^\times$. Let t be the number of orbits of pairs with stabilizers $\mathbb{F}_q^\times$; each such orbit consists of $\#(\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times) = q+1$ pairs (ϕ, C_y). Hence we have

$$(q+1)t + s = q^2 + 1 \quad \text{and} \quad t + s = q + 1.$$

This implies that $t = q-1$ and $s = 2$. Finally, as is explained in [11], the thickness of the singular point corresponding to an isomorphism class of (ϕ, C_y) is equal to $\#(\mathrm{Aut}(\phi, C_y)/\mathbb{F}_q^\times)$.

(ii) Similar to the previous case, $X_0(xy)_{\mathbb{F}_y}$ consists of two copies of $X_0(x)_{\mathbb{F}_y} \cong \mathbb{P}^1_{\mathbb{F}_{q^2}}$. The two copies of $X_0(x)_{\mathbb{F}_y}$ intersect exactly at the points corresponding to the isomorphism classes of pairs (ϕ, C_x) with ϕ supersingular. Again by Lemma 2.1, up to an isomorphism over $\bar{\mathbb{F}}_y$, there is a unique supersingular ϕ and $j(\phi) \neq 0$. Hence, by Lemma 2.3, $\mathrm{Aut}(\phi, C_x) \cong \mathbb{F}_q^\times$ for any C_x. There are $q_x + 1 = q + 1$ cyclic subgroups in ϕ isomorphic to A/x. The rest of the argument is the same as in the previous case. □

Theorem 5.2. *Let Φ_v denote the group of connected components of $J_0(xy)$ at $v \in |F|$. Let Z and Z' be the irreducible components in Proposition* 5.1 *with the convention that the reduction of* $[\infty]$ *lies on Z'. Let $z = Z - Z'$.*

(i) $\Phi_x \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$.
(ii) $\Phi_y \cong \mathbb{Z}/(q+1)\mathbb{Z}$.
(iii) *Under the canonical specialization map $\phi_x : \mathcal{C} \to \Phi_x$ we have*

$$\phi_x(c_x) = 0 \quad \textit{and} \quad \phi_x(c_y) = z.$$

In particular, q^2+1 divides the order of c_y.

(iv) *Under the canonical specialization map $\phi_y : \mathcal{C} \to \Phi_y$ we have*

$$\phi_y(c_x) = z \quad \textit{and} \quad \phi_y(c_y) = 0.$$

In particular, $q+1$ divides the order of c_x.

Proof. (i) and (ii) follow from Theorem 4.1 and Proposition 5.1.

(iii) The cusps reduce to distinct points in the smooth locus of $X_0(xy)_{\mathbb{F}_x}$, cf. [41]. Since by Theorem 4.1 we know that z has order q^2+1 in the component group Φ_x, it is enough to show that the reductions of $[y]$ and $[\infty]$ lie on distinct components Z and Z' in $X_0(xy)_{\mathbb{F}_x}$, but the reductions of $[x]$ and $[\infty]$ lie on the same component. The involution W_x interchanges the two components $X_0(y)_{\mathbb{F}_x}$, cf. [11, (5.3)]. Since $W_x([\infty]) = [y]$, the reductions of $[\infty]$ and $[y]$ lie on distinct components. On the other hand, W_y acts on $X_0(xy)_{\mathbb{F}_y}$ by acting on each component $X_0(y)_{\mathbb{F}_x}$ separately, without interchanging them. Since $W_y([\infty]) = [x]$, the reductions of $[\infty]$ and $[x]$ lie on the same component.

(iv) The argument is similar to (iii). Here W_y interchanges the two components $X_0(x)_{\mathbb{F}_y}$ of $X_0(xy)_{\mathbb{F}_y}$ and W_x maps the components to themselves. Hence $[\infty]$ and $[y]$ lie on one component and $[0]$ and $[x]$ on the other component. □

Theorem 5.3. *The cuspidal divisor group*

$$\mathcal{C} \cong \mathbb{Z}/(q+1)\mathbb{Z} \oplus \mathbb{Z}/\big(q^2+1\big)\mathbb{Z}$$

is the direct sum of the cyclic subgroups generated by c_x and c_y, which have orders $(q+1)$ *and* (q^2+1), *respectively.* (*Note that $\mathcal{C}$ is cyclic if q is even, but it is not cyclic if q is odd.*)

Proof. By Lemma 3.3 and Theorem 5.2, $\mathcal{C}$ is generated by c_x and c_y, which have orders $(q+1)$ and (q^2+1), respectively. If the subgroup of $\mathcal{C}$ generated by c_x non-trivially intersects with the subgroup generated by c_y, then, by Lemma 3.4, q must be odd and $\frac{q+1}{2}c_x = \frac{q^2+1}{2}c_y$. Applying ϕ_y to both sides of this equality, we get $\frac{q+1}{2}z = 0$, which is a contradiction since z generates $\Phi_y \cong \mathbb{Z}/(q+1)\mathbb{Z}$. □

Remark 5.4. The divisor class c_0 has order $(q+1)(q^2+1)$ (resp. $(q+1)(q^2+1)/2$) if q is even (resp. odd).

5.2. *Component group at* ∞

To obtain a model of $X_0(xy)_{F_\infty}$ over $\mathcal{O}_\infty$, instead of relying on the moduli interpretation of $X_0(xy)$, one has to use the existence of analytic uniformization for this curve; see [28, §4.2]. As far as the structure of the special fiber $X_0(xy)_{\mathbb{F}_\infty}$ is concerned, it is more natural to compute the dual graph of $X_0(xy)_{\mathbb{F}_\infty}$ directly using the quotient $\Gamma_0(xy) \setminus \mathcal{T}$ of the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(F_\infty)$. For the definition of $\mathcal{T}$, and more generally for the basic theory of trees and groups acting on trees, we refer to [40].

The quotient graph $\Gamma_0(xy) \setminus \mathcal{T}$ was first computed by Gekeler [10, (5.2)]. For our purposes we will need to know the relative position of the cusps on $\Gamma_0(xy) \setminus \mathcal{T}$ and also the stabilizers of the edges. To obtain this more detailed information, and for the general sake of completeness, we recompute $\Gamma_0(xy) \setminus \mathcal{T}$ in this subsection using the method in [16].

Denote

$$G_0 = \mathrm{GL}_2(\mathbb{F}_q)$$

and

$$G_i = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; \deg(b) \leqslant i \right\}, \quad i \geqslant 1.$$

As is explained in [16], $\Gamma_0(xy) \setminus \mathcal{T}$ can be constructed in "layers", where the vertices of the ith layer (in [16] called *type-i vertices*) are the orbits

$$X_i := G_i \setminus \mathbb{P}^1(A/xy)$$

and the edges connecting type-i vertices to type-$(i+1)$ vertices, called *type-i edges*, are the orbits

$$Y_i := (G_i \cap G_{i+1}) \setminus \mathbb{P}^1(A/xy).$$

There are obvious maps $Y_i \to X_i$, $Y_i \to X_{i+1}$ and $X_i \to X_{i+1}$ which are used to define the adjacencies of vertices in X_i and X_{i+1}; see [16, 1.7]. The graph $\Gamma_0(xy) \setminus \mathcal{T}$ is isomorphic to the graph with set of vertices $\bigsqcup_{i \geqslant 0} X_i$ and set of edges $\bigsqcup_{i \geqslant 0} Y_i$ with the adjacencies defined by these maps.

Note that $\mathbb{P}^1(A/xy) = \mathbb{P}^1(\mathbb{F}_x) \times \mathbb{P}^1(\mathbb{F}_y)$. We will represent the elements of $\mathbb{P}^1(A/xy)$ as couples $[P; Q]$ where $P \in \mathbb{P}^1(\mathbb{F}_x)$ and $Q \in \mathbb{P}^1(\mathbb{F}_y)$. With this notation, G_i acts diagonally on $[P; Q]$ via its images in $\mathrm{GL}_2(\mathbb{F}_x)$ and $\mathrm{GL}_2(\mathbb{F}_y)$, respectively.

The group G_0 acting on $\mathbb{P}^1(A/xy)$ has 3 orbits, whose representatives are

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \quad \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right], \quad \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} x \\ 1 \end{pmatrix}\right],$$

where in the last element we write x for the image in $\mathbb{F}_y$ of the monic generator of x under the canonical homomorphism $A \to A/y$. The orbit of $[\binom{1}{0}; \binom{1}{0}]$ has length $q+1$, the orbit of $[\binom{1}{0}; \binom{0}{1}]$ has length $q(q+1)$, and the orbit of $[\binom{1}{0}; \binom{x}{1}]$ has length $q(q^2-1)$, cf. [16, Prop. 2.10]. Next, note

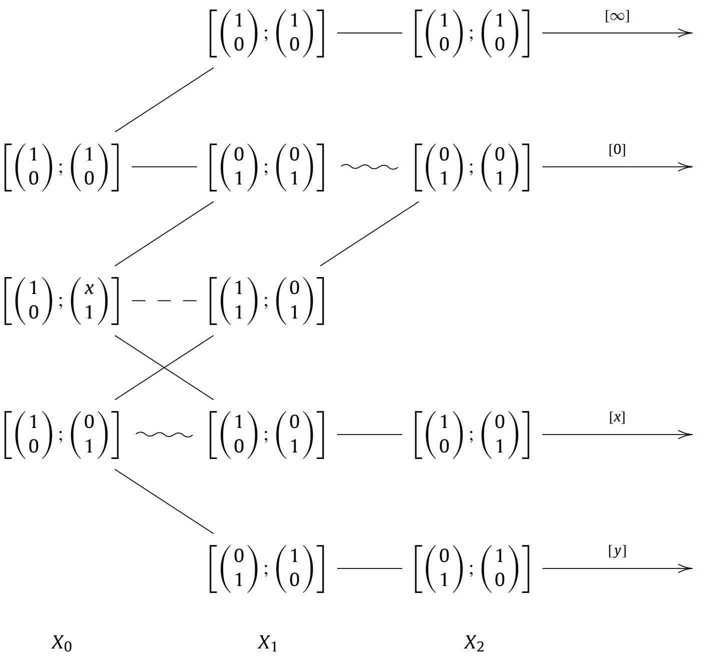

Fig. 2. $\Gamma_0(xy)\setminus\mathcal{T}$.

that $G_0\cap G_1$ is the subgroup B of the upper-triangular matrices in $\mathrm{GL}_2(\mathbb{F}_q)$. The G_0-orbit of $[\binom{1}{0};\binom{1}{0}]$ splits into two B-orbits with representatives:

$$\left[\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix}\right] \text{ and } \left[\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}\right]. \tag{5.1}$$

The lengths of these B-orbits are 1 and q, respectively. The G_0-orbit of $[\binom{1}{0};\binom{0}{1}]$ splits into three B-orbits with representatives:

$$\left[\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}\right], \quad \left[\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix}\right], \quad \left[\begin{pmatrix}1\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}\right]. \tag{5.2}$$

The lengths of these B-orbits are q, q, $q(q-1)$, respectively. Finally, the G_0-orbit of $[\binom{1}{0};\binom{x}{1}]$ splits into $(q+1)$ B-orbits each of length $q(q-1)$. The previous statements can be deduced from Proposition 2.11 in [16]. It turns out that the elements of $\mathbb{P}^1(\mathbb{F}_x)\times\mathbb{P}^1(\mathbb{F}_y)$ listed in (5.1) and (5.2) combined form a complete set of G_1-orbit representatives. For $i\geqslant 1$, the set of G_i-orbit representatives obviously contains a complete set of G_{i+1}-orbit representatives. A small calculation shows that

$$\left[\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix}\right], \quad \left[\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}\right], \quad \left[\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}0\\1\end{pmatrix}\right], \quad \left[\begin{pmatrix}0\\1\end{pmatrix};\begin{pmatrix}1\\0\end{pmatrix}\right] \tag{5.3}$$

is a complete set of G_i-orbit representatives for any $i\geqslant 2$. Moreover, the elements $[\binom{1}{1};\binom{0}{1}]$ and $[\binom{0}{1};\binom{0}{1}]$ are in the same G_2-orbit. We recognize the elements in (5.3) as the cusps $[\infty]$, $[0]$, $[x]$, $[y]$, respectively. Overall, the structure of $\Gamma_0(xy)\setminus\mathcal{T}$ is described by the diagram in Fig. 2. In the diagram the broken line $---$ indicates that there are $(q-1)$ distinct edges joining the corresponding vertices, and an arrow $\to$ indicates an infinite half-line.

Now we compute the stabilizers of the edges. Let e be an edge in $\Gamma_0(xy)\setminus\mathcal{T}$ of type i. Let

$$O(e)=(G_i\cap G_{i+1})[P;\,Q]$$

be its corresponding orbit in $(G_i\cap G_{i+1})\setminus\mathbb{P}^1(A/xy)$. Then for a preimage $\tilde{e}$ of e in $\mathcal{T}$ we have

$$\#\mathrm{Stab}_{\Gamma_0(xy)}(\tilde{e})=\#\mathrm{Stab}_{G_i\cap G_{i+1}}\big([P;\,Q]\big)=\frac{\#(G_i\cap G_{i+1})}{\#O(e)}.$$

Using this observation, we conclude from our previous discussion that the edges connecting $[\binom{1}{0};\binom{x}{1}]\in X_0$ to any vertex in X_1 have preimages whose stabilizers have order $\#B/q(q-1)=q-1$. The preimages of the edges connecting $[\binom{1}{0};\binom{0}{1}]\in X_0$ to $[\binom{1}{1};\binom{0}{1}]\in X_1$ and $[\binom{1}{0};\binom{0}{1}]\in X_1$ have stabilizers of orders $q-1$ and $(q-1)^2$, respectively. (Note that if a stabilizer has order $(q-1)$ then it is equal to the center $Z(\Gamma_0(xy))\cong\mathbb{F}_q^\times$ of $\Gamma_0(xy)$, as the center is a subgroup of any stabilizer.) The *valency* of a vertex v in a graph without loops is the number of distinct edges having v as an endpoint. (A loop is an edge whose endpoints are the same.) Consider the vertex $v=[\binom{1}{1};\binom{0}{1}]\in X_1$. Its valency is $(q+1)$. Let $\tilde{v}$ be a preimage of v in $\mathcal{T}$. Since the valency of $\tilde{v}$ is also $q+1$, $\mathrm{Stab}_{\Gamma_0(xy)}(\tilde{v})$ acts trivially on all edges having $\tilde{v}$ as an endpoint. Hence the stabilizer of any such edge is equal to $\mathrm{Stab}_{\Gamma_0(xy)}(\tilde{v})$. We already determined that the stabilizer of a preimage of an edge connecting v to a type-0 vertex is $\mathbb{F}_q^\times$. This implies that the stabilizer in $\Gamma_0(xy)$ of a preimage of the edge connecting v to $[\binom{0}{1};\binom{0}{1}]\in X_2$ is also $\mathbb{F}_q^\times$. Finally, consider the vertex $w=[\binom{0}{1};\binom{0}{1}]\in X_1$. Its valency is 3. Let S, S_1, S_2, S_3 be the orders of stabilizers in $\Gamma_0(xy)$ of a preimage $\tilde{w}$ of w in $\mathcal{T}$, and the edges connecting w to $[\binom{1}{0};\binom{1}{0}]\in X_0$, $[\binom{1}{0};\binom{x}{1}]\in X_0$, $[\binom{0}{1};\binom{0}{1}]\in X_2$, respectively. From our discussion of the lengths of orbits of type-0 edges, we have $S_1=(q-1)^2$ and $S_2=(q-1)$. Obviously, S_i's divide S. On the other hand, counting the lengths of orbits of $\mathrm{Stab}_{\Gamma_0(xy)}(\tilde{w})$ acting on the set of (non-oriented) edges in $\mathcal{T}$ having $\tilde{w}$ as an endpoint, we get

$$q+1=\frac{S}{S_1}+\frac{S}{S_2}+\frac{S}{S_3}=\frac{S}{(q-1)^2}+\frac{S}{(q-1)}+\frac{S}{S_3}.$$

This implies $S=S_3=(q-1)^2$. To summarize, in Fig. 2 a wavy line $\sim$ indicates that a preimage of the corresponding edge in $\mathcal{T}$ has a stabilizer in $\Gamma_0(xy)$ of order $(q-1)^2$. The edges connecting $[\binom{1}{0};\binom{x}{1}]$ or $[\binom{1}{1};\binom{0}{1}]$ to any other vertex have preimages in $\mathcal{T}$ whose stabilizers in $\Gamma_0(xy)$ are isomorphic to $\mathbb{F}_q^\times$.

Now from [28, §4.2] one deduces the following. The quotient graph $\Gamma_0(xy)\setminus\mathcal{T}$, without the infinite half-lines, is the dual graph of the special fiber of a semi-stable model of $X_0(xy)_{F_\infty}$ over $\mathrm{Spec}(\mathcal{O}_\infty)$. The special fiber $X_0(xy)_{\mathbb{F}_\infty}$ has 6 irreducible components Z, Z', E, E', G, G', all isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$, such that Z and Z' intersect in $q-1$ points, E intersects Z and E', E' intersects Z' and E, G intersects Z and G', G' intersects Z' and G. Moreover, all intersection points are ordinary double singularities. By [28, Prop. 4.3], the thickness of the singular point corresponding to an edge $e\in\Gamma_0(xy)\setminus\mathcal{T}$ is

$$\#\big(\mathrm{Stab}_{\Gamma_0(xy)}(\tilde{e})/\mathbb{F}_q^\times\big),$$

hence all intersection points on Z or Z' have thickness 1, but the intersection points of E and E', and of G and G' have thickness $(q-1)$, cf. Fig. 3. From the structure of $\Gamma_0(xy)\setminus\mathcal{T}$, one also concludes that the reductions of the cusps are smooth points in $X_0(xy)_{\mathbb{F}_\infty}$. Moreover, $[\infty]$, $[0]$, $[x]$, $[y]$ reduce to points on E, E', G, G' respectively.

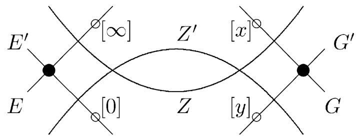

Fig. 3. $X_0(xy)_{\mathbb{F}_\infty}$ for $q = 3$.

Blowing up $X_0(xy)_{\mathcal{O}_\infty}$ at the intersection points of E, E', and G, G', $(q-2)$-times each, we obtain the minimal regular model of $X_0(xy)_F$ over $\operatorname{Spec}(\mathcal{O}_\infty)$. This is a curve of the type discussed in Section 4.2 with $m = n = (q+1)$, and we enumerate its irreducible components so that $E_1 = E$, $E_q = E'$, $G_1 = G$, $G_q = G'$.

Theorem 5.5. *Let $\phi_\infty : \mathcal{C} \to \Phi_\infty$ denote the canonical specialization map.*

(i) $\Phi_\infty \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$.
(ii) $\phi_\infty(c_x) = (q^2+1)e_q$ *and* $\phi_\infty(c_y) = -q(q+1)e_q = (q^3+1)e_q$.
(iii) *If q is even, then $\phi_\infty : \mathcal{C} \xrightarrow{\sim} \Phi_\infty$ is an isomorphism.*
(iv) *If q is odd, then there is an exact sequence*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{C} \xrightarrow{\phi_\infty} \Phi_\infty \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Proof. Part (i) is an immediate consequence of the preceding discussion and Theorem 4.1. We have determined the reductions of the cusps at ∞, so using Theorem 4.1, we get

$$\phi_\infty(c_x) = g_1 - e_1 = \big(q^2+q+1\big)e_q - qe_q = \big(q^2+1\big)e_q$$

and

$$\phi_\infty(c_y) = g_q - e_1 = -q^2 e_q - qe_q = -q(q+1)e_q,$$

which proves (ii). Since $\gcd(q^2+1, q(q+1)) = 1$ (resp. 2) if q is even (resp. odd), cf. Lemma 3.4, the subgroup of Φ_∞ generated by $\phi_\infty(c_x)$ and $\phi_\infty(c_y)$ is $\langle e_q\rangle$ (resp. $\langle 2e_q\rangle$) if q is even (resp. odd). On the other hand, we know that e_q generates Φ_∞. Therefore, if q is even, then ϕ_∞ is surjective, and if q is odd, then the cokernel of ϕ_∞ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The claims (iii) and (iv) now follow from Theorem 5.3. $\square$

Remark 5.6. We note that (iii) and a slightly weaker version of (iv) in Theorem 5.5 can be deduced from Theorem 5.3 and a result of Gekeler [14]. In fact, in [14, p. 366] it is proven that for an arbitrary $\mathfrak{n}$ the kernel of the canonical homomorphism from the cuspidal divisor group of $X_0(\mathfrak{n})_F$ to Φ_∞ is a quotient of $(\mathbb{Z}/(q-1)\mathbb{Z})^{c-1}$, where c is the number of cusps of $X_0(\mathfrak{n})_F$. In our case, this result says that $\ker(\phi_\infty)$ is a quotient of $(\mathbb{Z}/(q-1)\mathbb{Z})^3$. Now suppose q is even. Then $\mathcal{C} \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$. Since for even q, $\gcd(q-1, (q^2+1)(q+1)) = 1$, ϕ_∞ must be injective. But by (i), $\#\Phi_\infty = (q^2+1)(q+1) = \#\mathcal{C}$, so ϕ_∞ is also surjective. When $q = 2$, the fact that $\#\Phi_\infty = 15$ and ϕ_∞ is an isomorphism is already contained in [14, (5.3.1)].

Now suppose q is odd. Then $\mathcal{C} \cong \mathbb{Z}/(q^2+1)\mathbb{Z} \oplus \mathbb{Z}/(q+1)\mathbb{Z}$. Since

$$\gcd(q-1, q+1) = \gcd\big(q-1, q^2+1\big) = 2,$$

$\ker(\phi_\infty) \subset (\mathbb{Z}/2\mathbb{Z})^2$. Since Φ_∞ is cyclic but $\mathcal{C}$ is not, $\ker(\phi_\infty)$ is not trivial, hence it is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. (Theorem 5.5 implies that the second possibility does not occur.)

Notation 5.7. Let $\mathcal{C}_0$ be the subgroup of $\mathcal{C}$ generated by c_y.

Corollary 5.8. *The cyclic group $\mathcal{C}_0$ has order q^2+1. Under the canonical specializations $\mathcal{C}_0$ maps injectively into Φ_x and Φ_∞, and $\mathcal{C}_0$ is the kernel of ϕ_y.*

Proof. The claims easily follow from Theorems 5.2, 5.3 and 5.5. □

6. Component groups of J^{xy}

6.1. A class number formula

Let H be a quaternion algebra over F. Let $\mathrm{Ram} \subset |F|$ be the set of places where H ramifies. Assume $\infty \in \mathrm{Ram}$. Denote $\mathcal{R} = \mathrm{Ram} - \infty$. Note that $\mathcal{R} \neq \emptyset$ since $\#\mathrm{Ram}$ is even.

Let Θ be a hereditary A-order in H. Let $I_1, \dots, I_h$ be the isomorphism classes of left Θ-ideals. It is known that $h(\Theta) := h$, called the *class number* of Θ, is finite. For $i = 1, \dots, h$ we denote by Θ_i the right order of the respective I_i. (For the definitions see [42].) Denote

$$M(\Theta) = \sum_{i=1}^{h} \left(\Theta_i^\times : \mathbb{F}_q^\times\right)^{-1}.$$

It is not hard to show that each $\Theta_i^\times$ is isomorphic to either $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$; see [7, p. 383]. Let $U(\Theta)$ be the number of right orders Θ_i such that $\Theta_i^\times \cong \mathbb{F}_{q^2}^\times$. In particular,

$$h(\Theta) = M(\Theta) + U(\Theta)\left(1 - \frac{1}{q+1}\right).$$

Definition 6.1. For a subset S of $|F|$, let

$$\mathrm{Odd}(S) = \begin{cases} 1, & \text{if all places in } S \text{ have odd degrees;} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{S} \subset |F| - \infty$ be a finite (possibly empty) set of places such that $\mathcal{R} \cap \mathcal{S} = \emptyset$. Let $\mathfrak{n} \lhd A$ be the square-free ideal whose support is $\mathcal{S}$. Let Θ be an Eichler A-order of level $\mathfrak{n}$. (When $\mathcal{S} = \emptyset$, Θ is a maximal A-order in H.) The formulae that follow are special cases of (1), (4) and (6) in [7]:

$$M^{\mathcal{S}}(H) := M(\Theta) = \frac{1}{q^2-1} \prod_{v \in \mathcal{R}} (q_v - 1) \prod_{w \in \mathcal{S}} (q_w + 1),$$
$$U^{\mathcal{S}}(H) := U(\Theta) = 2^{\#\mathcal{R} + \#\mathcal{S} - 1}\, \mathrm{Odd}(\mathcal{R}) \prod_{w \in \mathcal{S}} \bigl(1 - \mathrm{Odd}(w)\bigr).$$

Denote

$$h^{\mathcal{S}}(H) = M^{\mathcal{S}}(H) + U^{\mathcal{S}}(H) \frac{q}{q+1}.$$

6.2. Component groups at x and y

Let D and R be as in Section 2.2. Recall that we assume $\infty \notin R$. Fix a place $w \in R$. Let D^w be the quaternion algebra over F which is ramified at $(R - w) \cup \infty$. Fix a maximal A-order $\mathfrak{D}$ in D^w, and denote

$$A^w = A[w^{-1}];$$
$$\mathfrak{D}^w = \mathfrak{D} \otimes_A A^w;$$
$$\Gamma^w = \{\gamma \in (\mathfrak{D}^w)^\times \mid \mathrm{ord}_w(\mathrm{Nr}(\gamma)) \in 2\mathbb{Z}\};$$

here w^{-1} denotes the inverse of a generator of the ideal in A corresponding to w, and Nr denotes the reduced norm on D^w.

By fixing an isomorphism $D^w \otimes_F F_w \cong \mathbb{M}_2(F_w)$, one can consider Γ^w as a subgroup of $\mathrm{GL}_2(F_w)$ whose image in $\mathrm{PGL}_2(F_w)$ is discrete and cocompact. Hence Γ^w acts on the Bruhat–Tits tree $\mathcal{T}^w$ of $\mathrm{PGL}_2(F_w)$. It is not hard to show that Γ^w acts without inversions, so the quotient graph $\Gamma^w \setminus \mathcal{T}^w$ is a finite graph without loops. We make $\Gamma^w \setminus \mathcal{T}^w$ into a graph with lengths by assigning to each edge e of $\Gamma^w \setminus \mathcal{T}^w$ the length $\#(\mathrm{Stab}_{\Gamma^w}(\tilde{e})/\mathbb{F}_q^\times)$, where $\tilde{e}$ is a preimage of e in $\mathcal{T}^w$. The graph with lengths $\Gamma^w \setminus \mathcal{T}^w$ does not depend on the choice of isomorphism $D^w \otimes_F F_w \cong \mathbb{M}_2(F_w)$, since such isomorphisms differ by conjugation.

As follows from the analogue of Cherednik–Drinfeld uniformization for $X^R_{F_w}$, proven in this context by Hausberger [22], $X^R_{F_w}$ is a twisted Mumford curve: Denote by $\mathcal{O}_w^{(2)}$ the quadratic unramified extension of $\mathcal{O}_w$ and denote by $\mathbb{F}_w^{(2)}$ the residue field of $\mathcal{O}_w^{(2)}$. Then X^R_F has a semi-stable model $X^R_{\mathcal{O}_w^{(2)}}$ over $\mathcal{O}_w^{(2)}$ such that the irreducible components of $X^R_{\mathbb{F}_w^{(2)}}$ are projective lines without self-intersections, and the dual graph $G(X^R_{\mathcal{O}_w^{(2)}})$, as a graph with lengths, is isomorphic to $\Gamma^w \setminus \mathcal{T}^w$.

On the other hand, as is done in [25] for the quaternion algebras over $\mathbb{Q}$, the structure of $\Gamma^w \setminus \mathcal{T}^w$ can be related to the arithmetic to D^w: The number of vertices of $\Gamma^w \setminus \mathcal{T}^w$ is $2h^{\emptyset}(D^w)$, the number of edges is $h^w(D^w)$, each edge has length 1 or $q+1$, and the number of edges of length $q+1$ is $U^w(D^w)$ (the notation here is as in Section 6.1). Hence, using the formulae in Section 6.1, we get the following:

Proposition 6.2. *X^R_F has a semi-stable model $X^R_{\mathcal{O}_w^{(2)}}$ over $\mathcal{O}_w^{(2)}$ such that $X^R_{\mathbb{F}_w^{(2)}}$ is a union of projective lines without self-intersections. The number of vertices of the dual graph $G(X^R_{\mathcal{O}_w^{(2)}})$ is*

$$\frac{2}{q^2-1}\prod_{v\in R-w}(q_v-1) + 2^{\#R-1}\,\mathrm{Odd}(R-w)\frac{q}{q+1};$$

the number of edges is

$$\frac{(q_w+1)}{q^2-1}\prod_{v\in R-w}(q_v-1) + 2^{\#R-1}\,\mathrm{Odd}(R-w)\big(1-\mathrm{Odd}(w)\big)\frac{q}{q+1}.$$

The edges of $G(X^R_{\mathcal{O}_w^{(2)}})$ have length 1 or $q+1$. The number of edges of length $q+1$ is

$$2^{\#R-1}\,\mathrm{Odd}(R-w)\big(1-\mathrm{Odd}(w)\big).$$

This proposition has an interesting corollary:

Corollary 6.3. *Let $g(R)$ be the genus of X^R_F. Then*

$$g(R) = 1 + \frac{1}{q^2-1}\prod_{v\in R}(q_v-1) - \frac{q}{q+1}2^{\#R-1}\,\mathrm{Odd}(R).$$

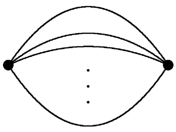

Fig. 4.

Proof. Let h_1 be the dimension of the first simplicial homology group of $G(X^R_{\mathcal{O}_w^{(2)}})$ with $\mathbb{Q}$-coefficients. Let V, E be the number of vertices and edges of this graph, respectively. By Euler's formula, $h_1 = E - V + 1$. Proposition 6.2 gives formulae for V and E from which it is easy to see that h_1 is given by the above expression. Since the irreducible components of $X^R_{\mathbb{F}_w^{(2)}}$ are projective lines, it is not hard to show that h_1 is the arithmetic genus of $X^R_{\mathbb{F}_w^{(2)}}$; cf. [21, p. 298]. On the other hand, $X^R_{\mathcal{O}_w^{(2)}}$ is flat over $\mathcal{O}_w^{(2)}$, so the genus $g(R)$ of its generic fiber is equal to the arithmetic genus of the special fiber; see [21, p. 263]. (Note that the special role of w in the formulae for V and E disappears in $g(R)$, as expected. This formula for $g(R)$ was obtained in [30] by a different argument.) $\square$

Theorem 6.4. *Let Φ'_v denote the group of connected components of J^{xy} at $v \in |F|$.*

(i) $\Phi'_x \cong \mathbb{Z}/(q+1)\mathbb{Z}$;
(ii) $\Phi'_y \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$.

Proof. In general, the information supplied by Proposition 6.2 is not sufficient for determining the graph $G(X^R_{\mathcal{O}_w^{(2)}})$ uniquely. Nevertheless, in the case when $R = \{x, y\}$ Proposition 6.2 does uniquely determine $G(X^R_{\mathcal{O}_w^{(2)}})$: $G(X^{xy}_{\mathcal{O}_x^{(2)}})$ is a graph without loops, which has 2 vertices, $q+1$ edges, and all edges have length 1. Similarly, $G(X^{xy}_{\mathcal{O}_y^{(2)}})$ is a graph without loops, which has 2 vertices, $q+1$ edges, two of the edges have length $q+1$ and all others have length 1. Hence, in both cases, the dual graph is the graph with two vertices and $q+1$ edges connecting them, cf. Fig. 4.

Now Theorem 4.1 can be used to conclude that the component groups are as stated. $\square$

6.3. Component group at ∞

Here we again rely on the existence of analytic uniformization. Let Λ be a maximal A-order in D. Let

$$\Gamma^\infty := \Lambda^\times.$$

Since D splits at ∞, by fixing an isomorphism $D \otimes F_\infty \cong \mathbb{M}_2(F_\infty)$, we get an embedding $\Gamma^\infty \hookrightarrow \mathrm{GL}_2(F_\infty)$. The group Γ^∞ is a discrete, cocompact subgroup of $\mathrm{GL}_2(F_\infty)$, well defined up to conjugation. Let $\mathcal{T}^\infty$ be the Bruhat–Tits tree of $\mathrm{PGL}_2(F_\infty)$. The group Γ^∞ acts on $\mathcal{T}^\infty$ without inversions, so the quotient $\Gamma^\infty \setminus \mathcal{T}^\infty$ is a finite graph without loops which we make into a graph with lengths by assigning to an edge e of $\Gamma^\infty \setminus \mathcal{T}^\infty$ the length $\#(\mathrm{Stab}_{\Gamma^\infty}(\tilde{e})/\mathbb{F}_q^\times)$, where $\tilde{e}$ is a preimage of e in $\mathcal{T}^\infty$. By a theorem of Blum and Stuhler [1, Thm. 4.4.11],

$$\left(X^R_{F_\infty}\right)^{\mathrm{an}} \cong \Gamma^\infty \setminus \Omega.$$

From this one deduces that X^R_F has a semi-stable model $X^R_{\mathcal{O}_\infty}$ over $\mathcal{O}_\infty$ such that the dual graph of $X^R_{\mathcal{O}_\infty}$, as a graph with lengths, is isomorphic to $\Gamma^\infty \setminus \mathcal{T}^\infty$, cf. [25]. The structure of $\Gamma^\infty \setminus \mathcal{T}^\infty$ can be related to the arithmetic of D; see [32].

Proposition 6.5. *X_F^R has a semi-stable model $X_{\mathcal{O}_\infty}^R$ over $\mathcal{O}_\infty$ such that the special fiber $X_{\mathbb{F}_\infty}^R$ is a union of projective lines without self-intersections. The number of vertices of the dual graph $G(X_{\mathcal{O}_\infty}^R)$ is*

$$\frac{2}{q-1}\big(g(R)-1\big)+\frac{q}{q-1}2^{\#R-1}\,\mathrm{Odd}(R);$$

the number of edges is

$$\frac{q+1}{q-1}\big(g(R)-1\big)+\frac{q}{q-1}2^{\#R-1}\,\mathrm{Odd}(R).$$

All edges have length 1.

Proof. See Proposition 5.2 and Theorem 5.5 in [32]. □

Theorem 6.6. $\Phi'_\infty \cong \mathbb{Z}/(q+1)\mathbb{Z}$.

Proof. Applying Proposition 6.5 in the case $R=\{x,y\}$, one easily concludes that X_F^{xy} has a semi-stable model over $\mathcal{O}_\infty$ whose dual graph looks like Fig. 4: it has 2 vertices, $q+1$ edges, and all edges have length 1. The structure of Φ'_∞ now follows from Theorem 4.1. □

7. Jacquet–Langlands isogeny

Let D and R be as in Section 2.2. Let $X:=X_F^R$, $X':=X_0(R)_F$, $J:=J^R$, $J':=J_0(R)$. Fix a separable closure F^{sep} of F and let $G_F:=\mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let p be the characteristic of F and fix a prime $\ell\neq p$. Denote by $V_\ell(J)$ the Tate vector space of J; this is a $\mathbb{Q}_\ell$-vector space of dimension $2g(R)$ naturally equipped with a continuous action of G_F. Let $V_\ell(J)^*$ be the linear dual of $V_\ell(J)$.

Theorem 7.1. *There is a surjective homomorphism $J'\to J$ defined over F.*

Proof. Let $\mathbb{A}=\prod'_{v\in|F|}F_v$ denote the Adele ring of F and let $\mathbb{A}^\infty=\prod'_{v\in|F|-\infty}F_v$, so $\mathbb{A}=\mathbb{A}^\infty\times F_\infty$. Fix a uniformizer π_∞ at ∞. Let $\mathcal{A}(D^\times(F)\setminus D^\times(\mathbb{A})/\pi_\infty^{\mathbb{Z}})$ be the space of $\overline{\mathbb{Q}}_\ell$-valued locally constant functions on $D^\times(\mathbb{A})/\pi_\infty^{\mathbb{Z}}$ which are invariant under the action of $D^\times(F)$ on the left. This space is equipped with the right regular representation of $D^\times(\mathbb{A})/\pi_\infty^{\mathbb{Z}}$. Since D is a division algebra, the coset space $D^\times(F)\setminus D^\times(\mathbb{A})/\pi_\infty^{\mathbb{Z}}$ is compact and decomposes as a sum of irreducible admissible representations Π with finite multiplicities $m(\Pi)>0$, cf. [26, §13]:

$$\mathcal{A}_D:=\mathcal{A}\big(D^\times(F)\setminus D^\times(\mathbb{A})/\pi_\infty^{\mathbb{Z}}\big)=\bigoplus_{\Pi}m(\Pi)\cdot\Pi. \tag{7.1}$$

Moreover, as follows from the Jacquet–Langlands correspondence and the multiplicity-one theorem for automorphic cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$, the multiplicities $m(\Pi)$ are all equal to 1; see [18, Thm. 10.10]. The representations appearing in the sum (7.1) are called *automorphic*. Each automorphic representation Π decomposes as a restricted tensor product $\Pi=\bigotimes_{v\in|F|}\Pi_v$ of admissible irreducible representations of $D^\times(F_v)$. We denote $\Pi^\infty=\bigotimes_{v\neq\infty}\Pi_v$, so $\Pi=\Pi^\infty\otimes\Pi_\infty$. If Π is finite dimensional, then it is of the form $\Pi=\chi\circ\mathrm{Nr}$, where χ is a Hecke character of $\mathbb{A}^\times$ and Nr is the reduced norm on $D^\times$, cf. [26, Lem. 14.8]. If Π is infinite dimensional, then Π_v is infinite dimensional for every $v\notin R$.

Let ψ_v be a character of $F_v^\times$. Denote by $\mathrm{Sp}_v\otimes\psi_v$ the unique irreducible quotient of the induced representation

$$\mathrm{Ind}_B^{\mathrm{GL}_2}\big(|\cdot|_v^{-\frac{1}{2}}\psi_v\oplus|\cdot|_v^{\frac{1}{2}}\psi_v\big),$$

where B is the subgroup of upper-triangular matrices in GL_2. The representation $\mathrm{Sp}_v \otimes \psi_v$ is called the *special representation* of $\mathrm{GL}_2(F_v)$ twisted by ψ_v. If $\psi_v = 1$, then we simply write Sp_v.

For $v \in R$, let $\mathcal{D}_v$ be the maximal order in $D(F_v)$. Let

$$\mathcal{K} := \prod_{v\in R} \mathcal{D}_v^\times \times \prod_{v\in|F|-R-\infty} \mathrm{GL}_2(\mathcal{O}_v) \subset D^\times\big(\mathbb{A}^\infty\big).$$

Taking the $\mathcal{K}$-invariants in Theorems 14.9 and 14.12 in [26], we get an isomorphism of G_F-modules

$$V_\ell(J)^* \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell = H^1_{\text{ét}}\big(X \otimes_F F^{\text{sep}}, \overline{\mathbb{Q}}_\ell\big) = \bigoplus_{\substack{\Pi\in\mathcal{A}_D\\ \Pi_\infty\cong\mathrm{Sp}_\infty}} \big(\Pi^\infty\big)^{\mathcal{K}} \otimes \sigma(\Pi), \tag{7.2}$$

where $\sigma(\Pi)$ is a 2-dimensional irreducible representation of G_F over $\overline{\mathbb{Q}}_\ell$ with the following property: If $(\Pi^\infty)^{\mathcal{K}} \neq 0$, then for all $v \in |F| - R - \infty$, $\sigma(\Pi)$ is unramified at v and there is an equality of L-functions

$$L\left(s - \frac{1}{2}, \Pi_v\right) = L\big(s, \sigma(\Pi)_v\big);$$

here $\sigma(\Pi)_v$ denotes the restriction of $\sigma(\Pi)$ to a decomposition group at v. This uniquely determines $\sigma(\Pi)$ by the Chebotarev density theorem [39, Ch. I, pp. 8–11]. Next, we claim that the dimension of $(\Pi^\infty)^{\mathcal{K}}$ is at most one. Indeed, if $v \in |F| - R - \infty$, then $\Pi_v^{\mathrm{GL}_2(\mathcal{O}_v)}$ is at most one-dimensional by [3, Thm. 4.6.2]. On the other hand, note that $\mathcal{D}_v^\times$ is normal in $D^\times(F_v)$ and $D^\times(F_v)/\mathcal{D}_v^\times \cong \mathbb{Z}$ for $v \in R$. Hence $\Pi_v^{\mathcal{D}_v^\times} \neq 0$ implies $\Pi_v = \psi_v \circ \mathrm{Nr}$ for some unramified character of $F_v^\times$ (ψ_v is unramified because the reduced norm maps $\mathcal{D}_v^\times$ surjectively onto $\mathcal{O}_v^\times$).

Let $\mathcal{I}_v$ be the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_v)$, i.e., the subgroup of matrices which maps to $B(\mathbb{F}_v)$ under the reduction map $\mathrm{GL}_2(\mathcal{O}_v) \to \mathrm{GL}_2(\mathbb{F}_v)$. Let

$$\mathcal{I} = \prod_{v\in R} \mathcal{I}_v \times \prod_{v\in|F|-R-\infty} \mathrm{GL}_2(\mathcal{O}_v) \subset \mathrm{GL}_2\big(\mathbb{A}^\infty\big).$$

Let $\mathcal{A}_0 := \mathcal{A}_0(\mathrm{GL}_2(F) \setminus \mathrm{GL}_2(\mathbb{A}))$ be the space of $\overline{\mathbb{Q}}_\ell$-valued cusp forms on $\mathrm{GL}_2(\mathbb{A})$; see [17, §4] or [3, §3.3] for the definition. Taking the $\mathcal{I}$-invariants in Theorem 2 of [8], we get an isomorphism of G_F-modules

$$V_\ell\big(J'\big)^* \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell = H^1_{\text{ét}}\big(X' \otimes_F F^{\text{sep}}, \overline{\mathbb{Q}}_\ell\big) = \bigoplus_{\substack{\Pi\in\mathcal{A}_0\\ \Pi_\infty\cong\mathrm{Sp}_\infty}} \big(\Pi^\infty\big)^{\mathcal{I}} \otimes \rho(\Pi), \tag{7.3}$$

where $\rho(\Pi)$ is 2-dimensional irreducible representation of G_F over $\overline{\mathbb{Q}}_\ell$ with the following property: If $(\Pi^\infty)^{\mathcal{I}} \neq 0$, then for all $v \in |F| - R - \infty$, $\rho(\Pi)$ is unramified at v and

$$L\left(s - \frac{1}{2}, \Pi_v\right) = L\big(s, \rho(\Pi)_v\big).$$

In this case, $(\Pi^\infty)^{\mathcal{I}}$ is finite dimensional, but its dimension might be larger than one (due to the existence of old forms).

The global Jacquet–Langlands correspondence [24, Ch. III] associates to each infinite dimensional automorphic representation Π of $D^\times(\mathbb{A})$ a cuspidal representation $\Pi' = \mathrm{JL}(\Pi)$ of $\mathrm{GL}_2(\mathbb{A})$ with the following properties:

(1) if $v \notin R$ then $\Pi_v \cong \Pi'_v$;
(2) if $v \in R$ and $\Pi_v \cong \psi_v \circ \mathrm{Nr}$ for a character ψ of $F_v^\times$, then

$$\Pi'_v \cong \mathrm{Sp}_v \otimes \psi_v.$$

As we observed above, for $\Pi \in \mathcal{A}_D$ such that $(\Pi^\infty)^{\mathcal{K}} \neq 0$, the characters ψ_v at the places in R are unramified. Thus, for $v \in R$, Π'_v is a twist of Sp_v by an unramified character. On the other hand, the representations of the form $\mathrm{Sp}_v \otimes \psi_v$, with ψ_v unramified, can be characterized by the property that they have a unique 1-dimensional $\mathcal{I}_v$-fixed subspace; see [4]. Hence if $(\Pi^\infty)^{\mathcal{K}} \neq 0$, then $((\Pi')^\infty)^{\mathcal{I}} \neq 0$.

Now using (7.2) and (7.3), one concludes that $V_\ell(J)$ is isomorphic with a quotient of $V_\ell(J')$ as a G_F-module. On the other hand, by a theorem of Zarhin (for $p > 2$) and Mori (for $p = 2$)

$$\mathrm{Hom}_F(J', J) \otimes \mathbb{Q}_\ell \cong \mathrm{Hom}_{G_F}\big(V_\ell(J'), V_\ell(J)\big). \tag{7.4}$$

Thus, there is a surjective homomorphism $J' \to J$ defined over F. □

Corollary 7.2. *$J_0(xy)$ and J^{xy} are isogenous over F.*

Proof. Since $\dim(J^{xy}) = q = \dim(J_0(xy))$, the claim follows from Theorem 7.1. □

Conjecture 7.3. *There exists an isogeny $J_0(xy) \to J^{xy}$ whose kernel is $\mathcal{C}_0$.*

As an initial evidence for the conjecture, note that $J_0(xy)/\mathcal{C}_0$ has component groups at x, y, ∞ of the same order as those of J^{xy}. This follows from Theorem 4.3, Corollary 5.8, and Table 1 in the introduction. We will show below that Conjecture 7.3 is true for $q = 2$.

Remark 7.4. The statement of Theorem 7.1 can be refined. The abelian variety J has toric reduction at every $v \in R$, so it is isogenous to an abelian subvariety of J' having the same reduction property. The new subvariety of J', J'^{new}, defined as in the case of classical modular Jacobians (cf. [35], [13, p. 248]), is the abelian subvariety of J' of maximal dimension having toric reduction at every $v \in R$. Hence J is isogenous to a subvariety of J'^{new}. By computing the dimension of J'^{new}, one concludes that J and J'^{new} are isogenous over F.

Remark 7.5. There is just one other case, besides the case which is the focus of this paper, when J and J' are actually isogenous. As one easily shows by comparing the genera of modular curves X^R and $X_0(R)$, the genera of these curves are equal if and only if $R = \{x, y\}$ and $\{\deg(x), \deg(y)\} = \{1, 1\}, \{1, 2\}, \{2, 2\}$. Assume $\deg(x) = \deg(y) = 2$. Then the genus of both X^{xy} and $X_0(xy)$ is q^2, but neither of these curves is hyperelliptic. The curve $X_0(xy)$ again has 4 cusps which can be represented as in Section 3. Calculations similar to those we have carried out in earlier sections lead to the following result:

(1) The cuspidal divisor group $\mathcal{C}$ is generated by c_0 and c_x. The order of c_0 is $q^2 + 1$. The order of c_x is divisible by $q^2 + 1$ and divides $q^4 - 1$. The order of c_y is divisible by $q^2 + 1$ and divides $q^4 - 1$.
(2) $\Phi_x \cong \Phi'_x \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$.
(3) $\Phi_y \cong \Phi'_y \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$.
(4) The canonical map $\phi_x : \mathcal{C} \to \Phi_x$ is surjective, and

$$\phi_x(c_0) = z, \qquad \phi_x(c_x) = 0, \qquad \phi_x(c_y) = z.$$

(5) The canonical map $\phi_y : \mathcal{C} \to \Phi_y$ is surjective, and

$$\phi_x(c_0) = z, \qquad \phi_y(c_x) = z, \qquad \phi_y(c_y) = 0.$$

The fact that $X_0(xy)$ is not hyperelliptic complicates the calculation of $\mathcal{C}$: just the relations between the cuspidal divisors arising from the Drinfeld discriminant function are not sufficient for pinning down the orders of c_x and c_y, cf. (3.3). Next, the calculations required for determining Φ_∞, Φ'_∞, and ϕ_∞ appear to be much more complicated than those in Sections 5.2 and 6.3. Nevertheless, based on the facts that we are able to prove, and in analogy with the case $\deg(x) = 1$, $\deg(y) = 2$, we make the following prediction: The cuspidal divisor group $\mathcal{C} \cong (\mathbb{Z}/(q^2+1)\mathbb{Z})^2$ is the direct sum of the cyclic subgroups generated by c_x and c_y both of which have order q^2+1, and there is an isogeny $J_0(xy) \to J^{xy}$ whose kernel is $\mathcal{C}$.

Definition 7.6. It is known that every elliptic curve E over F with conductor $\mathfrak{n}_E = \mathfrak{n}\cdot\infty$, $\mathfrak{n} \lhd A$, and split multiplicative reduction at ∞ is isogenous to a subvariety of $J_0(\mathfrak{n})$; see [17]. This follows from (7.3), (7.4), and the fact [6, p. 577] that the representation $\rho_E : G_F \to \mathrm{Aut}(V_\ell(E)^*)$ is automorphic (i.e., $\rho_E = \rho(\Pi)$ for some $\Pi \in \mathcal{A}_0$). The multiplicity-one theorem can be used to show that in the F-isogeny class of E there exists a unique curve E' which is isomorphic to a one-dimensional abelian subvariety of $J_0(\mathfrak{n})$, thus maps "optimally" into $J_0(\mathfrak{n})$. We call E' the $J_0(\mathfrak{n})$-*optimal* curve. Theorem 7.1 and Remark 7.4 imply that E with square-free conductor $R\cdot\infty$ and split multiplicative reduction at ∞ is also isogenous to a subvariety of J^R. Moreover, in the F-isogeny class of E there is a unique elliptic curve E'' which is isomorphic to a one-dimensional abelian subvariety of J^R. We call E'' the J^R-*optimal* curve.

Notation 7.7. Let E be an elliptic curve over F given by a Weierstrass equation

$$E\colon\ Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

Let $E^{(p)}$ be the elliptic curve given by the equation

$$E^{(p)}\colon\ Y^2 + a_1^pXY + a_3^pY = X^3 + a_2^pX^2 + a_4^pX + a_6^p.$$

There is a Frobenius morphism $\mathrm{Frob}_p : E \to E^{(p)}$ which maps a point (x_0, y_0) on E to the point (x_0^p, y_0^p) on $E^{(p)}$. It is clear that the j-invariants of these elliptic curves are related by the equation $j(E^{(p)}) = j(E)^p$. If E has semi-stable reduction at $v \in |F|$, then $\Phi_{E,v} \cong \mathbb{Z}/n\mathbb{Z}$, where $\Phi_{E,v}$ denotes the component group of E at v and $n = -\mathrm{ord}_v(j(E)) \geqslant 1$. In this case, $\Phi_{E^{(p)},v} \cong \mathbb{Z}/pn\mathbb{Z}$.

Definition 7.8. An elliptic curve E over F with j-invariant $j(E) \notin \mathbb{F}_q$ is said to be *Frobenius minimal* if it is not isomorphic to $\widetilde{E}^{(p)}$ for some other elliptic curve $\widetilde{E}$ over F. It is easy to check that this is equivalent to $j(E) \notin F^p$, cf. [38].

For q even, Schweizer has completely classified the elliptic curves over F having conductor of degree 4 in terms of explicit Weierstrass equations; see [37]. We are particularly interested in those curves which have conductor $xy\infty$ and split multiplicative reduction at ∞.

Theorem 7.9. *Assume $q = 2^s$. Elliptic curves over F with conductor $xy\infty$ exist only if there exists an $\mathbb{F}_q$-automorphism of F that transforms the conductor into $(T+1)(T^2+T+1)\infty$. In particular, s must be odd.*

If s is odd, then there exists two isogeny classes of elliptic curves over F with conductor $(T+1)(T^2+T+1)\infty$ and split multiplicative reduction at ∞. The Frobenius minimal curves in each isogeny class are listed in Tables 2 *and* 3*; the last three columns in the tables give the orders of the component groups $\Phi_{E,v}$ of the corresponding curve E at $v = x, y, \infty$.*

Proof. Theorem 4.1 in [37]. □

Table 2
Isogeny class I.

	Equation	x	y	∞
E_1	$Y^2+TXY+Y=X^3+T^3+1$	3	3	3
E_1'	$Y^2+TXY+Y=X^3+T^2(T^3+1)$	9	1	1
E_1''	$Y^2+TXY+Y=X^3$	1	1	9

Table 3
Isogeny class II.

	Equation	x	y	∞
E_2	$Y^2+TXY+Y=X^3+X^2+T$	5	1	5
E_2'	$Y^2+TXY+Y=X^3+X^2+T^5+T^2+T$	1	5	1

Next, [37, Prop. 3.5] describes explicitly the isogenies between the curves in classes I and II: There is an isomorphism of étale group-schemes over F

$$E_1[3]\cong H_1\oplus H_2,$$

where $H_1\cong\mathbb{Z}/3\mathbb{Z}$ and $H_2\cong\mu_3$. The subgroup-scheme H_1 is generated by $(T+1,1)$ and H_2 is generated by (T^2, sT^3+s^2), where s is a third root of unity. Then $E_1/H_1\cong E_1'$ and $E_1/H_2\cong E_1''$. (It is well known that an elliptic curve over F with conductor of degree 4 has rank 0, so in fact $E_1(F)=H_1\cong\mathbb{Z}/3\mathbb{Z}$.) Similarly, the subgroup-scheme H_3 of E_2 generated by $(1,1)$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$, $E_2/H_3\cong E_2'$, and $E_2(F)=H_3\cong\mathbb{Z}/5\mathbb{Z}$.

Proposition 7.10. *Assume $q=2^s$ and s is odd.*

(i) *E_1 and E_2 are the $J_0(xy)$-optimal curves in the isogeny classes I and II.*
(ii) *E_2' is the J^{xy}-optimal curve in the isogeny class II.*
(iii) *If Conjecture 7.3 is true, then E_1 is the J^{xy}-optimal curve in the isogeny class I.*

Proof. (i) There is a method due to Gekeler and Reversat [12, Cor. 3.19] which can be used to determine $\#\Phi_{E,\infty}$ of the $J_0(\mathfrak{n})$-optimal curve in a given isogeny class. This method is based on the study of the action of Hecke algebra on $H_1(\Gamma_0(\mathfrak{n})\setminus\mathcal{T},\mathbb{Z})$. For $\deg(\mathfrak{n})=3$ the Gekeler–Reversat method can be further refined [38, Cor. 1.2]. Applying this method for $\mathfrak{n}=xy$, one obtains $\#\Phi_{E,\infty}=3$ (resp. $\#\Phi_{E,\infty}=5$) for the $J_0(xy)$-optimal elliptic curve E in the isogeny class I (resp. II). Since there is a unique curve with this property in each isogeny class, we conclude that E_1 and E_2 are the $J_0(xy)$-optimal elliptic curves. (For $q=2$, this is already contained in [12, Ex. 4.4].)

(ii) Assume q is arbitrary. Let E be an elliptic curve over F which embeds into J^{xy}. Since J^{xy} has split toric reduction at ∞, [29, Cor. 2.4] implies that the kernel of the natural homomorphism

$$\Phi_{E,\infty}\to\Phi'_\infty\cong\mathbb{Z}/(q+1)\mathbb{Z}$$

is a subgroup of $\mathbb{Z}/(q_\infty-1)\mathbb{Z}$. Hence $\#\Phi_{E,\infty}$ divides (q^2-1). First, this implies that $\#\Phi_{E,\infty}$ is coprime to p, so E must be Frobenius minimal in its isogeny class. Second, if $q=2^s$ and s is odd, then 5 does not divide (q^2-1), so E_2 is not J^{xy}-optimal. This leaves E_2' as the only possible J^{xy}-optimal curve in the isogeny class II.

(iii) Let E be the J^{xy}-optimal curve in the isogeny class I. By the discussion in (ii), this curve is one of the curves in Table 2. Suppose there is an isogeny $\varphi: J_0(xy)\to J^{xy}$ whose kernel is $\mathcal{C}_0$. Restricting φ to $E_1\hookrightarrow J_0(xy)$, we get an isogeny $\varphi': E_1\to E$ defined over F whose kernel is a subgroup of $\mathcal{C}_0\cong\mathbb{Z}/(q^2+1)\mathbb{Z}$. Note that 3 does not divide q^2+1. On the other hand, any isogeny from E_1 to E_1' or E_1'' must have kernel whose order is divisible by 3. This implies that φ' has trivial kernel, so $E=E_1$. □

Remark 7.11. In the notation of the proof of Proposition 7.10, consider the restriction of φ to $E_2 \hookrightarrow J_0(xy)$. By part (ii) of the proposition, there results an isogeny $\varphi'' : E_2 \to E_2'$ whose kernel is a subgroup of $\mathbb{Z}/(q^2+1)\mathbb{Z}$. Since 5 divides q^2+1 when s is odd, part (ii) of Proposition 7.10 is compatible with Conjecture 7.3.

Theorem 7.12. *Conjecture* 7.3 *is true for* $q=2$.

Proof. Assume $q=2$. By Proposition 7.10, E_1 and E_2 are the $J_0(xy)$-optimal curves. Since the genus of $X_0(xy)$ is 2, it is hyperelliptic (this is true for general q by Schweizer's theorem which we used in Section 3). The genus being 2 also implies that a quotient of $X_0(xy)$ by an involution has genus 0 or 1. The Atkin–Lehner involutions form a subgroup in $\mathrm{Aut}(X_0(xy))$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Since the hyperelliptic involution is unique, each E_1 and E_2 can be obtained as a quotient of $X_0(xy)$ under the action of an Atkin–Lehner involution. Thus, there are degree-2 morphisms $\pi_i : X_0(xy) \to E_i$, $i=1,2$. In fact, one obtains the closed immersions $\pi_i^* : E_i \to J_0(xy)$ from these morphisms by Picard functoriality. Let $\widehat{\pi_i^*} : J_0(xy) \to E_i$ be the dual morphism. It is easy to show that the composition $\widehat{\pi_i^*} \circ \pi_i^* : E_i \to E_i$ is the isogeny given by multiplication by $2 = \deg(\pi_i)$. This implies that E_1 and E_2 intersect in $J_0(xy)$ in their common subgroup-scheme of 2-division points $S := \pi_1^*(E_1)[2] = \pi_2^*(E_2)[2]$, so

$$J_0(xy)(F) = H_1 \oplus H_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} = \mathcal{C}.$$

Let $\psi : J_0(xy) \to E_1 \times E_2$ be the isogeny with kernel S. Note that S is characterized by the non-split exact sequence of group-schemes over F:

$$0 \to \mu_2 \to S \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

By Proposition 7.10, E_2' is the J^{xy}-optimal elliptic curve in the isogeny class II. Let E be the J^{xy}-optimal elliptic curves in class I. From the proof of Proposition 7.10, we know that E is Frobenius minimal, so it is one of the curves listed in Table 2. There are also Atkin–Lehner involutions acting on X^{xy} and they form a subgroup in $\mathrm{Aut}(X^{xy})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$; see [31]. Now exactly the same argument as above implies that E and E_2' intersect in J^{xy} along their common subgroup-scheme of 2-division points $S' \cong S$. Let $\nu : J^{xy} \to E \times E_2'$ be the isogeny with kernel S'. Let $\hat{\nu} : E \times E_2' \to J^{xy}$ be the dual isogeny.

The following argument is motivated by [19]. Consider the composition

$$\phi : J_0(xy) \xrightarrow{\psi} E_1 \times E_2 \xrightarrow{\phi_1 \times \phi_2} E \times E_2' \xrightarrow{\hat{\nu}} J^{xy},$$

where ϕ_1 is either the identity morphism or has kernel H_1, H_2, and ϕ_2 has kernel H_3. Since $\phi_1 \times \phi_2$ has odd degree, this morphism maps the kernel of $\hat{\psi}$ to the kernel of $\hat{\nu}$. Indeed, both are the "diagonal" subgroups isomorphic to S in the corresponding group-schemes $(E_1 \times E_2)[2]$ and $(E \times E_2')[2]$. More precisely, $\mathcal{H} := \ker(\hat{\psi})$ is uniquely characterized as the subgroup-scheme of $\mathcal{G} := (E_1 \times E_2)[2]$ having the following properties: $\mathcal{H}^0$ is the image of the diagonal morphism $\mu_2 \to \mu_2 \times \mu_2 = \mathcal{G}^0$ and the image of $\mathcal{H}$ in $\mathcal{G}^{\mathrm{et}}$ under the natural morphism $\mathcal{G} \to \mathcal{G}^{\mathrm{et}}$ is the image of the diagonal morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A similar description applies to $\ker(\hat{\nu}) \subset (E \times E_2')[2]$. Thus, there is an isogeny $\phi' : J_0(xy) \to J^{xy}$ such that $\phi = \phi'[2]$ and $\ker(\phi') \cong \ker(\phi_1 \times \phi_2)$. We conclude that J^{xy} is isomorphic to the quotient of $J_0(xy)$ by one of the following subgroups

$$H_3, \qquad H_1 \oplus H_3, \qquad H_2 \oplus H_3.$$

Now note that H_1 and H_3 under the specialization map ϕ_∞ inject into Φ_∞, but H_2 maps to 0 (indeed, $H_2 \cong \mu_3$ has non-trivial action by $\mathrm{Gal}(\overline{\mathbb{F}}_\infty/\mathbb{F}_\infty)$ whereas Φ_∞ is constant). Hence Theorem 4.3 implies that the quotients of $J_0(xy)$ by the subgroups listed above have component groups at ∞ of orders 3, 1, 9, respectively. Since $\Phi'_\infty \cong \mathbb{Z}/3\mathbb{Z}$, we see that J^{xy} is the quotient of $J_0(xy)$ by H_3 which is $\mathcal{C}_0$ in this case. $\square$

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