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Indecomposable Decompositions of Pure-Injective Objects and the Pure-Semisimplicity

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We give a criterion for the existence of an indecomposable decomposition of pure-injective objects in a locally finitely presented Grothendieck category $\mathscr{A}$ (Theorem 2.5). As a consequence we get Theorem 3.2, asserting that an associative unitary ring R is right pure-semisimple if and only if every right R-module is a direct sum of modules that are pure-injective or countably generated. Some open problems are formulated in the paper. © 2001 Academic Press

1. INTRODUCTION

Let R be an associative ring with identity. We denote by $J(R)$ the Jacobson radical of R, by $\mathrm{Mod}(R)$ the category of right R-modules, and by

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mod(R) the full subcategory of Mod(R) formed by finitely generated right R-modules.

It was proved in [9] that a finitely presented pure-injective right R-module M has an indecomposable decomposition if and only if every pure submodule of M is pure-projective. This result has been partially extended in [5] by proving that a pure-projective right R-module M, which is pure-injective, has an indecomposable decomposition if and only if every pure submodule of M is pure-projective. Furthermore, it is shown in [5] that a pure-injective R-module M has an indecomposable decomposition if every pure submodule of M is a direct sum of countably generated modules. However, there are no general criteria for the existence of an indecomposable decomposition of an arbitrary pure-injective R-module (see [5, Remark, p. 3719]).

The main goal of this paper is to give a criterion of this kind for any pure-injective module, and more generally, for any pure-injective object M in a locally finitely presented Grothendieck category $\mathscr{A}$. Our main result asserts that a pure-injective object M of $\mathscr{A}$ has an indecomposable decomposition if and only if every directly pure subobject of M (in the sense of Definition 2.1 below) is a direct sum of objects that are pure-injective or countably generated (Theorem 2.4). As a consequence we get the main indecomposable decomposition results proved in [5, 9].

By applying Theorem 2.4 we show in Section 3 that a right R-module M is $\sum$-pure-injective if and only if every pure (or perfectly pure) submodule N of a pure-injective envelope of a direct sum of countably many copies of M is a direct sum of modules that are pure-injective or countably generated (see Theorem 3.1).

In Section 3 we apply our main results to the study of right pure-semisimple rings [17]. We show in Theorem 3.2 that a ring R is right pure-semisimple if and only if every right R-module is a direct sum of modules that are pure-injective or countably generated. We remark that if R is a ring for which every right R-module is a direct sum of modules that are pure-injective or pure-projective, then every indecomposable right R-module is pure-injective or pure-projective. Thus, Theorem 3.2 sheds a light on the following open question posed by Simson in [20, Problem 3.2]: *"Is a semiperfect ring R right artinian or right pure semisimple if every indecomposable right R-module is pure-injective or pure-projective?"*

This question is also discussed in Section 4, and a partial answer is given in Theorem 4.2.

2. THE MAIN DECOMPOSITION RESULTS

We recall that a Grothendieck category $\mathscr{A}$ is said to be locally finitely presented if there exists a set of finitely presented generators in $\mathscr{A}$ (see [14]).

$\mathscr{A}$ has enough injective objects and every object M of $\mathscr{A}$ admits an essential embedding in an injective object $E(M)$, called an **injective envelope** of M. The object $E(M)$ is uniquely determined by M up to isomorphism. It is well known that indecomposable injective objects of $\mathscr{A}$ have local endomorphism rings and the Azumaya's decomposition theorem remains valid for $\mathscr{A}$ (see [1, 14]).

Following [11], a concept of an algebraically compact object of $\mathscr{A}$ was introduced in [16, Sect. 4]. It was proved there that the algebraically compactness and the pure-injectivity in $\mathscr{A}$ coincide (see also [11]), and every object M of $\mathscr{A}$ admits a pure-essential embedding into a pure-injective object $E_{pure}(M)$. The object $E_{pure}(M)$ is uniquely determined by M up to isomorphism and is called a **pure-injective envelope** of M.

The following simple lemma is a consequence of [13, Theorem 2.17], but we are including a direct proof for the sake of completeness.

LEMMA 2.1. *Let $\mathscr{A}$ be a locally finitely presented Grothendieck category. Let E be a non-zero injective object of $\mathscr{A}$ such that any subobject E' of E is injective if E' is a direct sum of injective objects. Then E is a direct sum of indecomposable objects.*

Proof. First we claim that the injective envelope of any finitely generated subobject of E is a (finite) direct sum of indecomposable objects. Assume to the contrary, that M is a non-zero finitely generated subobject of E such that its injective envelope $E(M)$ is not a direct sum of finitely many indecomposable objects. It follows that $E(M)$ contains an infinite direct sum $\oplus_{j=1}^{\infty} Q_j$ of non-zero injective objects. Applying our hypothesis, we deduce that $\oplus_{j=1}^{\infty} Q_j$ is injective and thus, a direct summand of $E(M)$. Say that $E(M) = (\oplus_{j=1}^{\infty} Q_j) \oplus Q'$. Since M is finitely generated, $E(M) = Q_1 \oplus \cdots \oplus Q_m \oplus Q'$ for some m, and we get a contradiction.

Let now Ω_E be the set consisting of families $\{E_i\}_{i\in I}$ of indecomposable injective subobjects E_i of E such that $E \supseteq \sum_{i\in I} E_i = \oplus_{i\in I} E_i$. We view Ω_E as a partially ordered set with respect to the inclusion. By our claim, the set Ω_E is not empty. It is easy to check that Ω_E is inductive. By Zorn's lemma, there exists a maximal family $\{E_i\}_{i\in I}$ in Ω_E. Since the object $\oplus_{i\in I} E_i$ is injective by our hypothesis, then $E = \oplus_{i\in I} E_i$, because otherwise the family $\{E_i\}_{i\in I}$ is not maximal (by applying the above claim). ∎

The following definition will be useful throughout this paper.

DEFINITION 2.2. Let B be a subobject of an object A in a locally finitely presented Grothendieck category $\mathscr{A}$. Then B is called a **perfectly pure subobject** of A if $B = \bigcup_{\beta} B_{\beta}$ is a directed union of its subobjects B_{β} such that the composed monomorphism $B_{\beta} \subseteq B \subseteq A$ splits for all indices β.

It is easy to see that every perfectly pure subobject of A is a pure subobject of A. The following easy lemma will be useful later on.

LEMMA 2.3. *Let $\mathscr{A}$ be a locally finitely presented Grothendieck category. Let $\oplus_{i\in I}E_i$ be a direct sum of injective objects of $\mathscr{A}$ that is not injective. Let us denote by $\pi_j : \oplus_{i\in I}E_i \to E_j$ the canonical projections. Then there exist a subobject N of a finitely generated object M and a morphism $f : N \to \oplus_I E_i$ such that the set $\{j \in I \mid \pi_j \circ f \neq 0\}$ is infinite.*

Proof. By Baer's Injectivity Criterium for Grothendieck Categories (see [21, Proposition V.2.9]), there must exist a finitely generated object M of $\mathscr{A}$ and a morphism f from a subobject N of M to $\oplus_{i\in I}E_i$ that cannot be extended to M. Since finite direct sums of copies of injective objects are injectives, this means that $\mathrm{Im}(f)$ is not contained in any finite direct subsum of $\oplus_{i\in I}E_i$. Thus, the set $\{j \in I \mid \pi_j \circ f \neq 0\}$ must be infinite. ∎

The following result is basic for the proof of our main theorem.

THEOREM 2.4. *Let E be a non-zero injective object of a locally finitely presented Grothendieck category $\mathscr{A}$. The following conditions are equivalent*:

(a) *E is a direct sum of indecomposable objects.*

(b) *Every non-zero perfectly pure subobject of E is a direct sum of indecomposable injective objects.*

(c) *Every non-zero perfectly pure subobject of E is a direct sum of objects that are injective or countably generated.*

Proof. (a) $\Rightarrow$ (b). Suppose that $E = \oplus_{s\in S}E_s$ is a direct sum of indecomposable objects E_s. It follows that the endomorphism ring $\mathrm{End}(E_s)$ of E_s is local for any $s \in S$.

Let L be a non-zero perfectly pure subobject of E. By definition, $L = \bigcup_\beta L_\beta$ is a directed union of its subobjects L_β such that the composed monomorphism $L_\beta \subseteq L \subseteq E$ splits for all indices β. We shall show that L is direct sum of indecomposable injective objects.

Let Ω_L be the set consisting of families $\{Q_j\}_{j\in J}$ of indecomposable injective subobjects Q_j of L such that $L \supseteq \sum_{j\in J} Q_j = \oplus_{j\in J}Q_j$ and each Q_j is a subobject of some L_β. We view Ω_L as a partially ordered set with respect to the inclusion. It is easy to see that Ω_L is an inductive set.

Now we show that the set Ω_L is non-empty. Since L is not zero and is a directed union of injective subobjects L_β, then there exists a non-zero finitely generated subobject X of L. This means that X embeds in L_β for some ordinal β and thus, the injective envelope $E(X)$ of X is also a subobject of L_β. By our assumption, $E(X)$ is a direct sum of (finitely many) indecomposable subobjects $Q_1, \dots, Q_r$ of $L_\beta \subseteq L$ and therefore the family $\{Q_1, \dots, Q_r\}$ belongs to Ω_L.

By Zorn's lemma, there exists a maximal element $\{Q_j\}_{j\in J}$ in Ω_L. We shall show that $L = \sum_{j\in J} Q_j = \oplus_{j\in J}Q_j$. By [1, Proposition 25.5] the decomposition $E = \oplus_{s\in S}E_s$ complements direct summands. Since $\oplus_{j\in J'}Q_j$ is injective for any finite subset J' of J then the composed monomorphism $\oplus_{j\in J}Q_j \subseteq L \subseteq E$ splits (see, e.g., [5, Theorem 3.4]). Consequently, $L = (\oplus_{j\in J}Q_j) \oplus L'$ for some subobject L' of L. We claim that L' is zero. If L' is not zero then L' contains a finitely generated subobject M. Its injective envelope $E(M)$ is a subobject of some L_γ. By assumption, the object $E(M)$ has an indecomposable direct summand Q' contained in L_γ. Since $Q' \cap (\oplus_{j\in J}Q_j) \subseteq E(M) \cap (\oplus_{j\in J}Q_j) = 0$ then the family $\{Q'\} \cup \{Q_j\}_{j\in J}$ belongs to Ω_L, contrary to the maximality of $\{Q_j\}_{j\in J}$. Consequently L' is zero and $L = \oplus_{j\in J}Q_j$.

(b) ⇒ (c). Obvious.

(c) ⇒ (a). Suppose that (c) holds but E is not a direct sum of indecomposable objects. By Lemma 2.1, there exists a non-injective subobject E' of E such that $E' = \oplus_{i\in I}E_i$ is a direct sum of non-zero injective subobjects E_i of E, for each $i \in I$, where I is an infinite set. By Lemma 2.3, there exists a subobject N of a finitely generated object M and a morphism $f : N \to \oplus_{i\in I}E_i$ such that the set $I' = \{j \in I \mid \pi_j \circ f \neq 0\}$ is infinite, where $\pi_j : \oplus_{i\in I}E_i \to E_j$ denotes the canonical projection. Let J be an infinite countable subset of I'.

Choose a finitely generated subobject X_j of E_j with the property that $0 \neq X_j \subseteq \mathrm{Im}(N_j \cdot f)$ for any $j \in J$. Denote by Q_j an injective envelope of X_j in E_j. Since the set J is countable and each X_j is finitely generated then $\oplus_{j\in J}X_j$ is countably generated, and obviously it is essential in $\oplus_{j\in J}Q_j$. Let $Q = E(\oplus_{j\in J}Q_j)$ be an injective envelope of $\oplus_{j\in J}Q_j$ in E, and let

$$\pi : \bigoplus_{i\in I} E_i \to \bigoplus_{j\in J} Q_j$$

be the epimorphism that carries E_i to zero if $i \in I\backslash J$, whereas the restriction of π to E_i is the composition of the natural direct summand projection on $E_i \to Q_i$ with the canonical monomorphism $Q_i \to \oplus_{j\in J}Q_j$ for all $i \in J$. We claim that the composed morphism

$$g = \pi \circ f : N \longrightarrow \bigoplus_{j\in J} Q_j$$

admits no extensions to a morphism $h : M \to \oplus_{j\in J}Q_j$ along the monomorphism $u : N \to M$. In particular, $\oplus_{j\in J}Q_j$ is not injective. Suppose to the contrary that g admits such an extension h. Since M is finitely generated then $\mathrm{Im}(h)$, and so $\mathrm{Im}(g)$, is contained in $\oplus_{j\in F}Q_j$ for some finite subset F of J. Thus, $\pi_j \circ g = 0$ for each $j \in J\backslash F$, but this contradicts our construction of each object Q_j.

Consider the set Ω of subobjects L of Q satisfying the following three conditions:

(1) $\oplus_{j\in J}Q_j \subseteq L \subseteq Q$,

(2) L is a direct sum of injective subobjects of Q,

(3) the morphism $g = \pi \circ f : N \to \oplus_{j\in J} Q_j \subseteq L$ admits no extensions to a morphism $h : M \to L$ along the monomorphism $u : N \to M$.

It is clear that Ω is not empty, because the object $\oplus_{j\in J} Q_j$ belongs to Ω. We shall show that Ω is an inductive set with respect to the inclusion. Let $\{L_k\}_{k\in K}$ be a chain in Ω and let $L = \bigcup_{k\in K} L_k$. Since L is a directed union of direct sums of injective subobjects of Q, it is a perfectly pure subobject of E (since it is the direct union of the finite direct sums of copies of these injective objects). Thus, by hypothesis,

$$L = \left(\bigoplus_{u\in U} Y_u\right) \oplus \left(\bigoplus_{v\in V} Z_v\right)$$

is a direct sum of injective objects Y_u and countably generated objects Z_v. Moreover, U and V are countable sets, because L contains a countably generated subobject $\oplus_{j\in J} X_j$ that it is essential in it, as it was so in Q. Thus, $Z = \oplus_{v\in V} Z_v$ is countably generated. We can perform the object $Z = \sum_{n\in\mathbb{N}} Z'_n$ as a countable sum of finitely generated subobjects.

Since Z'_1 is finitely generated, it is contained in $\bigcup_{k\in F} L_k$ for some finite subset $F \subseteq K$. Furthermore, since each L_k is a direct sum of injective objects, then L contains an injective envelope $E(Z'_1)$ of Z'_1. Moreover, $E(Z'_1) \cap (\oplus_{u\in U} Y_u) = 0$, because $Z'_1 \cap (\oplus_{u\in U} Y_u) = 0$. Thus,

$$E(Z'_1) \cong \frac{(\oplus_{u\in U} Y_u) \oplus E(Z'_1)}{\oplus_{u\in U} Y_u} \subseteq \frac{(\oplus_{u\in U} Y_u) \oplus (\oplus_{v\in V} Z_v)}{\oplus_{u\in U} Y_u} \cong \bigoplus_{v\in V} Z_v = Z$$

and it is clear that the above isomorphism fixes Z'_1. Thus, Z contains the injective envelope $E(Z'_1)$ of Z'_1, and therefore there is a decomposition $Z = E(Z'_1) \oplus Z''_1$. Denote by $Z'_{1,n}$ the image of Z'_n under the natural projection on Z''_1 for $n \geq 2$. Further, we set $Z'_{1,1} = Z'_1$ for simplicity.

It is easy to check that $Z = E(Z'_1) \oplus \sum_{n\geq 2} Z'_{1,n}$, and therefore we get a decomposition $L = (\oplus_{u\in U} Y_u) \oplus E(Z'_1) \oplus \sum_{n\geq 2} Z'_{1,n}$.

By applying the same construction to L and $Z'_{1,2}$ we get

$$L = \left(\bigoplus_{u\in U} Y_u\right) \oplus E(Z'_{1,1}) \oplus E(Z'_{2,2}) \oplus \sum_{n\geq 3} Z'_{2,n}.$$

Repeating this process, we construct an infinite set $\{E(Z'_{n,n})\}_{n\in\mathbb{N}}$ of injective subobjects of L such that for each $m \in \mathbb{N}$, we have that $(\oplus_{u\in U} Y_u) \oplus (\oplus_{n=1}^{m} E(Z'_{n,n})) \subseteq L$.

Moreover, by construction, $Z'_m \subseteq \oplus_{n=1}^{m} Z'_{n,n}$, for each $m \in \mathbb{N}$. As a consequence, $Z \subseteq \oplus_{n\in\mathbb{N}} E(Z'_{n,n})$, the object L admits a decomposition

$$L = \left(\bigoplus_{u\in U} Y_u\right) \oplus \left(\bigoplus_{n\in\mathbb{N}} E(Z'_{n,n})\right)$$

and we have proved that L satisfies (2).

Finally, we have to show that the morphism $g = \pi \circ f : N \to \oplus_{j\in J} Q_j \subseteq L$ admits no extensions to a morphism $h : M \to L$ along the monomorphism $u : N \to M$.

Suppose to the contrary that g admits such an extension h. Then $\operatorname{Im}(h)$ is finitely generated, since so is M. We deduce that there exists a $k \in K$ such that $\operatorname{Im}(h) \subseteq L_k$. But this is a contradiction, because $L_k \in \Omega$ and therefore the morphism g cannot be extended to a morphism $M \to L_k$ by hypothesis.

We have proved that $L \in \Omega$, and so Ω is an inductive set. By Zorn's lemma, there exists a maximal element L_0 in Ω. By our hypothesis, the object L_0 is a direct sum of injective objects. Let

$$L_0 = \bigoplus_{t\in T} W_t,$$

where W_t is injective for any $t \in T$. Denote by $q_t : L_0 \to W_t$, the canonical projections. Since g cannot be extended to a morphism $h : M \to L_0$, there exists an infinite subset $T' \subseteq T$ such that $q_t \circ g \neq 0$ for each $t \in T'$ (because otherwise $\operatorname{Im}(g)$ would be contained in a finite subsum of the W_t's, say $\oplus_{t\in F} W_t$, that would be injective, and g would extend to a morphism $M \to \oplus_{t\in F} W_t \subseteq L_0$).

Let us write the set T' as a disjoint union $T' = T_1 \cup T_2$ of infinite subsets T_1 and T_2. Denote by $q_{T_1} : \oplus_{t\in T} W_t \to \oplus_{t\in T_1} W_t$ and $q_{T_2} : \oplus_{t\in T} W_t \to \oplus_{t\in T_2} W_t$ the canonical projections. It is clear that the morphism $q_{T_i} \circ g$ cannot be extended to a morphism $h : M \to \oplus_{t\in T_i} W_t$, for $i = 1, 2$, because otherwise $\operatorname{Im}(h)$ would be contained in some finite subsum $\oplus_{t\in F} W_t$ of $\oplus_{t\in T_i} W_t$, as M is finitely generated. It follows that $q_t \circ g = q_t \circ q_{T_i} \circ g = 0$ for each $t \in T_i \backslash F$, and we get a contradiction.

Let us choose an injective envelope $E(\oplus_{t\in T_1} W_t)$ of $\oplus_{t\in T_1} W_t$ in Q. Note that $\oplus_{t\in T_1} W_t \neq E(\oplus_{t\in T_1} W_t)$, because $q_{T_1} \circ g$ has no extension to a morphism $M \to \oplus_{t\in T_1} W_t$. Thus, $L_0 = \oplus_{t\in T} W_t$ is strictly contained in $E(\oplus_{t\in T_1} W_t) \oplus (\oplus_{t\in T_2} W_t)$ and the morphism g has no extension to a morphism $M \to E(\oplus_{t\in T_1} W_t) \oplus (\oplus_{t\in T_2} W_t)$, because otherwise $q_{T_2} \circ g$ extends to a morphism $M \to \oplus_{t\in T_2} W_t$, and consequently the object $E(\oplus_{t\in T_1} W_t) \oplus (\oplus_{t\in T_2} W_t)$ belongs to Ω, a contradiction with the maximality of L_0. This finishes the proof of the theorem. ∎

Now we are able to prove the main result of this section.

THEOREM 2.5. *Let E be a non-zero pure-injective object of a locally finitely presented Grothendieck category $\mathcal{A}$. The following conditions are equivalent*:

(a) *E is a direct sum of indecomposable objects.*

(b) *Every perfectly pure subobject of E is a direct sum of indecomposable pure-injective objects.*

(c) *Every perfectly pure subobject of E is a direct sum of objects that are pure-injective or countably generated.*

Proof. It was shown in [18] that there exists a locally finitely presented Grothendieck category $D(\mathcal{A})$ and a fully faithful additive functor

$$t : \mathcal{A} \longrightarrow D(\mathcal{A}) \tag{2.5}$$

with the following properties:

(i) The functor $\mathbf{t}$ admits a right adjoint functor $\mathbf{g} : D(\mathcal{A}) \to \mathcal{A}$.

(ii) A short exact sequence $\mathbf{X} : 0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ is pure if and only if the sequence $\mathbf{t}(\mathbf{X})$ is exact in $D(\mathcal{A})$, or equivalently, if and only if the sequence $\mathbf{t}(\mathbf{X})$ is pure exact in $D(\mathcal{A})$.

(iii) $\mathbf{t}$ carries finitely generated objects to finitely generated ones.

(iv) The image of $\mathcal{A}$ under the functor $\mathbf{t}$ is the full subcategory of $D(\mathcal{A})$ formed by all FP-injective objects.

(v) An object A of $\mathcal{A}$ is pure-injective if and only if $\mathbf{t}(A)$ is an injective object of $D(\mathcal{A})$.

It follows that A is a perfectly pure subobject of E if and only if $\mathbf{t}(A)$ is a perfectly pure subobject of $\mathbf{t}(E)$.

Consequently, the conditions (a), (b), and (c) are equivalent to the corresponding conditions (a), (b), and (c) for $\mathbf{t}(E)$ in $\mathbf{t}(E)$ with "pure-injective" and "injective" interchanged. Thus, the result is an immediate consequence of Theorem 2.3 applied to the injective object $\mathbf{t}(E)$ of the category $D(\mathcal{A})$. ∎

As an immediate consequence of Theorem 2.5 we get

COROLLARY 2.6. *Let E be a pure-injective object of a locally finitely presented Grothendieck category $\mathcal{A}$. If every pure subobject of E is a direct sum of countably generated objects, then E is a direct sum of indecomposable objects.*

By applying Theorem 2.4 to the category $\mathcal{A} = \mathrm{Mod}(R)$ of right R-modules we get [6, Theorem 2.5], which is the main result of [6].

3. Σ-PURE-INJECTIVITY AND THE PURE SEMISIMPLICITY

We recall that a module M is Σ-pure-injective if any direct sum of copies of M is a pure injective module (see [10, 23]).

An interesting consequence of Theorem 2.4 is the following characterization of Σ-pure-injective modules (compare with [23]).

COROLLARY 3.1. *Let M be a right R-module. The following conditions are equivalent*:

(a) *M is Σ-pure-injective.*

(b) *Every pure submodule of a pure-injective envelope of a direct sum of arbitrary many copies of M is a direct sum of modules that are pure-injective or countably generated.*

(b′) *Every pure submodule of a pure-injective envelope of a direct sum of countably many copies of M is a direct sum of modules that are pure-injective or countably generated.*

(c) *Every perfectly pure submodule of a pure-injective envelope of a direct sum of copies of M is a direct sum of modules that are pure-injective or countably generated.*

(c′) *Every perfectly pure submodule of a pure-injective envelope of a direct sum of countably many copies of M is a direct sum of modules that are pure-injective or countably generated.*

Proof. (a) ⇒ (b). It is well known that every pure-submodule of a Σ-pure-injective module is again Σ-pure-injective (see, e.g., [8, Corollary 1.4]).

The implications (b) ⇒ (b′) ⇒ (c′) and (b) ⇒ (c) ⇒ (c′) are trivial.

(c′) ⇒ (a). Let $E = E_{pure}(M^{(\mathbb{N})})$ be a pure-injective envelope of a direct sum of countably many copies of M. By Theorem 2.4, E is a direct sum of indecomposable modules. On the other hand, E contains a direct sum $E_{pure}(M)^{(\mathbb{N})}$ of countably many copies of the pure-injective envelope $E_{pure}(M)$ of M as a pure submodule. Then $E_{pure}(M)^{(\mathbb{N})}$ is a direct summand of E by [5, Theorem 3.4]. Consequently, $E_{pure}(M)$ is Σ-pure-injective, and in view of [8, Corollary 1.4], M is Σ-pure-injective, because M is a pure submodule of $E_{pure}(M)$. ∎

We recall from [17] that a ring R is called **right pure semisimple** if every right R-module is a direct sum of finitely presented modules, or equivalently, if every right R-module is algebraically compact (i.e., pure-injective) [11]. These rings are always right artinian (see, e.g., [4, Proposition 5]). The reader is referred to [10, 19, 25] for a background on right pure semisimple rings.

The above corollary yields to the following characterization of pure-semisimple rings.

THEOREM 3.2. *Let R be a ring. The following conditions are equivalent*:

(a) *R is right pure semisimple.*

(b) *Every right R-module is a direct sum of modules that are pure-injective or pure-projective.*

(c) *Every right R-module is a direct sum of modules that are pure-injective or countably generated.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (a). Suppose that (c) holds. By Corollary 3.1, every right R-module is Σ-pure-injective. Thus, R is right pure-semisimple (see, e.g., [4, Proposition 5]). ∎

REMARK 3.3. (a) The implication (b) $\Rightarrow$ (c) of Theorem 3.2 sheds a light on the following open question posed in [20, Problem 3.2] (compare with [23]): *"Is a semiperfect ring R right artinian or right pure semisimple if every indecomposable right R-module is pure-injective or pure-projective?"*

(b) A characterization of rings R for which every indecomposable right R-module is pure-injective or pure-projective remains also an open problem (see [20, Problem 3.2]).

It was pointed out by N. V. Dung that this class of rings contains a large class of non-noetherian rings R having no indecomposable decomposition. Namely, let R be a right semi-artinian V-ring, that is, every non-zero right R-module contains a non-zero injective submodule (see [7]). It follows that every indecomposable right R-module is simple and injective. The results of Dung and Smith in [7] show that there are many non-noetherian algebras which are semi-artinian V-rings. ∎

We recall from [18] that a locally finitely presented Grothendieck category $\mathscr{A}$ is pure-semisimple if every object of $\mathscr{A}$ is a direct sum of finitely presented objects.

In relation with Theorem 3.2 and the discussion above the following result proved in [18] would be of some interest.

THEOREM 3.4. *A locally finitely presented Grothendieck category $\mathscr{A}$ is pure-semisimple if and only if there exists a cardinal number $\aleph$ such that every pure-injective object of $\mathscr{A}$ is a direct sum of $\aleph$-generated objects.*

Proof. We recall from [18] that $\mathscr{A}$ is pure-semisimple if and only if the category $D(\mathscr{A})$(2.5) is locally noetherian. On the other hand, by the properties of the functor (2.5) listed in the proof of Theorem 2.4, a cardinal number $\aleph$ such that every pure-injective object of $\mathscr{A}$ is a direct sum of $\aleph$-generated objects does exist if and only if every injective object of $D(\mathscr{A})$ is a direct sum of $\aleph$-generated objects. By Roos [15] the last condition holds if and only if $D(\mathscr{A})$ is locally noetherian, and we are done. ∎

Thus, the following questions related with our previous results arise naturally.

QUESTION 3.5. *Let R be a ring and suppose that there exists a cardinal number $\aleph$ such that every right R-module is a direct sum of modules that are pure-injective or $\aleph$-generated. Is R right pure-semisimple?*

We do not know the answer even for $\aleph = \aleph_1$.

QUESTION 3.6. *Let R be a ring which is right noetherian (or right artinian) and suppose that the isomorphism classes of the indecomposable right R-modules form a set. Is R right pure-semisimple?*

4. WHEN ARE ALL STRICTLY INDECOMPOSABLE COUNTABLY GENERATED OBJECTS PURE-PROJECTIVE?

We finish this paper by a discussion of a problem close to that one presented in Remark 3.3(b).

Following [4, 24] we call a non-zero object T of $\mathcal{A}$ *strictly indecomposable* if the intersection of all non-zero pure subobjects of T is non-zero. It is easy to see that strictly indecomposable objects are indecomposable.

The proof of Theorem 4.2 below will depend on the following simple but useful observation.

LEMMA 4.1. *For every non-zero object M of a locally finitely presented Grothendieck category $\mathcal{A}$ there exists a pure epimorphism $v : M \to T$, where T is a strictly indecomposable object.*

Proof. We shall follow an idea in [4, Proposition 1; 22, 36.4; 24].

Let M be a non-zero object of $\mathcal{A}$. If M is strictly indecomposable we set $T = M$. Assume that M is not strictly indecomposable. Then there exists a non-zero pure subobject N of M. Fix a non-zero finitely generated subobject X of N and consider the family $\mathcal{F}$ of all non-zero pure subobjects L of M such that there is no monomorphism $X \to L$. Since M is not strictly indecomposable and N is a pure subobject of M containing X then there exists a non-zero pure subobject L of M which does not contain X, and therefore L belongs to $\mathcal{F}$. Since obviously $\mathcal{F}$ is an inductive family then by Zorn's lemma there exists a maximal object L in $\mathcal{F}$. It follows that X belongs to all pure subobjects of M properly containing L. We set $T = M/L$ and we take for $v : M \to T$ the natural epimorphism. It is easy to check that T is strictly indecomposable and the lemma follows. ∎

The following theorem answers partially the question stated in Remark 3.3(b). On the other hand, it generalizes the results given in [3, Theorem 4.5; 4, Propositions 4 and 5; 16, Theorem 6.3; 18, Theorem 1.3].

THEOREM 4.2. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. The following conditions are equivalent*:

(a) *Every indecomposable object of $\mathcal{A}$ is pure-projective*.

(b) *Every strictly indecomposable countably generated object of $\mathcal{A}$ is pure-projective*.

(c) *The category $\mathcal{A}$ is pure-semisimple*.

Proof. The implications (c)⇒(a)⇒(b) are obvious.

(b) ⇒ (c). Suppose that every strictly indecomposable countably generated object of $\mathcal{A}$ is pure-projective.

First we shall prove that every non-zero countably generated object M of $\mathcal{A}$ is a continuous well-ordered union (in the sense of [12])

$$M = \bigcup_{\xi<\gamma} M_\xi$$

of subobjects M_ξ of M such that the following four conditions are satisfied:

(0) $\xi < \gamma$ are ordinal numbers and γ is at most the minimal uncountable number,

(1) the embedding $M_\xi \subseteq M$ is pure and the object M_ξ is countably generated for every $\xi < \gamma$,

(2) the object $M_{\xi+1}/M_\xi$ is strictly indecomposable and countably generated for every $\xi < \gamma$,

(3) $M_\beta = \bigcup_{\xi<\beta} M_\beta$ for any limit ordinal number $\beta < \gamma$.

Let M be a non-zero countably generated object of $\mathcal{A}$. By Lemma 4.1, there exists a pure epimorphism $v : M \to T_1$, where T_1 is strictly indecomposable and countably generated. By (b), the object T_1 is pure-projective and therefore the pure epimorphism v splits. Consequently M contains a countably generated strictly indecomposable pure-projective direct summand T_1' isomorphic with T_1. We take for M_1 the object T_1'.

Assume that the object M_ξ is defined. If $M_\xi = M$ we set $\gamma = \xi + 1$. If $M_\xi \neq M$ we define $M_{\xi+1} \subseteq M$ as follows. By Lemma 4.1 applied to the non-zero countably generated object $M_\xi' = M/M_\xi$ there exists a pure epimorphism $v_\xi : M_\xi' \to T_\xi$, where T_ξ is strictly indecomposable and countably generated. By (b), the object T_ξ is pure-projective and therefore the composed pure epimorphism $M \longrightarrow M/M_\xi \longrightarrow^{v_\xi} T_\xi$ splits. Consequently, there exists a direct summand T_ξ' of M isomorphic with T_ξ such that $M_\xi \bigcap T_\xi' = 0$. We take for $M_{\xi+1}$ the object $M_\xi \oplus T_\xi' \subseteq M$. It is not difficult to check that the condition (2) is satisfied and (1) is satisfied with ξ and $\xi + 1$ interchanged.

If β is a limit ordinal number and the objects M_ξ are defined for all ordinals $\xi < \beta$ such that (1) and (2) hold for $\xi < \beta$, we set $M_\beta = \cup_{\xi<\beta} M_\beta$. Obviously the condition (1) is satisfied with ξ and β interchanged.

Since M is countably generated then obviously there exists an ordinal number γ, which is at most the minimal uncountable number, such that M is a continuous well-ordered union $M = \cup_{\xi<\gamma} M_\xi$ of the subobjects M_ξ of M constructed above and the conditions (0)–(3) are satisfied.

By the well-known theorem of Auslander in [2] (see also [12, 16]) there exists an isomorphism $M \cong \oplus_{\xi<\gamma} M_{\xi+1}/M_\xi$, and therefore the object M is pure-projective, because according to (2) the non-zero countably generated objects $M_{\xi+1}/M_\xi$ are strictly indecomposable, and therefore they are pure-projective by our hypothesis (b).

Consequently, every countably generated object M of $\mathcal{A}$ is pure-projective. It follows from [16, Theorem 6.3] that every object of $\mathcal{A}$ is pure-projective, that is, the category $\mathcal{A}$ is pure semisimple. This finishes the proof. ∎

REMARK 4.3. We hope that the condition (b) in Theorem 4.2 is equivalent to the following one:

(b′) *Every strictly indecomposable countably presented object of* $\mathcal{A}$ *is pure-projective*.

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REFERENCES

1. F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," Springer-Verlag, New York/Berlin, 1991.
2. M. Auslander, On the dimension of modules and algebras, III, *Nagoya Math. J.* **9** (1955), 67–77.
3. G. Azumaya, Countable generatedness version of rings of pure global dimension zero, *in* London Math. Soc. Lecture Notes Series, Vol. 168, pp. 43–79, Cambridge Univ. Press, Cambridge, UK, 1992.
4. G. Azumaya and A. Facchini, Rings of pure global dimension zero and Mittag–Leffler modules, *J. Pure Appl. Algebra* **62** (1989), 109–122.
5. N. V. Dung, Modules with indecomposable decompositions that complement maximal direct summands, *J. Algebra* **197** (1997), 449–467.
6. N. V. Dung, Indecomposable decompositions of pure-injective modules, *Comm. Algebra* **26** (1998), 3709–3725.

7. N. V. Dung and P. F. Smith, On semi-artinian V-modules, *J. Pure Appl. Algebra* **82** (1992), 27–37.
8. J. L. García and N. V. Dung, Some decomposition properties of injective and pure-injective modules, *Osaka J. Math.* **31** (1994), 95–98.
9. J. L. Gómez Pardo and P. A. Guil Asensio, Indecomposable decompositions of finitely presented pure-injective modules, *J. Algebra* **192** (1997), 200–208.
10. C. U. Jensen and H. Lenzing, "Model Theoretic Algebra with Particular Emphasis on Fields, Rings, Modules," Algebra, Logic and Applications, Vol. 2, Gordon & Breach, New York, 1989.
11. R. Kielpiński, On Γ-pure injective modules, *Bull. Polon. Acad. Sci. Ser. Math.* **15** (1967), 127–131.
12. B. Mitchell, Rings with several objects, *Adv. Math.* **8** (1972), 1–161.
13. S. H. Mohamed and B. J. Müller, "Continuous and Discrete Modules," Cambridge Univ. Press, Cambridge, UK, 1990.
14. N. Popescu, "Abelian Categories with Applications to Rings and Modules," Academic Press, San Diego, 1973.
15. J. E. Roos, Locally noetherian categories and linearly compact rings: Applications, *in* Lecture Notes in Math., Vol. 92, pp. 197–277, Springer-Verlag, New York/Berlin, 1969.
16. D. Simson, On pure global dimension of locally finitely presented Grothendieck categories, *Fund. Math.* **96** (1977), 91–116.
17. D. Simson, Pure semisimple categories and rings of finite representation type, *J. Algebra* **48** (1977), 290–296; Corrigendum, **67** (1980), 254–256.
18. D. Simson, On pure semi-simple Grothendieck categories, I, *Fund. Math.* **100** (1978), 211–222.
19. D. Simson, A class of potential counter-examples to the pure semisimplicity conjecture, *in* "Advances in Algebra and Model Theory" (M. Droste and R. Göbel, Eds.), Algebra, Logic and Applications Series, Vol. 9, pp. 345–373, Gordon & Breach, Australia, 1997.
20. D. Simson, Dualities and pure semisimple rings, *in* "Proc. Conference, Abelian Groups, Module Theory and Topology, University of Padova, June 1997," *Lecture Notes in Pure and Appl. Math.,* Vol. 201, pp. 381–388, Dekker, New York, 1998.
21. B. Stenström, "Rings of Quotients," Springer-Verlag, New York/Berlin, 1975.
22. R. Wisbauer, "Foundations of Module and Ring Theory," Algebra, Algebra, Logic and Applications, Vol. 3, Gordon & Breach, New York, 1988.
23. B. Zimmermann-Huisgen, Rings whose right modules are direct sums of indecomposable modules, *Proc. Amer. Math. Soc.* **77** (1979), 191–197.
24. W. Zimmermann, Einige Charakterisierung der Ringe über denen reine Untermoduln direkte Summanden sind, *Bayer. Akad. Wiss. Math.-Natur.* **2** (1973), 77–79.
25. W. Zimmermann and B. Zimmermann-Huisgen, On the sparsity of representations of rings of pure global dimension zero, *Trans. Amer. Math. Soc.* **320** (1990), 695–711.