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Stability criteria for set dynamic equations on time scales☆

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ABSTRACT

Very recently, a new theory known as set dynamic equations on time scales has been built. In this paper, notions of stability for the solutions of set dynamic equations on time scales, using Lyapunov-like functions are considered. Criteria for the equistability, equiasymptotic stability, uniform and uniform asymptotic stability are developed.

1. Introduction

The interesting feature of the set differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single valued mapping. Also, we have only semilinear complete metric space to work with, in the present setup, compared to the complete normed linear space that one employs in the usual study of the ordinary differential equations. The study of set differential equations has been initiated as an independent subject and several results of interest can be found in [1–9]. The basic theory, comparison results and the stability considerations for hybrid dynamical systems were discussed in [10]. Since then, much progress has been made in studying various fundamental aspects of the stability of set differential equations (see [11–16]). For instance, certain Lyapunov-like functions were used to study their stability criteria by Lakshmikantham, Leela and Devi [12]. In [13], Bhaskar and Devi studied the Lyapunov stability for the solutions of set differential equations, using Lyapunov-like functions which are continuous. Moreover, an important comparison result in the light of Lyapunov functions was employed in [13] to investigate the qualitative behaviour of the solutions of the following set differential equation

$$D_H U = F(t, U), \qquad U(t_0) = U_0 \in K_c(\mathbb{R}^n),$$

such as, the equistability, equiasymptotic stability, uniform and uniform asymptotic stability. Here $K_c(\mathbb{R}^n)$ denotes the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Gnana Bhaskar and Shaw explored the stability criteria for set difference equations in [16].

On the other hand, a theory known as dynamic systems on time scales has been built which incorporates both continuous and discrete times, namely, time as arbitrary closed sets of reals, and permits us to handle both systems simultaneously (see [17,18]). This theory allows one to get some insight into and better understanding of the subtle difference between discrete and continuous systems. The theory of dynamical systems on time scales recently received much attention and is

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undergoing rapid development (see [17–20]). Recently, many monographs to investigate the stability criteria for the solution of dynamic systems on time scales can be found (see [21–28]).

Based on the above mentioned notions, Hong [29] introduced a class of new derivatives of multivalued functions on time scales and developed a new theory of set dynamic equations (SDEs) by extending set valued differential equations onto time scales, also, author provided some results on the existence of their solution. In this paper, provided inspiration by [13,16], we shall consider the stability criteria for the solutions of set dynamic equations as in the original Lyapunov results for ordinary differential equations. Our purpose is to explore such a stability criteria in a unified way of the study in set differential equations and set difference equations and to offer more general conclusions. We employ an important comparison result in terms of Lyapunov functions and investigate the qualitative behaviour of the solutions of the initial value problem for the following SDE

$$\Delta_H U = F(t, U), \qquad U(t_0) = U_0 \in K_c(\mathbb{R}), \tag{1}$$

where Δ_H denotes the derivative of multivalued functions defining on the time scales (see [29, Definition 3.1]). As a result, the mapping $U \in \mathcal{C}^1_{rd}(J_{\mathbf{T}}, K_c(\mathbb{R})), J_{\mathbf{T}} = [t_0, t_0 + a]_{\mathbf{T}}$ $(a > 0)$ is said to be a solution of (1) on $J_{\mathbf{T}}$ if it satisfies (1) on $J_{\mathbf{T}}$. Since $U \in \mathcal{C}^1_{rd}(J_{\mathbf{T}}, K_c(\mathbb{R}))$ we have $U(t) = U_0 + \int_{t_0}^{t} \Delta_H U(s)\Delta s$, $t \in J_{\mathbf{T}}$. Thus we associate with the initial value problem (1) the following

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s))\Delta s, \quad t \in J_{\mathbf{T}}.$$

As an analogue of [2], these Lyapunov-like functions serve as a vehicle to transform the set dynamic equations into scalar comparison differential equations, and therefore, it is enough to consider the qualitative properties of the simpler comparison equation under suitable conditions for Lyapunov-like functions.

2. Preliminaries

In this section we give all the necessary background material needed for a self-contained presentation of our study.

We begin with a brief but complete description of the basic known results for Hausdorff metrics, continuity and differentiability for multivalued mappings on time scales, also, the concept of time scales, single-valued functions and their corresponding properties within the framework of time scales.

Let $K_c(\mathbb{R}^n)$ denote the collection of nonempty, compact and convex subsets of $\mathbb{R}^n$. The following operations can be naturally defined on it:

$$X + Y = \{x + y : x \in X, y \in Y\}, \qquad \lambda \cdot X = \{\lambda \cdot x : x \in X\}, \quad \lambda \in \mathbb{R}_+.$$
$$XY = \{xy : x \in X, y \in Y\} \quad \text{for } X, Y \in K_c(\mathbb{R}).$$

In addition, the set $Z \in K_c(\mathbb{R}^n)$ satisfying $X = Y + Z$ is known as the geometric difference of the sets X and Y and is denoted by the symbol $X - Y$. It is worthy to note that the geometric difference of two sets does not always exist but if it does it is unique.

We define the Hausdorff metric as

$$D[X, Y] = \max\left\{\sup_{y\in Y} d(y, X), \sup_{x\in X} d(x, Y)\right\},$$

where $d(x, Y) = \inf\{d(x, y) : y \in Y\}$ and X, Y are bounded subsets of $\mathbb{R}^n$. The Hausdorff metric satisfies the following relations:

$$D[X, Y] \geq 0 \quad \text{with } D[X, Y] = 0 \text{ if and only if } X = Y,$$
$$D[X, Y] = D[Y, X],$$
$$D[X, Y] \leq D[X, Z] + D[Z, Y],$$

for any $X, Y, Z \in K_c(\mathbb{R}^n)$.

Notice that $K_c(\mathbb{R}^n)$ with the metric is a complete metric space. Moreover $K_c(\mathbb{R}^n)$ equipped with the above-mentioned natural algebraic operations of addition and nonnegative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [4]. On the other hand, the Hausdorff metric D is compatible with the operations defined on it as described by the following properties: for any $X, Y, Z, W \in K_c(\mathbb{R}^n)$ and $\mu, \nu \in \mathbb{R}_+$,

$$D[X \pm Z, Y \pm Z] = D[X, Y],$$
$$D[\mu X, \mu Y] = \mu D[X, Y],$$
$$D[X \pm Z, Y \pm W] \leq D[X, Y] + D[Z, W],$$
$$D[\mu X, \nu Y] \leq \max\{\mu, \nu\} \cdot D[X, Y] + |\mu - \nu| \cdot (\|X\| + \|Y\|),$$

where $\|V\| = D[V, \{0\}]$ for $V \in K_c(\mathbb{R}^n)$. Here we assume that the differences appearing in the above formulas exist.

A multivalued mapping $F : I \to K_c(\mathbb{R}^n)$, where $I \subset \mathbb{R}^n$, is said to have the limit at $x_0 \in I$ if there exists a element $A \in K_c(\mathbb{R}^n)$ such that, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, x_0) > 0$ such that $D[F(x), A] < \varepsilon$, for all $x \in I$ with $\|x - x_0\| < \delta$. We denote the limit by $\lim_{x\to x_0} F(x)$, that is, $A = \lim_{x\to x_0} F(x)$. Let $F(x_0)$ be well defined. F is called continuous at $x_0 \in I$ if its limit at x_0 exists and $\lim_{x\to x_0} F(x) = F(x_0)$.

Alternatively, we may write, in terms of the convergence of sequences that

$$\lim_{x_n\to x_0} D[F(x_n), F(x_0)] = 0$$

for all sequences $\{x_n\}$ in I with $\lim x_n \to x_0$.

It is significant to refer that if we restrict ourselves to single valued mappings, then the previous notions reduce to their classical counterparts, i.e. to ordinary continuity in $\mathbb{R}^n$.

We recall also briefly the notions of time scales and Hilger derivative on them.

Let $\mathbf{T}$ be a closed nonempty subset of real number set $\mathbb{R}$. In the light of some of the current literature, $\mathbf{T}$ is called a time scale or measure chain. The calculus of time scales we refer readers to Bohner and Peterson [18]. Here we introduce the basic notions connected to time scales and differentiability of functions on them. Let us start by defining the forward and backward jump operators.

Let $\mathbf{T}$ be a time scale. For $t \in \mathbf{T}$ we define the forward jump operator $\sigma : \mathbf{T} \to \mathbf{T}$ by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} : \tau > t\}$$

and the graininess function $\mu : \mathbf{T} \to \mathbb{R}^+$ by

$$\mu(t) = \sigma(t) - t,$$

while the backward jump operator $\rho : \mathbf{T} \to \mathbf{T}$ is defined by

$$\rho(r) = \sup\{\tau \in \mathbf{T} : \tau < r\}.$$

In this definition we put $\inf \emptyset = \sup \mathbf{T}$ (i.e. $\sigma(t) = t$ if $\mathbf{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbf{T}$ (i.e. $\rho(t) = t$ if $\mathbf{T}$ has a minimum t), where $\emptyset$ denotes the empty set. t is said to be right scattered if $\sigma(t) > t$ and t is said to be right dense (rd) if $\sigma(t) = t$. t is said to be left scattered if $\rho(t) < t$ and t is said to be left dense (ld) if $\rho(t) = t$. A point is said to be isolated (dense) if it is right-scattered (right-dense) and left-scattered (left-sense) at the same time. We introduce the sets $\mathbf{T}^k$ and $\mathbf{T}_k$ which are derived from the time scale $\mathbf{T}$ as follows. If $\mathbf{T}$ has a right scattered minimum m, then $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, then $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

A function f is left(right)-dense continuous (ld(rd)-c, for short) if f is continuous at each left(right) dense point in $\mathbf{T}$ and its right(left)-sided limits exist at each right(left) dense points in $\mathbf{T}$. By $C_{ld}(\mathbf{T}, \mathbb{R})$ and $C_{rd}(\mathbf{T}, \mathbb{R})$ we denote the set of all left and right dense continuous functions from $\mathbf{T}$ to $\mathbb{R}$, respectively.

The set of functions $f : \mathbf{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbf{T}, \mathbb{R}).$$

For $f : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}^k$, S. Hilger defined the delta (or Hilger) derivative of $f(t)$, $f^{\Delta}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there exists a neighborhood U of t (i.e. $U = (t - \delta, t + \delta) \cap \mathbf{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. We say f is Δ-differentiable at t if its delta derivative exists at t. Moreover, we say f is Δ-differentiable on $\mathbf{T}^k$ if its delta derivative exists at each $t \in \mathbf{T}^k$. The function $f^{\Delta} : \mathbf{T}^k \to \mathbb{R}$ is then called the delta (or Hilger) derivative of f on $\mathbf{T}^k$.

For $f : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}_k$, we define the nabla derivative of $f(t)$, $f^{\nabla}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \le \varepsilon|\rho(t) - s|,$$

for all $s \in U$.

If $\mathbf{T} = \mathbb{R}$ then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbf{T} = \mathbb{Z}$ then $f^{\Delta}(t) = f(t+1) - f(t)$ is the forward difference operator while $f^{\nabla}(t) = f(t) - f(t-1)$ is the backward difference operator.

A continuous function $f : \mathbf{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbf{T}^k$, $\mathbf{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbf{T}$, and f is differentiable at each $t \in D$.

For an interval of real numbers I, by $I_{\mathbf{T}}$ we mean the set $I \cap \mathbf{T}$ and by $C^{\Delta}_{ld}(I_{\mathbf{T}})$ we mean the set of all functions from $I_{\mathbf{T}}$ to $[0, \infty)$ which are Δ-differentiable on $I_{\mathbf{T}^k}$.

If $F^{\Delta}(t) = f(t)$ for $t \in [a, T]_{\mathbf{T}}$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^t f(s)\nabla s = F(t) - F(a).$$

By $L^{\Delta}(I_{\mathbf{T}})$, $L^{\nabla}(I_{\mathbf{T}})$ we denote the set consisting of all functions which are delta integrable on $I_{\mathbf{T}}$ and the set consisting of all functions which are nabla integrable on $I_{\mathbf{T}}$, respectively. Some useful relationships concerning the Hilger derivative are given next.

A function $f : \mathbf{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (belong to $\mathbb{R}$) at all right-dense points in $\mathbf{T}$ and its left-sided limits exist (belong to $\mathbb{R}$) at all left-dense points in $\mathbf{T}$.

Lemma 2.1 ([18, Theorem 8.12]). *Suppose $f_n : \mathbf{T} \to \mathbb{R}$ is pre-differentiable with D for each $n \in \mathbb{N}$ (a set consisting of all nature numbers). Assume that for each $t \in \mathbf{T}^k$ there exists a compact interval neighborhood $U(t)$ such that the sequence $\{f_n^{\Delta}\}_{n\in\mathbb{N}}$ converges uniformly on $U(t) \cap D$.*

(i) *If $\{f_n\}$ converges at some $t_0 \in U(t)$ for some $t \in \mathbf{T}^k$, then it converges uniformly on $U(t)$.*
(ii) *If $\{f_n\}$ converges at some $t_0 \in \mathbf{T}$, then it converges uniformly on $U(t)$ for all $t \in \mathbf{T}^k$.*
(iii) *The limit mapping $f = \lim_{n\to\infty} f_n$ is pre-differentiable with D and we have*

$$f^{\Delta}(t) = \lim_{n\to\infty} f_n(t)$$

for all $t \in D$.

Lemma 2.2 ([18, Theorem 8.13]). *Let $t_0 \in \mathbf{T}$, $c \in \mathbb{R}$, and a regulated map $f : \mathbf{T}^k \to \mathbb{R}$ be given. Then there exists exactly one pre-differentiable function F satisfying*

$$\begin{cases} F^{\Delta}(t) = f(t) & \forall t \in D, \\ F(t_0) = c. \end{cases}$$

Definition 2.1 ([29]). Assume $F : \mathbf{T} \to K_c(\mathbb{R})$ is a multivalued function and let $t \in \mathbf{T}^k$. Let $\Delta_H F(t)$ be an element of $K_c(\mathbb{R})$ (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood $U_{\mathbf{T}}$ of t (i.e. $U_{\mathbf{T}} = (t-\delta, t+\delta) \cap \mathbf{T}$ for some $\delta > 0$) such that

$$D[F(t+h) - F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \le \varepsilon(h - \mu(t)),$$
$$D[F(\sigma(t)) - F(t-h), \Delta_H F(t)(\mu(t) + h)] \le \varepsilon(\mu(t) + h)$$

for all $t-h, t+h \in U_{\mathbf{T}}$ with $0 \le h < \delta$, where $\mu(t)$ is the graininess function. We call $\Delta_H F(t)$ the Δ_H-derivative of F at t. We say that F is Δ_H-differentiable at t if its Δ_H-derivative exists at t. Moreover, we say F is Δ_H-differentiable on $\mathbf{T}^k$ if its Δ_H-derivative exists at each $t \in \mathbf{T}^k$. The multivalued function $\Delta_H F : \mathbf{T}^k \to K_c(\mathbb{R})$ is then called the Δ_H-derivative of F on $\mathbf{T}^k$.

Proposition 2.1 ([29]). *Some easy and useful relationships concerning the Δ_H-derivative are given next.*

(i) *If the Δ_H-derivative of F at t exists, then it is unique. Hence, the Δ_H-derivative is well defined.*
(ii) *Assume $F : \mathbf{T} \to K_c(\mathbb{R})$ is a multivalued function and let $t \in \mathbf{T}^k$. Then we have the following:*
   (1) *If F is Δ_H-differentiable at t, then F is continuous at t.*
   (2) *If F is continuous at t and t is right scattered, then F is Δ_H-differentiable at t with*
   $$\Delta_H F(t) = \frac{F(\sigma(t)) - F(t)}{\mu(t)}.$$
   (3) *If t is right-dense, then F is Δ_H-differentiable at t iff the limits*
   $$\lim_{h\to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t) - F(t-h)}{h}$$
   exist and satisfy the equations
   $$\lim_{h\to 0^+} \frac{F(t+h) - F(t)}{h} = \lim_{h\to 0^+} \frac{F(t) - F(t-h)}{h} = \Delta_H F(t).$$
   (4) *If F is differentiable at t, then*
   $$F(\sigma(t)) = F(t) + \mu(t)\Delta_H F(t).$$
(iii) *Assume that multivalued functions $F, G : \mathbf{T} \to K_c(\mathbb{R})$ are Δ_H-differentiable at $t \in \mathbf{T}^k$. Then*
   (1) *The sum $F + G : \mathbf{T} \to K_c(\mathbb{R})$, defined by $(F+G)(t) = F(t) + G(t) = \{x + y : x \in F(t), y \in G(t)\}$ for each $t \in \mathbf{T}$, are Δ_H-differentiable at $t \in \mathbf{T}^k$ with*
   $$\Delta_H(F+G)(t) = \Delta_H F(t) + \Delta_H G(t).$$
   (2) *For any nonnegative constant λ, $\lambda F : \mathbf{T} \to K_c(\mathbb{R})$ is Δ_H-differentiable at t with*
   $$\Delta_H(\lambda F)(t) = \lambda \Delta_H F(t).$$
(iv) *Assume that multivalued functions $F, G : \mathbf{T} \to K_c(\mathbb{R})$ are Δ_H-differentiable at $t \in \mathbf{T}^k$. Then the product function FG defined by $(FG)(t) = F(t)G(t)$ for $t \in \mathbf{T}$ is Δ_H-differentiable at $t \in \mathbf{T}^k$ with*

$$\Delta_H(FG)(t) = F(\sigma(t))\Delta_H G(t) + G(t)\Delta_H F(t) = F(t)\Delta_H G(t) + G(\sigma(t))\Delta_H F(t).$$

In what follows, in order to define the integral of multivalued functions on time scales, we first need the following notions. Let $\mathbf{D} \subset \mathbf{T}$. A function $f : \mathbf{D} \to \mathbb{R}$ is called a sector of the multivalued function $F : \mathbf{D} \to K_c(\mathbb{R})$ if $f(t) \in F(t)$ for all $t \in \mathbf{D}$. By $S_F(\mathbf{D})$ we mean the set of all Δ-integrable sectors of F on $\mathbf{D}$.

Definition 2.2 ([29]). A multivalued function $F : \mathbf{T} \to K_c(\mathbb{R})$ is called Δ_H-integrable on $\mathbf{D} \subset \mathbf{T}$ if F has a Δ-integrable sector on $\mathbf{D}$. In this case, we define the Δ_H-integral of F on $\mathbf{D}$, denoted by $\int_{\mathbf{D}} F(s)\Delta s$, as the set

$$\int_{\mathbf{D}} F(s)\Delta s = \left\{ \int_{\mathbf{D}} f(s)\Delta s : f \in S_F(\mathbf{D}) \right\}.$$

Proposition 2.2 ([29]). *Assume that $t_0, T, \in \mathbf{T}$ and $F, G : [t_0, T]_{\mathbf{T}} \to K_c(\mathbb{R})$ are Δ_H-integrable and have rd-continuous sectors, then we have*

(i) $\int_{t_0}^T [F(s) + G(s)]\Delta s = \int_{t_0}^T F(s)\Delta s + \int_{t_0}^T G(s)\Delta s.$

(ii) $\int_{t_0}^t \lambda F(s)\Delta s = \lambda \int_{t_0}^t F(s)\Delta s, \lambda \in \mathbb{R}_+, t \in [t_0, T]_{\mathbf{T}}.$

(iii) $\int_{t_0}^T F(s)\Delta s = \int_{t_0}^t F(s)\Delta s + \int_t^T F(s)s\Delta s, t \in [t_0, T]_{\mathbf{T}}$ *with* $t_0 \le t \le T.$

(iv) $\int_{t_0}^{t_0} F(s)\Delta s = \{0\}.$

(v) *If* $f \in S_F([t_0, T]_{\mathbf{T}})$ *implies that* $f \in C_{rd}([t_0, T]_{\mathbf{T}})$, *then* $\|F(\cdot)\| : [t_0, T]_{\mathbf{T}} \to \mathbb{R}_+$ *is Δ-integrable and*

$$\left\| \int_{t_0}^T F(s)\Delta s \right\| \le \int_{t_0}^T \|F(s)\|\Delta s.$$

(vi) *If* $f \in S_F([t_0, T]_{\mathbf{T}})$ *and* $g \in S_G([t_0, T]_{\mathbf{T}})$ *imply that* $f \in C_{rd}([t_0, T]_{\mathbf{T}})$ *and* $g \in C_{rd}([t_0, T]_{\mathbf{T}})$, *respectively, then* $D[F(\cdot), G(\cdot)] : [t_0, T]_{\mathbf{T}} \to \mathbb{R}_+$ *is Δ-integrable and*

$$D\left[\int_{t_0}^T F(s)\Delta s, \int_{t_0}^T G(s)\Delta s \right] \le \int_{t_0}^T D[F(s), G(s)]\Delta s.$$

Definition 2.3. A multivalued function $F : \mathbf{T} \to K_c(\mathbb{R})$ is called regulated provided its regulated sectors exist.

A multivalued function $F : \mathbf{T} \to K_c(\mathbb{R})$ is called rd-continuous provided its rd-continuous sectors exist.

In this paper, the set of rd-continuous multivalued functions $F : \mathbf{D} \subset \mathbf{T} \to K_c(\mathbb{R})$ will be denoted by

$$\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbf{D}) = \mathcal{C}_{rd}(\mathbf{D}, K_c(\mathbb{R})).$$

The set of multivalued functions $F : \mathbf{D} \subset \mathbf{T} \to K_c(\mathbb{R})$ that are Δ_H-differentiable and whose Δ_H-derivative is rd-continuous is denoted by

$$\mathcal{C}_{rd}^1 = \mathcal{C}_{rd}^1(\mathbf{D}) = \mathcal{C}_{rd}^1(\mathbf{D}, K_c(\mathbb{R})).$$

Definition 2.4. A continuous multivalued function $F : \mathbf{T} \to K_c(\mathbb{R})$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbf{T}^k$, $\mathbf{T}^k \backslash D$ is countable and contains no right-scattered elements of $\mathbf{T}$, and F is Δ_H-differentiable at each $t \in D$.

Proposition 2.3 ([29]). (i) *Let $I \subset \mathbb{R}$ is an interval. If $t_0 \in \mathbf{T}$, then $\mathscr{F}$ defined by*

$$\mathscr{F}(t) = X_0 + \int_{t_0}^t F(s)\Delta s, \quad \text{for } t \in I_{\mathbf{T}} \text{ and } X_0 \in K_c(\mathbb{R}),$$

where $F : I_{\mathbf{T}} \to K_c(\mathbb{R})$ is rd-continuous, is Δ_H-differentiable and one has the equality

$$\Delta_H \mathscr{F}(t) = F(t), \quad \text{a.e. on } I_{\mathbf{T}}$$

(ii) *If F is rd-continuous and $t \in \mathbf{T}^k$, then*

$$\int_t^{\sigma(t)} F(s)\Delta s = \mu(t)F(t).$$

3. Comparison results

In this section, we will formulate a comparison theorem for the solutions of SDE (1). As an application of the comparison result, we also prove a global existence result. In what follows, we always assume that $\mathbf{T}$ is the time scale with $t_0 \ge 0$ as the minimal element and has no maximal element. Throughout this paper, unless otherwise mentioned, $U(t, t_0, U_0)$ always stands for the solution of SDE (1) on $J_{\mathbf{T}}$ corresponding the initial value (t_0, U_0).

Theorem 3.1 (*Comparison Result*). *Assume that $F \in \mathcal{C}_{rd}(\Omega, K_c(\mathbb{R}))$, where $\Omega = J_{\mathbf{T}} \times \{U \in K_c(\mathbb{R}) : D[U, U_0] \le b\}$ with $J_{\mathbf{T}} = [t_0, t_0 + a]_{\mathbf{T}}$ and $a, b > 0$, and for $t \in \mathbf{T}$, $U(t) = U(t, t_0, U_0)$, $V(t) = V(t, t_0, U_0) \in K_c(\mathbb{R}^n)$,*

$$D[F(t, U(t)), F(t, V(t))] \le g(t, D[U(t), V(t)]),$$

where $g \in C_{rd}(\Omega_0, \mathbb{R}_+)$ *with* $\Omega_0 = J_{\mathbf{T}} \times \{w \in \mathbb{R} : |w - w_0| \le b\}$ *and* $g(t, w)$ *is nondecreasing in* w *for each* $t \in J_{\mathbf{T}}$*. Moreover, we require that there exists the maximal solution* $r(t, t_0, w_0)$ *of the scalar equation*

$$w^{\Delta}(t) = g(t, w), \qquad w(t_0) = w_0 \ge 0, \quad t \in J_{\mathbf{T}}. \tag{2}$$

Then we have

$$D[U(t), V(t)] \le r(t, t_0, w_0), \quad t \in J_{\mathbf{T}},$$

provided that $D[U_0, V_0] \le w_0$*.*

Proof. Since $U(t), V(t)$ are solutions of SDE (1), the differences $U(s) - U(t), V(s) - V(t)$ exist for small $s - t > 0$. Set $m(t) = D[U(t), V(t)]$ for $t \in \mathbf{T}$. As an application of the properties of Hausdorff metric, we obtain the estimation

$$\begin{aligned} m(s) - m(t) &= D[U(s), V(s)] - D[U(t), V(t)] \\ &\le D[U(s), U(t) + (s-t)F(t, U)] + D[U(t) + (s-t)F(t, U), V(t) + (s-t)F(t, V)] \\ &\quad + D[V(t) + (s-t)F(t, V), V(s)] - D[U(t), V(t)] \\ &\le D[U(s), U(t) + (s-t)F(t, U)] + D[V(t) + (s-t)F(t, V), V(s)] + (s-t)D[F(t, U), F(t, V)] \\ &= D[U(s) - U(t), (s-t)F(t, U)] + D[(s-t)F(t, V), V(s) - V(t)] + (s-t)D[F(t, U), F(t, V)]. \end{aligned}$$

This implies that

$$\frac{m(s) - m(t)}{s - t} \le D\left[\frac{U(s) - U(t)}{s - t}, F(t, U)\right] + D\left[F(t, V), \frac{V(s) - V(t)}{s - t}\right] + D[F(t, U), F(t, V)]. \tag{3}$$

If t is a right-dense point, taking $\overline{\lim}\sup$ as $s \to t^+$, then inequality (3), together with Proposition 2.1(3), yields

$$\begin{aligned} m_+^{\Delta}(t) &= \overline{\lim}\sup_{s \to t^+} \frac{m(s) - m(t)}{s - t} \\ &\le D[\Delta_H U(t), F(t, U)] + D[\Delta_H V(t), F(t, V)] + D[F(t, U), F(t, V)] \\ &= D[F(t, U), F(t, V)], \end{aligned}$$

where $m_+^{\Delta}(t)$ is the right-derivative of $m(t)$. On the other hand, if t is a right-scattered point, let us take $s = \sigma(t)$ in (3). From Proposition 2.1(3) it follows that

$$\begin{aligned} m^{\Delta}(t) &= \frac{m(\sigma(t)) - m(t)}{\mu(t)} \\ &\le D\left[\frac{U(\sigma(t)) - U(t)}{\sigma(t) - t}, F(t, U)\right] + D\left[F(t, V), \frac{V(\sigma(t)) - V(t)}{\sigma(t) - t}\right] + D[F(t, U), F(t, V)] \\ &= D[\Delta_H U(t), F(t, U)] + D[\Delta_H V(t), F(t, V)] + D[F(t, U), F(t, V)] \\ &= D[F(t, U), F(t, V)]. \end{aligned}$$

In addition, the fact that $D[U_0, V_0] \le w_0$ implies that $m(t_0) \le w_0$. Consequently, the comparison theorem for ordinary dynamic equations (see [25, Theorem 5.2]) gives

$$D[U(t), V(t)] \le r(t, t_0.w_0), \quad t \in J_{\mathbf{T}}.$$

This proof is complete. □

Remark 3.1. Assume F satisfies the hypotheses of Theorem 3.1 and $\|F(t, U)\| \le M_0$ on Ω. In addition, g is assumed to satisfy that $g \in C(J_{\mathbf{T}} \times [0, 2b], \mathbb{R}_+)$, $g(t, w) \le M_1$ on $[0, 2b]$, $g(t, 0) \equiv 0$, $g(t, w)$ is increasing in w for each $t \in J_{\mathbf{T}}$ and $w(t) \equiv 0$ is the only solution of

$$w^{\Delta}(t) = g(t, w), \qquad w(t_0) = 0.$$

Then the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F(s, U_n(s))\Delta s, \quad n = 0, 1, 2, \ldots.$$

exist on $I_{\mathbf{T}} = [t_0, t_0 + \alpha]$ with $\alpha = \min(a, \frac{b}{M}]$, $M = \max\{M_0, M_1\}$, as continuous functions and converge uniformly to the unique solution $U(t) = U(t, t_0, U_0)$ of SDE (1) on $I_{\mathbf{T}}$.

The proof of Remark 3.1 exhibits the idea of the comparison result.

Theorem 3.2. *Assume that*

(s1) $F \in \mathcal{C}_{rd}(\mathbf{T} \times K_c(\mathbb{R}), K_c(\mathbb{R}))$ *satisfies*

$$\|F(t, \Phi)\| \leq g(t, \|\Phi\|)$$

for each $(t, \Phi) \in \mathbf{T} \times K_c(\mathbb{R})$*, where* $g \in C_{rd}(\mathbf{T} \times \mathbb{R}_+, \mathbb{R}_+)$ *and* $g(t, w)$ *is nondecreasing in* w *for each* $t \in \mathbf{T}$.

(s2) *The solution* $w(t, t_0, w_0)$ *of the equation*

$$w^{\Delta}(t) = g(t, w(t)), \qquad w(t_0) \geq w_0,$$

exists for $t \in \mathbf{T}$.

If F *is smooth enough to ensure the local existence, then the largest interval of existence of any solution* $U(t, t_0, w_0)$ *of* (1) *with* $\|U_0\| \leq w_0$ *is* $[t_0, \infty)_{\mathbf{T}}$.

Proof. Let us assume that $V(t) = V(t, t_0, U_0)$ is a solution of (1) with $\|U_0\| \leq w_0$ existing on the largest interval $[t_0, \tau)_{\mathbf{T}}$. We will show that τ is infinity. On the contrary, there exists $\beta \leq \tau$ such that $V(t)$ is a solution of (1) existing on interval $[t_0, \beta]_{\mathbf{T}}$ and β cannot be increased. For $t \in [t_0, \beta]_{\mathbf{T}}$, set $m(t) = D[V(t), \{0\}]$. Employing the procedure used in the proof of the Theorem 3.1, we obtain the differential inequality

$$m^{\Delta}(t) \leq g(t, m(t)), \quad t \in [t_0, \beta]_{\mathbf{T}}.$$

Note that $\|U_0\| \leq w_0$, we have

$$D[V(t), \{0\}] \leq r(t, t_0, w_0), \quad t \in [t_0, \beta]_{\mathbf{T}}. \tag{4}$$

Now we select that $t_1, t_2 \in t \in [t_0, \beta]_{\mathbf{T}}$ such that $t_1 < t_2$. By means of the properties of Hausdorff metric, we have

$$\begin{aligned} D[V(t_1), V(t_2)] &= D\left[V(t_1), V(t_1) + \int_{t_1}^{t_2} F(s, U)\Delta s\right] \\ &= D\left[\{0\}, \int_{t_1}^{t_2} F(s, U)\Delta s\right] \leq \int_{t_1}^{t_2} D[\{0\}, F(s, U(s))]\Delta s \\ &\leq \int_{t_1}^{t_2} g(s, D[U(s), \{0\}])\Delta s. \end{aligned}$$

In view of the nondecreasing of g in w and from (4) it follows that

$$\begin{aligned} D[V(t_1), V(t_2)] &\leq \int_{t_1}^{t_2} g(s, r(s, t_0, w_0))\Delta s \\ &= r(t_2, t_0, w_0) - r(t_1, t_0, w_0). \end{aligned}$$

If β is a left-dense point, we allow $t_1, t_2 \to \beta$ in the above relation, since $\lim_{t\to\beta^-} r(t)$ exists and is finite, from the above inequality it follows that the limit of $V(t)$ exists and is finite as t tends to β^- by Cauchy's criterion of convergence. We can define $V(\beta) = \lim_{t\to\beta^-} V(t)$. We observe that a solution can have finite escape time only before left-dense points $t \in \mathbf{T}$, since their neighborhoods contain infinitely many points to the left of t. Hence it is sufficient that we are only allowed to suppose that β is a left-dense point of $\mathbf{T}$. Consequently, we consider $W_0 = V(\beta)$ as a new initial function at $t = \beta$. Then, by the assumption of local existence, there exists a solution $V(t, \beta, W_0)$ of (1) on the interval $[\beta, \gamma]$ with $\gamma > \beta$. This implies that the solution $V(t)$ can be continued beyond β, which contradicts our assumption that β cannot be increased. This completes the proof of the theorem. □

Following the idea in [12, Theorem 3.1], we present the following comparison theorem in terms of Lyapunov-like functions on time scales is very important to investigate the stability criteria of SDE (1). In order to discuss the stability criteria of the solution of SDE (1), we state some notions and definitions.

On the Lyapunov-like function on time scales, Kaymakcalan [21] defined the Dini derivative of the function $V \in C_{rd}(\mathbf{T} \times \mathbb{R}^n, \mathbb{R}_+)$ along the solutions of SDE (1) when we restrict ourselves into single valued mappings $U = u$ and $F = f$ by

$$\begin{aligned} D_{\Delta}^{-}V(t, u) &= \lim_{\mu(t)\to 0} \inf \frac{V(t, u) - V(t - \mu(t), u - \mu(t)f(t, u))}{\mu(t)}, \\ D_{\Delta}^{+}V(t, u) &= \lim_{\mu(t)\to 0} \sup \frac{V(t + \mu(t), u + \mu(t)f(t, u)) - V(t, u)}{\mu(t)}. \end{aligned}$$

Now, to avoid the nonexistence of the above derivative when $\mu(t) \geq h$ (a positive constant), we present a class of new generalized Dini derivatives of the Lyapunov-like function on $\mathbf{T}$ as follows

Definition 3.1. For $A \in \mathcal{C}_{rd}(\mathbf{T}, K_c(\mathbb{R}))$, $t \in \mathbf{T}$ and $V \in C_{rd}(\mathbf{T} \times K_c(\mathbb{R}), \mathbb{R}_+)$, we call $\Delta^r V(t, A)$ and $\Delta_r V(t, A)$ the right upper (ru) and the right lower (rl) derivatives of the function V at $(t, A(t))$, respectively, if

$$\Delta^r V(t, A(t)) = \begin{cases} \dfrac{V(\sigma(t), A(\sigma(t))) - V(t, A(t))}{\mu(t)}, & \sigma(t) > t, \\ \limsup\limits_{s \to t^+} \dfrac{V(s, A(t) + (s-t)F(t, A(t))) - V(t, A(t))}{s-t}, & \sigma(t) = t. \end{cases}$$

$$\Delta_r V(t, A(t)) = \begin{cases} \dfrac{V(\sigma(t), A(\sigma(t))) - V(t, A(t))}{\mu(t)}, & \sigma(t) > t, \\ \liminf\limits_{s \to t^+} \dfrac{V(s, A(t) + (s-t)F(t, A(t))) - V(t, A(t))}{s-t}, & \sigma(t) = t. \end{cases}$$

Similarly, we call $\Delta^l V(t, A(t))$ and $\Delta_l V(t, A(t))$ the left upper (lu) and left lower (ll) derivatives of the function V at $(t, A(t))$, respectively, if

$$\Delta^l V(t, A(t)) = \begin{cases} \dfrac{V(t, A(t)) - V(\rho(t), A(\rho(t)))}{t - \rho(t)}, & t > \rho(t), \\ \limsup\limits_{s \to t^-} \dfrac{V(s, A(t) + (s-t)F(t, A(t))) - V(t, A(t))}{s-t}, & \rho(t) = t. \end{cases}$$

$$\Delta_l V(t, A(t)) = \begin{cases} \dfrac{V(t, A(t)) - V(\rho(t), A(\rho(t)))}{\mu(t)}, & t - \rho(t), \\ \liminf\limits_{s \to t^-} \dfrac{V(s, A(t) + (s-t)F(t, A(t))) - V(t, A(t))}{s-t}, & \rho(t) = t. \end{cases}$$

Theorem 3.3. *Assume that V given as in Definition 3.1 satisfies*

$$\begin{aligned} &\Delta^r V(t, U(t)) \le g(t, \|U(t)\|), \quad t \in \mathbf{T}, \\ &|V(t, U(t)) - V(t, V(t))| \le LD[U(t), V(t)], \quad L \ge 0, t \in \mathbf{T}, \end{aligned}$$

where $g \in C_{rd}(\mathbf{T} \times \mathbb{R}_+, \mathbb{R})$ and $U(t) = U(t, t_0, U_0)$, $V(t) = V(t, t_0, V_0)$. Then, if $U(t)$ is such that $V(t_0, U_0) \le w_0$ on $[t_0, \infty)_{\mathbf{T}}$, we have

$$V(t, U(t)) \le r(t, t_0, w_0), \quad t \in [t_0, \infty)_{\mathbf{T}}, \tag{5}$$

where $r(t, t_0, w_0)$ is the maximal solution of DE (2) *existing on $[t_0, \infty)_{\mathbf{T}}$.*

Proof. Define $m(t) = V(t, U(t))$ so that $m(t_0) = V(t_0, U_0) \le w_0$, $U(t) = U(t, t_0, U_0)$ is any solution of SDE (1) existing on $[t_0, \infty)_{\mathbf{T}}$. Now for $s \in \mathbf{T}$ with $s > t$, by our assumptions it follows that

$$\begin{aligned} m(s) - m(t) &= V(s, U(s)) - V(t, U(t)) \\ &\le V(s, U(s)) - V(s, U(t) + (s-t)F(t, U(t))) + V(s, U(t) + (s-t)F(t, U(t))) - V(t, U(t)) \\ &\le LD[U(s), U(t) + (s-t)F(t, U(t))] + V(s, U(t) + (s-t)F(t, U(t))) - V(t, U(t)). \end{aligned} \tag{6}$$

Since $U(\cdot)$ is the solution of SDE (1), the Hukuhara difference $Z(t) = U(s) - U(t)$ exists for $s, t \in \mathbf{T}$ and small $s - t > 0$. Hence employing the properties of Hausdorff metric, we have

$$\begin{aligned} D[U(s), U(t) + (s-t)F(t, U(t))] &= D[U(t) + Z(t), U(t) + (s-t)F(t, U(t))] \\ &= D[Z(t), (s-t)F(t, U(t))] = D[U(s) - U(t), (s-t)F(t, U(t))]. \end{aligned}$$

This shows that

$$\frac{1}{s-t} D[U(s), U(t) + (s-t)F(t, U(t))] = D\left[\frac{U(s) - U(t)}{s-t}, F(t, U(t))\right].$$

Lending this to (6), we obtain

$$\frac{m(s) - m(t)}{s-t} \le LD\left[\frac{U(s) - U(t)}{s-t}, F(t, U(t))\right] + \frac{V(s, (s-t)F(t, U(t))) - V(t, U(t))}{s-t}.$$

Employing the procedure used in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} \limsup_{s \to t^+} \frac{1}{s-t} D[U(s), U(t) + (s-t)F(t, U(t))] &= \limsup_{s \to t^+} D\left[\frac{U(s) - U(t)}{s-t}, F(t, U(t))\right] \\ &= D[\Delta_H U(t), F(t, U(t))] = 0. \end{aligned}$$

Employing the procedure used in the proof of Theorem 3.1, again, we find from Definition 3.1 and Proposition 2.1(ii)(4) that

$$m_+^{\Delta}(t) = \Delta^r(t, U(t)) \le g(t, m(t)), \quad t \in [t_0, \infty)_{\mathbf{T}}.$$

Consequently, by [24, Theorem 5.2] we arrive at the estimate

$$m(t) \le r(t, t_0, w_0), \quad t \in [t_0, \infty)_{\mathbf{T}},$$

where $r(t, t_0, w_0)$ is the maximal solution of DE (2) existing on $[t_0, \infty)_{\mathbf{T}}$. This proof is complete. □

4. Stability criteria

In order to discuss the stability criteria of the solutions of SDE (1), we state some notions and definitions.

Since the cause of the problem in SDE is due to the requirement of the existence of Hukuhara difference in the SDE, we may need to incorporate the Hukuhara difference in the initial conditions also, in order to match the behaviour of solutions of SDE with the corresponding DE (see (2), for example), we assume, following the idea in [12,13], as a standard hypothesis that the Hukuhara difference exists for any given initial values $U_0, V_0 \in K_c(\mathbb{R})$ such that $U_0 - V_0 = W_0$ is defined. Then we consider the solutions $U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$ of SDE (1).

We are now in a position to formulate the stability criteria for the solution of SDE (1). In this section we assume that $F(t, \{0\}) = \{0\}$ and the solutions are unique and exist for all $t \in \mathbf{T}$ with $t \ge t_0$. Let us first define the stability of the trivial solution and set our notations.

Definition 4.1. Let $U(t) = U(t, t_0, U_0)$. The trivial solution $U(t) \equiv \{0\}$ is said to be

(I) equi-stable if for each $\varepsilon > 0$ and $t_0 \in \mathbf{T}$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|U_0\| < \delta$ implies that

$$\|U(t)\| < \varepsilon, \quad t \in [t_0, \infty)_{\mathbf{T}}. \tag{7}$$

(II) uniformly stable if in (I) the $\delta = \delta(\varepsilon) > 0$ is independent of t_0.

(III) equi-asymptotically stable if (I) holds and for any $\varepsilon > 0$ there exists a $T > 0$ such that (7) holds for all $t \in [t_0 + T, \infty)_{\mathbf{T}}$.

(IV) uniformly asymptotically stable if (II) and (III) hold simultaneously.

Let $S(b) = \{U \in K_c(\mathbb{R}) : \|U\| \le b\}$ for $b > 0$. Let $\mathscr{K}$ be the class consisting of functions $\varphi \in C[[0, b), \mathbb{R}_+]$ such that $\varphi(0) = 0$ and $\varphi(w)$ is increasing in w. For the sake of convenient, the following basic notions are needed:

Definition 4.2. The function $V \in C_{rd}(\mathbf{T} \times K_c(\mathbb{R}), \mathbb{R})$ is said to be positive definite if

(i) $V(t, \{0\}) = 0$ for all $t \in \mathbf{T}$.

(ii) There exists the function $\varphi \in \mathscr{K}$ such that $V(t, A) \ge \varphi(\|A\|)$ for each $(t, A) \in \mathbf{T} \times S(b)$.

V is said to be negative definite if $-V$ is positive definite.

We begin the following result that follows the corresponding result of [25, Theorem 6.2] with appropriate modifications and offers the first stability criteria of this paper.

Theorem 4.1. *Suppose that $F \in \mathcal{C}_{rd}(\mathbf{T} \times K_c(\mathbb{R}), K_c(\mathbb{R}))$, $g \in C_{rd}(\mathbf{T} \times \mathbb{R}_+, \mathbb{R})$ with $g(t, 0) \equiv 0$ for all $t \in \mathbf{T}$. Moreover,*

(i) *There exists V given as in Definition 3.1 such that $V(t, \{0\}) = 0$ for all $t \in \mathbf{T}$ and*

$$\Delta^r V(t, U(t)) \le g(t, \|U(t)\|), \quad t \in \mathbf{T}.$$

$$|V(t, U(t)) - V(t, W(t))| \le LD[U(t), W(t)], \quad L \ge 0,\ t \in \mathbf{T}$$

with $U(t) = U(t, t_0, U_0)$, $W(t) = W(t, t_0, W_0)$.

(ii) *$g(t, w)$ is nondecreasing in w for each $t \in \mathbf{T}$.*

Then the equi-stability properties of the trivial solution of DE (2), *in which $t \in \mathbf{T}$ instead of $t \in J_{\mathbf{T}}$, implies the corresponding stability properties of the trivial solution of* SDE (1).

Proof. Since $V(t_0, \{0\}) = 0$ and $V(t_0, A)$ is continuous with respect to A, for any $\varepsilon > 0$, there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that $\|U_0\| < \delta_1$ implies that $V(t_0, U_0) \le w_0$.

Let the trivial solution of DE (2) be equi-stable. Then, given $\varepsilon > 0$ and $t_0 \ge 0$, there exists a $\delta_2 = \delta_2(t_0, \varepsilon) > 0$ such that

$$0 \le w_0 < \delta_2 \text{ implies } w(t) < \varphi(\varepsilon), \quad t \in \mathbf{T}, \tag{8}$$

where $w(t) = w(t, t_0, w_0)$ is any solution of (2). We claim that with these ε and $\delta = \min\{\delta_1, \delta_2\}$, the trivial solution of SDE (1) is also equi-stable. Suppose that this were false, there would exist a solution $U(t) = U(t, t_0, U_0)$ of SDE (1) with $\|U_0\| < \delta$ and a $t_1 \in \mathbf{T}$, $t_1 > t_0$, such that

$$\varphi(\varepsilon) \le \|U(t_1)\| \quad \text{and} \quad \|U(t)\| < \varphi(\varepsilon), \quad t \in [t_0, t_1)_{\mathbf{T}}.$$

On the other hand, using the inequality (5) at $t = t_1$, we arrive at the contradiction

$$\varphi(\varepsilon) \le \|U(t_1)\| \le r(t_1, t_0, w_0) < \varphi(\varepsilon),$$

which proves our claim. This proof is complete. □

Theorem 4.2. *Suppose that $F \in \mathcal{C}_{rd}(\mathbf{T} \times K_c(\mathbb{R}), K_c(\mathbb{R}))$. Moreover, there exists V given as in Definition 3.1 such that V is positive definite and*

$$\Delta^r V(t, U(t)) \le 0, \quad t \in \mathbf{T}.$$
$$|V(t, U(t)) - V(t, W(t))| \le LD[U(t), W(t)], \quad L \ge 0,\ t \in \mathbf{T},$$

where $U(t) = U(t, t_0, U_0)$, $W(t) = W(t, t_0, U_0)$. Then the trivial solution of SDE (1) *is equi-stable.*

Proof. Define $m(t) = V(t, U(t))$, we have, from the fact that V is positive definite, $m(t) = V(t, U(t)) \ge \varphi(\|U(t)\|)$ for some $\varphi \in \mathcal{K}$. Since $V(t_0, \{0\}) = 0$ and $V(t_0, A)$ is continuous with respect to A, for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ $(\delta \le \varphi(\varepsilon))$ such that $\|U_0\| < \delta$ implies that $V(t_0, U_0) < \varphi(\varepsilon)$, that is, $m(t_0) = V(t_0, U_0) < \varphi(\varepsilon)$.

On the other hand, the equation

$$u^\Delta = 0, \qquad u(t_0) = V(t_0, U_0)$$

has only a solution $u \equiv V(t_0, U_0)$. Now employing the procedure used in the proof of (5), the hypothesis $\Delta^r V(t, A) \le 0$ guarantees

$$m(t) = V(t, U(t)) \le V(t_0, U_0), \quad t \in [t_0, \infty)_\mathbf{T}. \tag{9}$$

Therefore, for all $t \in [t_0, \infty)_\mathbf{T}$, we have

$$\varphi(\|U(t)\|) \le V(t, U(t)) \le V(t_0, U_0) < \varphi(\varepsilon).$$

Note that φ is increasing, we infer U satisfies (7), i.e., the trivial solution of SDE (1) is equi-stable as desired. □

We will next consider the uniform stability criteria.

Theorem 4.3. *Suppose that $F \in \mathcal{C}_{rd}(\mathbf{T} \times K_c(\mathbb{R}), K_c(\mathbb{R}))$. Moreover, there exists V given as in Definition 3.1 such that V is positive definite and*

$$\Delta^r V(t, U(t)) \le 0, \quad t \in \mathbf{T}.$$
$$|V(t, U(t)) - V(t, W(t))| \le LD[U(t), W(t)], \quad L \ge 0,\ t \in \mathbf{T},$$

where $U(t) = U(t, t_0, U_0)$, $W(t) = W(t, t_0, U_0)$. Also, there exists $\psi \in \mathcal{K}$ such that $V(t, A) \le \psi(\|A\|)$ $(t, A) \in \mathbf{T} \times S(b)$. Then the trivial solution of SDE (1) *is uniformly stable.*

Proof. Theorem 4.2 guarantees the stability criteria. we next prove the uniform stability criteria. There exists $\varphi \in \mathcal{K}$ such that

$$\varphi(\|A\|) \le V(t, A) \le \psi(\|A\|), \quad (t, A) \in \mathbf{T} \times S(b).$$

For any $\varepsilon > 0$ satisfying $\psi^{-1}(\varphi(\varepsilon)) < b$, set $\delta = \psi^{-1}(\varphi(\varepsilon))$. Since $\varphi, \psi \in \mathcal{K}$, $\delta = \delta(\varepsilon)$ exists uniquely and is independent of t_0. Hence if $\|U_0\| < \delta$, from (9) it follows that

$$\varphi(\|U(t)\|) \le V(t, U(t)) \le V(t_0, U_0) \le \psi(\|U_0\|) < \psi(\delta), \quad t \in [t_0, \infty)_\mathbf{T}.$$

This yields that

$$\|U(t)\| < \varphi^{-1}(\psi(\varepsilon)) = \varepsilon \quad \text{for all } t \in [t_0, \infty)_\mathbf{T}.$$

Hence the trivial solution of SDE (1) is uniformly stable and the proof is complete. □

The next result provides sufficient conditions for equi-asymptotic stability criteria.

Theorem 4.4. *Let the assumptions of Theorem 4.2 hold except that the estimate $\Delta^r V(t, A) \le 0$ be strengthened to*

$$\Delta^r V(t, A) \le -\phi(\|A\|) \tag{10}$$

for $(t, A) \in \mathbf{T} \times K_c(\mathbb{R})$, where, $\phi \in \mathcal{K}$ is given. Then the trivial solution of SDE (1) *is equi-asymptotically stable.*

Proof. Clearly, Theorem 4.2 guarantees that the trivial solution of SDE (1) is equi-stable. Thus, for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|U_0\| < \delta$ implies that

$$\|U(t)\| < \varepsilon, \quad t \in [t_0, \infty)_\mathbf{T}.$$

From our assumptions it follows that

$$V(t, U(t)) + \int_{t_0}^{t} \phi(\|U(s)\|)\Delta s \le V(t_0, U_0), \quad t \in [t_0, \infty)_\mathbf{T}.$$

Note that $V(t, U(t)) \geq 0$ and $\Delta^r V(t, A) < 0$ guarantees that the function $V_U(t) = V(t, U(t))$ is decreasing in $t \in \mathbf{T}$. It is easy to see that the limit $\lim_{t\to\infty} V_U(t)$ exists. Let α denote the limit.

We next prove that $\alpha = 0$. On the contrary, we have $\alpha > 0$. By means of the decrease of $V_U(t)$, we have $V_U(t) \geq \alpha > 0$ for all $t \in [t_0, \infty)_{\mathbf{T}}$. On the other hand, from the continuity of $V_U(t) = V(t, U(t))$ with respect to U and $V(t, 0) = 0$, there exists a positive constant $\xi > 0$ such that $\|U(t)\| > \xi$ for each $t \in [t_0, \infty)_{\mathbf{T}}$. This, together with $\psi \in \mathscr{K}$, yields that

$$V(t, U(t)) \leq V(t_0, U_0) - \int_{t_0}^{t} \phi(\|U(s)\|)\Delta s \leq V(t_0, U_0) - \phi(\xi)(t - t_0), \quad t \in [t_0, \infty)_{\mathbf{T}}.$$

Let $t \in \mathbf{T}$ be large enough, we obtain that $V_U(t) \leq 0$, a contradiction. Hence $\alpha = 0$, that is, $\lim_{t\to\infty} V_U(t) = 0$.

Finally, we will prove that $\lim_{t\to\infty} \|U(t)\| = 0$. If this were false, there would exists positive number $\varepsilon_0 > 0$ such that, for any natural number k, $\|U(t_k)\| > \varepsilon_0$ for some $t_k \in \mathbf{T}$ with $t_k \geq k$. From this, combining the fact that V is positive definite, there would exist $\varphi \in \mathscr{K}$ such that $V(t_k, U(t_k)) \geq \varphi(\|U(t_k)\|) \geq \varphi(\varepsilon_0) > 0$ for $k = 1, 2, \ldots$. This contradicts $\lim_{t\to\infty} V(t, U(t)) = 0$. Hence $\lim_{t\to\infty} \|U(t)\| = 0$ which guarantees that the trivial solution of SDE (1) is equi-asymptotically stable and the proof is complete. □

The next theorem presents the sufficient conditions for the uniformly asymptotic stability.

Theorem 4.5. *If the assumptions of Theorem 4.3 hold except that the estimate $\Delta^r V(t, A) \leq 0$ be strengthened to (10), then the trivial solution of* SDE (1) *is uniformly asymptotically stable.*

Proof. Theorem 4.3 guarantees that the uniformly stability follows. By our assumptions, there exist functions $\phi, \varphi, \psi \in \mathscr{K}$ such that

$$\varphi(\|A\|) \leq V(t, A) \leq \psi(\|A\|), \quad (t, A) \in \mathbf{T} \times S(b), \quad \text{and} \tag{11}$$

$$\Delta^r V(t, U(t)) \leq -\phi(\|U(t)\|) \leq -\phi(\psi^{-1}(V(t, U(t)))) < 0. \tag{12}$$

(12) shows

$$\frac{\Delta^r V_U(t)}{\phi(\psi^{-1}(V_U(t)))} \leq -1.$$

This yields that

$$\int_{V_U(t_0)}^{V_U(t)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) \leq -(t - t_0).$$

i.e.

$$\int_{V_U(t)}^{V_U(t_0)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) \geq (t - t_0).$$

From (11) it follows that $V_U(t_0) \leq \psi(\|U_0\|) \leq \psi(b)$. For any $\varepsilon > 0$ ($\varepsilon < b$), from (11) we have

$$\begin{aligned}\int_{\varphi(\|U(t)\|)}^{\psi(b)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) &= \int_{\varphi(\|U(t)\|)}^{\psi(\varepsilon)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) + \int_{\varphi(\varepsilon)}^{\psi(b)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) \\ &\geq \int_{V_U(t)}^{V_U(t_0)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) \geq (t - t_0).\end{aligned}$$

Let us take $T = T(\varepsilon, b) > \int_{\varphi(\varepsilon)}^{\psi(b)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s)$, then T is independent of t_0 and U_0 and

$$\begin{aligned}\int_{\varphi(\|U(t)\|)}^{\psi(b)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) &\geq t - t_0 - \int_{\varphi(\varepsilon)}^{\psi(b)} \frac{1}{\phi(\psi^{-1}(V_U(s)))} \Delta V_U(s) \\ &> t - t_0 - T \geq 0, \quad t \in (t_0 + T, \infty)_{\mathbf{T}}.\end{aligned}$$

This reduces that $\varphi(\|U(t)\|) < \varphi(\varepsilon)$ for all $t \in (t_0 + T, \infty)_{\mathbf{T}}$. In view of the monotonicity of φ we obtain $\|U(t)\| < \varepsilon$ for all $t \in (t_0 + T, \infty)_{\mathbf{T}}$. This guarantees the uniformly asymptotic stability criteria as desired and the proof is complete. □

Finally, we consider the unstable criteria of of the trivial solution of SDE (1).

Theorem 4.6. *Suppose that V given as in Definition 3.1 such that $V(t, \{0\}) = 0$ and for any $c > 0$ there exists $A \in S(c)$ such that $V(t, A) > 0$, where $t \in \mathbf{T}$. Moreover, if $\Delta^r V(t, A)$ is positive definite, also, there exists $\psi \in \mathscr{K}$ such that $V(t, A) \leq \psi(\|A\|)$ $(t, A) \in \mathbf{T} \times S(b)$, then the trivial solution of* SDE (1) *is unstable.*

Proof. For any $\delta > 0$, by our assumptions, there exists $W_0 \in S(\delta)$ (i.e. $\|W_0\| < \delta$) such that $V(t_0, W_0) > 0$. If the trivial solution of SDE (1) is stable, then for any $\varepsilon > 0$ ($\varepsilon < b$), there exists $\delta > 0$ such that $\|W_0\| < \delta$ implies $\|U(t)\| < \varepsilon$ for each $t \in [t_0, \infty)_{\mathbf{T}}$, where $U(t) = U(t, t_0, W_0)$. Since $\Delta^r V(t, U)$ is positive definite, $V(t, U(t))$ is increasing. We have

$$V(t, U(t)) \geq V(t_0, W_0) > 0, \quad \text{for all } t > t_0.$$

This yields that $\psi(\|U(t)\|) \geq V(t, U(t)) \geq V(t_0, W_0) > 0$, namely,

$$\|U(t)\| \geq \psi^{-1}(V(t_0, W_0)) := \alpha > 0.$$

Again, applying the hypothesis that $\Delta^r V(t, U)$ is positive definite, there exists $\varphi \in \mathscr{K}$ such that

$$\Delta^r V(t, U(t)) \geq \varphi(\|U(t)\|)$$

for $(t, U(t)) \in \mathbf{T} \times K_c(\mathbb{R})$. Integrating this inequality with respect to $t > t_0$ yields

$$V(t, U(t)) \geq V(t_0, W_0) + \int_{t_0}^{t} \varphi(\|U(s)\|)\Delta s \geq V(t_0, W_0) + \varphi(\alpha)(t - t_0).$$

By means of $\|U(t)\| < \varepsilon$, we have

$$\psi(\varepsilon) \geq V(t_0, W_0) + \varphi(\alpha)(t - t_0).$$

This a contradiction for t approaching infinity. Hence the trivial solution of SDE (1) is unstable. This proof is complete. □

5. Examples

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^k.$$

For given regressive functions p, q, Bohner and Peterson in [18] defined the delta exponential function $e_p(\cdot, s)$ as the unique solution of the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$ with $s \in \mathbb{T}$. Furthermore, the "circle plus" and "circle minus" are defined as, respectively,

$$p \oplus q = p + q + \mu pq, \qquad p \ominus q = \frac{p - q}{1 + \mu q} \quad \text{and} \quad \ominus p = 0 \ominus p.$$

It is easy to check that $p \oplus (\ominus q) = p \ominus q$, $\ominus(\ominus p) = p$, $p \ominus q = \ominus(q \ominus p)$ and $p \ominus p = 0$.

If $p \equiv 1$, let $\ominus = \ominus 1$, $e(t, s) = e_1(t, s)$ for $t, s \in \mathbf{T}$. We will start with a simple example of a SDE to illustrate our approach, which is easy to be solved analytically.

Example 5.1. Let us consider the DE

$$u^\Delta = \ominus u, \qquad u(0) = u_0 \in \mathbb{R}, \quad 0 \in \mathbf{T}$$

and the corresponding SDE

$$\Delta_H U = \ominus U, \qquad U(0) = U_0 \in K_c(\mathbb{R}). \tag{13}$$

By means of Proposition 2.3 and the properties of the Hausdorff metric we see easily that the solution of (13) is unique if it exists. We next prove that the values of the solution of (13) are interval function. In fact, let $U = [u_1, u_2]$ with

$$u_1(t) = \frac{1}{2}[u_{10} + u_{20}]e(0, t) + \frac{1}{2}[u_{10} \ominus u_{20}]e(t, 0),$$
$$u_2(t) = \frac{1}{2}[u_{10} + u_{20}]e(0, t) + \frac{1}{2}[u_{20} \ominus u_{10}]e(t, 0).$$

In virtue of [18, Theorem 2.36], we have

$$u_1^\Delta(t) = \ominus\frac{1}{2}[u_{10} + u_{20}]e(0, t) + \frac{1}{2}[u_{10} \ominus u_{20}]e(t, 0) = \ominus u_2(t),$$
$$u_2^\Delta(t) = \ominus\frac{1}{2}[u_{10} + u_{20}]e(0, t) + \frac{1}{2}[u_{20} \ominus u_{10}]e(t, 0) = \ominus u_1(t).$$

This shows that the vector function (u_1, u_2) is the solution of the system of equations

$$u_1^\Delta(t) = \ominus u_2(t), \qquad u_1(0) = u_{10},$$
$$u_2^\Delta(t) = \ominus u_1(t), \qquad u_2(0) = u_{20}.$$

Conclusively, $U = [u_1, u_2]$ is a solution of (13) with the initial value $U_0 = [u_{10}, u_{20}]$.

Conclusion 1. Given $U_0 \in K_c(\mathbb{R})$, let us choose $V(t, A(t)) = \|A(t)\|$ for $t \in \mathbf{T}$. Then, for $U(t) = U(t, t_0, U_0)$, a solution of (13) corresponding the initial value $(0, U_0)$, we have

$$\Delta^r V(t, U(t)) = \begin{cases} \dfrac{\|U(\sigma(t))\| - \|U(t)\|}{\mu(t)}, & \sigma(t) > t, \\ \limsup\limits_{s\to t^+} \dfrac{\|U(t) + (s-t)(\ominus U(t))\| - \|U(t)\|}{s-t} = -\dfrac{1}{1+\mu(t)}\|U(t)\| = -\|U(t)\|, & \sigma(t) = t. \end{cases}$$

Now, let us take $g(t, w) = \frac{1}{1+\mu(t)} w$. If $t \in \mathbf{T}$ is right dense, then $\Delta^r V(t, U(t)) = -\|U(t)\| \le g(t, \|U(t)\|)$. If t is right scattered, then $\Delta^r V(t, U(t)) = \frac{\|U(\sigma(t))\| - \|U(t)\|}{\mu(t)} \le \frac{\|U(\sigma(t)) - U(t)\|}{\mu(t)} = \|\Delta_H U(t)\| = \| \ominus U(t)\| = \frac{1}{1+\mu(t)}\|U(t)\| = g(t, \|U(t)\|)$. From Theorem 4.1 it follows that the equi-stability properties of the trivial solution of the DE (2) with $g(t, w) = \frac{1}{1+\mu(t)} w$ and $w_0 = \|U_0\|$ implies the corresponding stability properties of the trivial solution of SDE (13).

Conclusion 2. By the above discussion we obtain assuming $u_{10} \ne -u_{20}$ that

$$U(t) = \frac{1}{2}[u_{10} - u_{20}, u_{20} - u_{10}]\, e(t, 0) + \frac{1}{2}[u_{10} + u_{20}, u_{20} + u_{10}]\, e(0, t).$$

By the analogical argument as in [12], for any general initial value U_0, the solution of SDE (13) contains both the desired and the undesired parts compared to the solution of the corresponding DE. Let us choose the appropriate initial value $U_0 = [u_{10}, u_{20}]$, say $U_0 = [c, c]$ for some real number c, such that the term with $e(t, 0)$ is eliminated and only the desirable part of the solution compared with the DE is retained. In this case, note that $1 + \mu(t) > 0$, from [18, Theorem 2.36(viii) and Theorem 2.48(1)] we have $e(0, t) > 0$ and $e^{\Delta}(0, \cdot) < 0$. Hence $\Delta^r V(t, U(t)) \le 0$ with $V(t, A(t)) = \|A(t)\|$. Now Theorem 4.2 guarantees that the trivial solution of SDE (13) is equi-stable.

Let $\psi \in \mathscr{K}$ with $\psi(w) = 2w$ for $w \in \mathbb{R}_+$. Then Theorem 4.3 guarantees that the trivial solution of SDE (13) is uniformly stable.

From the above discussion it follows that $\|U(t)\| = |c|e(0, t) > 0$ and $\|U(t)\|$ is decreasing. Now we take

$$\phi(t) = \begin{cases} \dfrac{\|U(t)\| - \|U(\sigma(t)\|)}{\mu(t)}, & \sigma(t) > t, \\ \dfrac{1}{2}\|U(t)\|, & \sigma(t) = t. \end{cases}$$

Then $\phi \in \mathscr{K}$ and the inequality (10) holds. Therefore, Theorem 4.5 guarantees that the trivial solution of SDE (13) is uniformly asymptotically stable.

Example 5.2. Consider the SDE as

$$\Delta_H U = \lambda(t)U, \qquad U(0) = U_0, \tag{14}$$

which is generated by

$$u^{\Delta} = \lambda(t)u, \qquad u(0) = u_0,$$

where $\lambda(t) > 0$ is a real-valued function from $\mathbf{T}_+ =: \mathbb{R}_+ \cap \mathbf{T}$ into $\mathbb{R}$ such that $\lambda \in L^1(\mathbf{T})$ is rd-continuous and $1 + \mu(t)\lambda(t) > 0$ for all $t \in \mathbf{T}$, then we see that, with similar computation,

$$U(t) = U_0 e_{\lambda}(t, 0),\ t \in \mathbf{T}_+,$$

is the unique solution of (14) corresponding the initial value $(0, U_0)$ for any $U_0 \in K_c(\mathbb{R})$. Note that $e_{\lambda}(0, 0) = 1$ and $e_{\lambda}^{\Delta}(t, 0) = \lambda(t)e_{\lambda}(t, 0) > 0$, we obtain that $e_{\lambda}(t, 0)$ is increasing and $e_{\lambda}(t, 0) \ge 1$ on $\mathbf{T}_+$.

Let $V(t, A) = \|A\|$ for $A \in K_c(\mathbb{R})$. Then $V(t, U(t)) = \|U_0\| e_{\lambda}(t, 0)$ if $U(t)$ is a solution of (14). It is easy to see that the conditions of Theorem 4.6 are satisfied if $\|U_0\| > 0$. consequently, the trivial solution of SDE (14) is unstable.

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