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A note on the fractional Cauchy problems with nonlocal initial conditions[☆]

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ABSTRACT

Of concern is the Cauchy problems for fractional integro-differential equations with nonlocal initial conditions. Using a new strategy in terms of the compactness of the semigroup generated by the operator in the linear part and approximating technique, a new existence theorem for mild solutions is established. An application to a fractional partial integro-differential equation with a nonlocal initial condition is also considered.

1. Introduction

Since the differential equations involving fractional derivatives in time, compared with those of integer order in time, are more realistic to describe many phenomena in nature (for instance, to describe the memory and hereditary properties of various materials and processes), the study of such equations has become an object of extensive study during recent years (see, e.g., [1–10] and references therein).

On the other hand, the nonlocal Cauchy problems, in many cases, take much better effect in applications than the traditional problems with a local initial datum. For more detailed information about the importance of nonlocal initial conditions in applications, we refer to, e.g., [11–17].

Let $-A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$ and let $0 \in \rho(A)$. Denote by X_α the Banach space $D(A^\alpha)$ endowed with the graph norm

$$\|u\|_\alpha = \|A^\alpha u\| \quad \text{for } u \in X_\alpha.$$

Consider the following Cauchy problem for fractional integro-differential equations in X_α with nonlocal initial conditions

$$\begin{cases} {}^cD_t^\beta u(t) + Au(t) = f(t, u(t)) + \displaystyle\int_0^t K(t-s)g(s, u(s))\mathrm{d}s, \quad t \in [0, T], \\ u(0) = H(u) \end{cases} \tag{1.1}$$

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where ${}^cD_t^\beta$, $0 < \beta < 1$, is the Caputo fractional derivative of order β, $K : [0, T] \to \mathbb{R}^+$ is a continuous function, and $f,\ g : [0, T] \times X_\alpha \to X$, $H : C([0, T]; X_\alpha) \to X_\alpha$ are given operators to be specified later. As can be seen, H constitutes a nonlocal condition.

In some existing articles, the fractional Cauchy problems with nonlocal initial conditions were treated under the hypothesis that the nonlocal item is compact or Lipschitz continuous. To make things more applicable, in this work we will prove the existence of mild solutions to (1.1) under a much weaker hypothesis in which H has not Lipschitz continuity nor the compactness. More precisely, H has only continuity and some growth condition. A new strategy which relies on the compactness of the semigroup generated by A and approximating technique is used to obtain the existence results. The theorems formulated are extensions of many previous results on the fractional Cauchy problems with nonlocal initial conditions.

The paper is organized as follows. In Section 2, some required notations, definitions and lemmas are given. In Section 3, we present our main results. An example in Section 3 is given to illustrate our abstract results.

2. Results and proofs

Throughout this paper, let $C([0, T]; X_\alpha)$ be the Banach space of all continuous functions from $[0, T]$ into X_α with the norm

$$|u|_\alpha = \sup\{\|u(t)\|_\alpha,\ t \in [0, T]\}.$$

$\mathcal{L}(X)$ stands for the Banach space of all linear and bounded operators on X. Let

$$M = \sup\{\|T(t)\|_{\mathcal{L}(X)},\ t \in [0, \infty)\}.$$

For any $r > 0$, write

$$\Omega_r = \{u \in C([0, T]; X_\alpha);\ |u|_\alpha \le r\}.$$

The following are basic properties of $A^\alpha (0 \le \alpha < 1)$.

Theorem 2.1 (*[18], pp. 69–75*)**.**

(a) *$T(t) : X \to X_\alpha$ for each $t > 0$, and $A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in X_\alpha$ and $t \ge 0$.*
(b) *$A^\alpha T(t)$ is bounded on X for every $t > 0$ and there exist $M_\alpha > 0$ and $\delta > 0$ such that*

$$\|A^\alpha T(t)\|_{\mathcal{L}(X)} \le \frac{M_\alpha}{t^\alpha} e^{-\delta t}.$$

(c) *$A^{-\alpha}$ is a bounded linear operator on X with $D(A^\alpha) = Im(A^{-\alpha})$.*
(d) *If $0 < \alpha_1 \le \alpha_2$, then $X_{\alpha_2} \hookrightarrow X_{\alpha_1}$.*

Lemma 2.1 ([19])**.** *The restriction of $T(t)$ to X_α is exactly the part of $T(t)$ in X_α and is an immediately compact semigroup in X_α, and hence it is immediately norm-continuous.*

For $x \in X$, define two families $\{\mathcal{S}_\beta(t)\}_{t\ge 0}$ and $\{\mathcal{P}_\beta(t)\}_{t\ge 0}$ of operators by

$$\mathcal{S}_\beta(t)x = \int_0^\infty \Psi_\beta(s)T(t^\beta s)x\mathrm{d}s, \qquad \mathcal{P}_\beta(t)x = \int_0^\infty \beta s\Psi_\beta(s)T(t^\beta s)x\mathrm{d}s, \quad 0 < \beta < 1,$$

where

$$\Psi_\beta(s) = \frac{1}{\pi\beta}\sum_{n=1}^{\infty}(-s)^{n-1}\frac{\Gamma(1+\beta n)}{n!}\sin(n\pi\beta), \quad s \in (0, \infty)$$

is the function of Wright type defined on $(0, \infty)$ which satisfies

$$\Psi_\beta(s) \ge 0,\ s \in (0, \infty), \quad \int_0^\infty \Psi_\beta(s)\mathrm{d}s = 1, \quad \text{and}$$
$$\int_0^\infty s^\zeta \Psi_\beta(s)\mathrm{d}s = \frac{\Gamma(1+\zeta)}{\Gamma(1+\beta\zeta)}, \quad \zeta \in [0, 1].$$

The following lemma follows from the results in [20,21].

Lemma 2.2. *The following properties are valid.*

(1) *For every $t \ge 0$, $\mathcal{S}_\beta(t)$ and $\mathcal{P}_\beta(t)$ are linear and bounded operators on X, i.e.,*

$$\|\mathcal{S}_\beta(t)x\| \le M\|x\|, \qquad \|\mathcal{P}_\beta(t)x\| \le \frac{\beta M}{\Gamma(1+\beta)}\|x\| \quad \text{for all } x \in X \text{ and } 0 \le t < \infty.$$

(2) *For every* $x \in X$, $t \to \mathcal{S}_\beta(t)x$, $t \to \mathcal{P}_\beta(t)x$ *are continuous functions from* $[0, \infty)$ *into* X.
(3) $\mathcal{S}_\beta(t)$ *and* $\mathcal{P}_\beta(t)$ *are compact operators on* X *for* $t > 0$.
(4) *For all* $x \in X$, $\|A^\alpha \mathcal{P}_\beta(t)x\| \le C_\alpha t^{-\alpha\beta}\|x\|$, *where* $C_\alpha = \frac{M_\alpha \beta \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}$.
(5) *The restriction of* $\mathcal{S}_\beta(t)$ *to* X_α *and the restriction of* $\mathcal{P}_\beta(t)$ *to* X_α *are norm-continuous.*

We can also prove the following compactness criterion.

Lemma 2.3. *For every* $t > 0$, *the restriction of* $\mathcal{S}_\beta(t)$ *to* X_α *and the restriction of* $\mathcal{P}_\beta(t)$ *to* X_α *are compact operators in* X_α.

Proof. First consider the restriction of $\mathcal{S}_\beta(t)$ to X_α. For any $r > 0$ and $t > 0$, it is sufficient to show that the set $\{\mathcal{S}_\beta(t)u;\ u \in B_r\}$ is relatively compact in X_α, where $B_r := \{u \in X_\alpha;\ \|u\|_\alpha \le r\}$.

Since by Lemma 2.1, the restriction of $T(t)$ to X_α is compact for $t > 0$ in X_α, for each $t > 0$ and $\varepsilon \in (0, t)$,

$$\left\{\int_\varepsilon^\infty \Psi_\beta(s)T(t^\beta s)u\mathrm{d}s;\ u \in B_r\right\} = \left\{T(t^\beta \varepsilon)\int_\varepsilon^\infty \Psi_\beta(s)T(t^\beta s - t^\beta \varepsilon)u\mathrm{d}s;\ u \in B_r\right\}$$

is relatively compact in X_α. Also, for every $u \in B_r$, as

$$\int_\varepsilon^\infty \Psi_\beta(s)T(t^\beta s)u\mathrm{d}s \to \mathcal{S}_\beta(t)u, \quad \varepsilon \to 0$$

in X_α, we conclude, using the total boundedness, that the set $\{\mathcal{S}_\beta(t)u;\ u \in B_r\}$ is relatively compact, which implies that the restriction of $\mathcal{S}_\beta(t)$ to X_α is compact. The same idea can be used to prove that the restriction of $\mathcal{P}_\beta(t)$ to X_α is also compact. □

Based on an overall observation of the previous related literature, in this paper we adopt the following definition of mild solution of (1.1).

Definition 2.1. By a mild solution of (1.1), we mean a function $u \in C([0, T]; X_\alpha)$ satisfying

$$u(t) = \mathcal{S}_\beta(t)H(u) + \int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)\left(f(s, u(s)) + \int_0^s K(s-\tau)g(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s$$

for $t \in [0, T]$.

For the sake of convenience, we write

$$\widetilde{k} := \int_0^T K(t)\mathrm{d}t, \qquad C_{\alpha,\gamma} := \left(\frac{1-\gamma}{(1-\alpha)\beta - \gamma}\right)^{1-\gamma}.$$

Here, we will obtain mild solutions under the following assumptions.

(H$_1$) $f,\ g : I \times X_\alpha \to X$ are continuous, for some $r > 0$ there exist a constant $\gamma \in [0, \beta(1-\alpha))$ and functions $\varphi_r(\cdot) \in L^{1/\gamma}(0, T; \mathbb{R}^+)$, $\phi_r(\cdot) \in L^\infty(0, T; \mathbb{R}^+)$ such that for all $t \in [0, T]$ and $x \in X_\alpha$ satisfying $\|x\|_\alpha \le r$,

$$\|f(t, x)\| \le \varphi_r(t), \qquad \|g(t, x)\| \le \phi_r(t),$$

and

$$\liminf_{r\to+\infty} \frac{\|\varphi_r\|_{L^{1/\gamma}(0,T)}}{r} = \sigma_1 < \infty, \qquad \liminf_{r\to+\infty} \frac{\|\phi_r\|_{L^\infty(0,T)}}{r} = \sigma_2 < \infty.$$

(H$_2$) (i) $H : C([0, T]; X_\alpha) \to X_\alpha$ is continuous, there exists a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $u \in \Omega_r$,

$$\|H(u)\|_\alpha \le \Phi(r), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\Phi(r)}{r} = \mu < \infty.$$

(ii) There is a $\eta \in (0, T)$ such that for any $u, w \in C([0, T]; X_\alpha)$ satisfying $u(t) = w(t)$ $(t \in [\eta, T])$, $H(u) = H(w)$.

Remark 2.1. Let us note that (H$_2$) (ii) is the case when the values of the solution $u(t)$ for t near zero do not affect $H(u)$. A typical example of the operator H is that $H(u) = \sum_{i=1}^p C_i u(t_i)$, where C_i $(i = 1, \dots, p)$ are given constants and $0 < t_1 < \cdots < t_{p-1} < t_p < +\infty$ $(p \in \mathbb{N})$, which is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

We are now ready to state our main result.

Theorem 2.2. *Let* (H$_1$) *and* (H$_2$) *hold. Then Cauchy problem* (1.1) *has at least one mild solution provided that*

$$M\mu + C_\alpha C_{\alpha,\gamma}\sigma_1 T^{(1-\alpha)\beta-\gamma} + \frac{C_\alpha \widetilde{k}\sigma_2 T^{(1-\alpha)\beta}}{(1-\alpha)\beta} < 1. \tag{2.1}$$

Proof. We proceed in three steps.

Step 1. Let $m \geq 1$ be fixed. Consider the Cauchy problem of the form

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + f(t, u(t)) + \int_0^t K(t-s)g(s, u(s))\mathrm{d}s, & t \in [0, T], \\ u(0) = T\left(\frac{1}{m}\right)H(u). \end{cases} \tag{2.2}$$

With the help of (H_1), (H_2)(i) and (2.1), we prove that for every $m \geq 1$, Cauchy problem (2.2) has at least a mild solution u_m.

To this end, we define an operator on $C([0, T]; X_\alpha)$ by

$$(\Gamma^\beta u)(t) = \mathcal{S}_\beta(t)T\left(\frac{1}{m}\right)H(u) + \int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)\left(f(s, u(s)) + \int_0^s K(s-\tau)g(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0, T].$$

Then, it is sufficient to show that Γ^β has a fixed point. Note first that Γ^β is well defined. In what follows, we prove that there is a positive number k_0 such that Γ^β maps Ω_{k_0} into itself. If this is not the case, then for each $k > 0$, there would exist $u_k \in \Omega_k$ and $t_k \in [0, T]$ such that $\|(\Gamma^\beta u_k)(t_k)\|_\alpha > k$. Thus, we see, from Lemma 2.2(4), (H_1), (H_2) (i) and the Hölder inequality, that

$$\begin{aligned} k &< \|(\Gamma^\beta u_k)(t_k)\|_\alpha \\ &\leq \left\|\mathcal{S}_\beta(t_k)T\left(\frac{1}{m}\right)H(u_k)\right\|_\alpha + \int_0^{t_k}(t_k-s)^{\beta-1}\left\|\mathcal{P}_\beta(t_k-s)\left(f(s, u_k(s)) + \int_0^s K(s-\tau)g(\tau, u_k(\tau))\mathrm{d}\tau\right)\right\|_\alpha \mathrm{d}s \\ &\leq \left\|\mathcal{S}_\alpha(t_k)T\left(\frac{1}{m}\right)\right\|_{\mathcal{L}(X)}\|H(u_k)\|_\alpha + \int_0^{t_k}(t_k-s)^{\beta-1}\|\mathcal{P}_\beta(t_k-s)\|_\alpha\left\|f(s, u_k(s)) + \int_0^s K(s-\tau)g(\tau, u_k(\tau))\mathrm{d}\tau\right\|\mathrm{d}s \\ &\leq M\|H(u_k)\|_\alpha + C_\alpha\int_0^{t_k}(t_k-s)^{\beta-1-\alpha\beta}(\varphi_k(s) + \widetilde{k}\|\phi_k\|_{L^\infty(0,T)})\mathrm{d}s \\ &\leq M\Phi(k) + C_\alpha C_{\alpha,\gamma}T^{(1-\alpha)\beta-\gamma}\|\varphi_k\|_{L^{1/\gamma}(0,T)} + \frac{C_\alpha\widetilde{k}T^{(1-\alpha)\beta}}{(1-\alpha)\beta}\|\phi_k\|_{L^\infty(0,T)}. \end{aligned}$$

Dividing on both sides by k and taking the lower limit as $k \to \infty$, we have

$$1 \leq M\mu + C_\alpha C_{\alpha,\gamma}\sigma_1 T^{(1-\alpha)\beta-\gamma} + \frac{C_\alpha\widetilde{k}\sigma_2 T^{(1-\alpha)\beta}}{(1-\alpha)\beta}.$$

This contradicts (2.1).

Next, we prove that Γ^β is continuous on Ω_{k_0}. Taking $u_1, u_2 \in \Omega_{k_0}$, we note, from ($H_1$), that

$$\begin{aligned} \int_0^t (t-s)^{\beta-1-\alpha\beta}\|f(s, u_1(s)) - f(s, u_2(s))\|\mathrm{d}s &\leq 2\int_0^t (t-s)^{\beta-1-\alpha\beta}\varphi_{k_0}(s)\mathrm{d}s \\ &\leq 2C_{\alpha,\gamma}T^{(1-\alpha)\beta-\gamma}\|\varphi_{k_0}\|_{L^{1/\gamma}(0,T)}, \end{aligned}$$

and

$$\begin{aligned} &\int_0^t (t-s)^{\beta-1-\alpha\beta}\left\|\int_0^s K(s-\tau)g(\tau, u_1(\tau))\mathrm{d}\tau - \int_0^s K(s-\tau)g(\tau, u_2(\tau))\mathrm{d}\tau\right\|\mathrm{d}s \\ &\quad\leq 2\widetilde{k}\|\phi_{k_0}\|_{L^\infty(0,T)}\int_0^t (t-s)^{\beta-1-\alpha\beta}\mathrm{d}s \\ &\quad\leq \frac{2\widetilde{k}T^{(1-\alpha)\beta}}{(1-\alpha)\beta}\|\phi_{k_0}\|_{L^\infty(0,T)}. \end{aligned}$$

This together with the Lebesgue dominated convergence theorem gives that

$$\begin{aligned} \|(\Gamma^\beta u_1)(t) - (\Gamma^\beta u_2)(t)\|_\alpha \leq &\left\|\mathcal{S}_\beta(t)T\left(\frac{1}{m}\right)\right\|_{\mathcal{L}(X)}\|H(u_1) - H(u_2)\|_\alpha \\ &+ \int_0^t (t-s)^{\beta-1}\|\mathcal{P}_\beta(t-s)\|_\alpha\|f(s, u_1(s)) - f(s, u_2(s))\|\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t (t-s)^{\beta-1} \|\mathcal{P}_\beta(t-s)\|_\alpha \left\| \int_0^s K(s-\tau) g(\tau, u_1(\tau)) \mathrm{d}\tau \right. \\
&\left. - \int_0^s K(s-\tau) g(\tau, u_2(\tau)) \mathrm{d}\tau \right\| \mathrm{d}s \\
\leq\ & M \|H(u_1) - H(u_2)\|_\alpha + C_\alpha \int_0^t (t-s)^{\beta-1-\alpha\beta} \|f(s, u_1(s)) - f(s, u_2(s))\| \mathrm{d}s \\
&+ C_\alpha \int_0^t (t-s)^{\beta-1-\alpha\beta} \left\| \int_0^s K(s-\tau) g(\tau, u_1(\tau)) \mathrm{d}\tau - \int_0^s K(s-\tau) g(\tau, u_2(\tau)) \mathrm{d}\tau \right\| \mathrm{d}s \\
\to\ & 0, \quad \text{as } u_1 \to u_2 \text{ in } \Omega_{k_0},
\end{aligned}
$$

in view of the continuity of operator H and operators $f,\ g$ with respect to second variables. That is to say that Γ^β is continuous.

In what follows we show that Γ^β is a compact operator on Ω_{k_0}. For this purpose we introduce the following two operators:

$$
(\Gamma_1^\beta u)(t) = \mathcal{S}_\beta(t) T\left(\frac{1}{m}\right) H(u), \quad t \in [0, T],
$$

$$
(\Gamma_2^\beta u)(t) = \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s) \left(f(s, u(s)) + \int_0^s K(s-\tau) g(\tau, u(\tau)) \mathrm{d}\tau \right) \mathrm{d}s, \quad t \in [0, T].
$$

Since the compactness of $T(t)$ for every $t > 0$ implies that the restriction of $T(t)$ to X_α is an immediately compact semigroup in X_α (see Lemma 2.1), for every $m \geq 1$ we can deduce, by the boundedness of $\mathcal{S}_\beta(t)$ in X_α and (H_2)(i), that

Γ_1^β mapping Ω_{k_0} into $C([0, T]; X_\alpha)$ is compact.

Moreover, by (H_1), Lemmas 2.1 and 2.2, an argument similar to that in [5, Theorem 3.2] shows that for all $t \in (0, T]$,

the set $\{(\Gamma_2^\beta u)(t);\ u \in \Omega_{k_0}\}$ is relatively compact in X_α

and the set $\{(\Gamma_2^\beta u)(\cdot);\ u \in \Omega_{k_0}\}$ is equicontinuous on $[0, T]$. Hence by the Arzela–Ascoli theorem one has that Γ_2^β is compact. Consequently, we have proved that

$\Gamma^\beta = \Gamma_1^\beta + \Gamma_2^\beta$ is compact.

At the end of this step, applying Schauder fixed point theorem we obtain that for each $m \geq 1$, Γ^β has at least a fixed point in Ω_{r_0}, denoted by u_m. Furthermore, it is clear that for each $m \geq 1$, u_m is a mild solution of Cauchy problem (2.2).

Step 2. We show that

the set $\{u_m\}_{m=1}^\infty \subset \Omega_{r_0}$ is precompact in $C([0, T]; X_\alpha)$.

Assume that the operators Γ_1^β and Γ_2^β are defined the same as in Step 1. Therefore, it is sufficient to show that the sets $\{\Gamma_1^\beta u_m;\ m \geq 1\}$ and $\{\Gamma_2^\beta u_m;\ m \geq 1\}$ are precompact in $C([0, T]; X_\alpha)$.

Firstly, noticing (H_1), Lemmas 2.1 and 2.2, it is not difficult to prove, by the arguments similar to those for [5, Theorem 3.2], that the set $\{\Gamma_2^\beta u_m;\ m \geq 1\}$ is precompact in $C([0, T]; X_\alpha)$. Let $\xi \in (0, \eta)$ be fixed, where η is the constant in (H_2)(ii). Note in particular that the set $\{\Gamma_2^\beta u_m;\ m \geq 1\}|_{[\xi, T]}$ is precompact in $C([\xi, T]; X_\alpha)$.

Next, we consider the set $\{\Gamma_1^\beta u_m;\ m \geq 1\}$. To prove that the set $\{\Gamma_1^\beta u_m;\ m \geq 1\}$ is precompact in $C([0, T]; X_\alpha)$, we only need to prove that the set $\{\Gamma_1^\beta u_m;\ m \geq 1\}|_{[0, \xi]}$ and the set $\{\Gamma_1^\beta u_m;\ m \geq 1\}|_{[\xi, T]}$ are precompact in $C([0, \xi]; X_\alpha)$ and in $C([\xi, T]; X_\alpha)$, respectively.

In view of (H_2)(i) and Lemma 2.3 we have that for all $t \in [\xi, T]$, the set

$$
\{\Gamma_1^\beta u_m(t);\ m \geq 1\} = \left\{ \mathcal{S}_\alpha(t) T\left(\frac{1}{m}\right) H(u_m);\ m \geq 1 \right\}
$$

is relatively compact in X_α. On the other hand, for $t_1, t_2 \in [\xi, T]$ with $t_1 \leq t_2$, by means of Lemma 2.2(5) and (H_2)(i) one has

$$
\begin{aligned}
&\left\| \mathcal{S}_\alpha(t_2) T\left(\frac{1}{m}\right) H(u_m) - \mathcal{S}_\alpha(t_1) T\left(\frac{1}{m}\right) H(u_m) \right\|_\alpha = \left\| (A^\alpha \mathcal{S}_\alpha(t_2) - A^\alpha \mathcal{S}_\alpha(t_1)) T\left(\frac{1}{m}\right) H(u_m) \right\| \\
&\to 0, \quad \text{as } t_2 \to t_1,
\end{aligned}
$$

uniformly for $m \geq 1$. Hence, an application of Arzela–Ascoli's theorem justifies that

the set $\{\Gamma_1^\beta u_m;\ m \geq 1\}|_{[\xi, T]}$ is precompact in $C([\xi, T]; X_\alpha)$.

Consequently, it is proved that

the set $\{u_m; m \geq 1\}|_{[\xi,T]}$ is precompact in $C([\xi, T]; X_\alpha)$.

Without loss of generality, we let

$$u_m \to u_0 \quad \text{in } C([\xi, T]; X_\alpha)$$

as $m \to \infty$, which implies that

$$u_m^\eta \to \widetilde{u}_0 \quad \text{in } C([0, T]; X_\alpha)$$

as $m \to \infty$, where

$$u_m^\eta(t) = \begin{cases} u_m(t) & \text{if } t \in [\eta, T], \\ u_m(\eta) & \text{if } t \in [0, \eta]. \end{cases}, \qquad \widetilde{u}_0 = \begin{cases} u_0(t) & \text{if } t \in [\eta, T], \\ u_0(\eta) & \text{if } t \in [0, \eta]. \end{cases}$$

This together with the strong continuity of $T(t)$ on X and (H_2)(i) yields that

$$\begin{aligned} \left\| T\left(\frac{1}{m}\right) H(u_m) - H(u_0) \right\|_\alpha &= \left\| A^\alpha T\left(\frac{1}{m}\right) H(u_m^\eta) - A^\alpha H(\widetilde{u}_0) \right\| \\ &\leq \left\| T\left(\frac{1}{m}\right) A^\alpha H(\widetilde{u}_0) - A^\alpha H(\widetilde{u}_0) \right\| + \left\| T\left(\frac{1}{m}\right) (A^\alpha H(\widetilde{u}_0) - A^\alpha H(u_m^\eta)) \right\| \\ &\leq \left\| \left(T\left(\frac{1}{m}\right) - I \right) A^\alpha H(\widetilde{u}_0) \right\| + M \| H(\widetilde{u}_0) - H(u_m^\eta) \|_\alpha \\ &\to 0 \quad \text{as } m \to \infty, \end{aligned}$$

from which we see that

$$\text{the set } \left\{ T\left(\frac{1}{m}\right) H(u_m); m \geq 1 \right\} \text{ is relatively compact in } X_\alpha,$$

and so is the set $\{\mathcal{S}_\alpha(t) T(\frac{1}{m}) H(u_m)\}$ for all $t \in [0, \xi]$. Noticing this and Lemma 2.2(3), we have that for $t_1, t_2 \in [0, \xi]$ with $t_1 \leq t_2$,

$$\begin{aligned} \left\| \mathcal{S}_\alpha(t_2) T\left(\frac{1}{m}\right) H(u_m) - \mathcal{S}_\alpha(t_1) T\left(\frac{1}{m}\right) H(u_m) \right\|_\alpha &= \left\| (\mathcal{S}_\alpha(t_2) - \mathcal{S}_\alpha(t_1)) A^\alpha T\left(\frac{1}{m}\right) H(u_m) \right\| \\ &\to 0, \quad \text{as } t_2 \to t_1, \end{aligned}$$

uniformly for $m \geq 1$. That is to say,

the set $\{\Gamma_1^\beta u_m(\cdot); m \geq 1\}|_{[0,\xi]}$ on $[0, \xi]$ is equicontinuous.

Therefore, again by Arzela–Ascoli's theorem one can conclude that $\{\Gamma_1^\beta u_m; m \geq 1\}|_{[0,\xi]}$ is precompact in $C([0, \xi]; X_\alpha)$. Summarizing the above, we have that the set $\{u_m\}_{m=1}^\infty$ is precompact in $C([0, T]; X_\alpha)$.

Step 3. From *Step* 2, it follows that the set $\{u_m\}_{m=1}^\infty$ is precompact in $C([0, T]; X_\alpha)$. Without loss of generality, we let

$$u_m \to u \quad \text{in } C([0, T]; X_\alpha)$$

as $m \to \infty$. Note that for each $m \geq 1$, u_m is a solution of the following integral equation

$$u_m(t) = \mathcal{S}_\beta(t) T\left(\frac{1}{m}\right) H(u_m) + \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s) \left(f(s, u_m(s)) + \int_0^s K(s-\tau) g(\tau, u_m(\tau)) \mathrm{d}\tau \right) \mathrm{d}s, \quad t \in [0, T].$$

Then letting $m \to \infty$ on both sides one has

$$u(t) = \mathcal{S}_\beta(t) H(u) + \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s) \left(f(s, u(s)) + \int_0^s K(s-\tau) g(\tau, u(\tau)) \mathrm{d}\tau \right) \mathrm{d}s, \quad t \in [0, T].$$

This yields that $u \in C([0, T]; X_\alpha)$ is a mild solution of Cauchy problem (1.1) and the proof is then complete. □

Corollary 2.1. *Let* (H_2) (ii) *and the following hypotheses hold.*

(H_1') $f, g : I \times X_\alpha \to X$ *are continuous, for some* $r > 0$ *there exist positive functions* φ_r *satisfying* $\frac{\varphi_r(\cdot)}{(t-\cdot)^{1-\beta(1-\alpha)}} \in L^1(0, t; \mathbb{R}^+)$ *and* $\phi_r \in L^\infty(0, T; \mathbb{R}^+)$ *such that for all* $t \in [0, T]$ *and* $x \in X_\alpha$ *satisfying* $\|x\|_\alpha \leq r$,

$$\|f(t, x)\| \leq \varphi_r(t), \qquad \|g(t, x)\| \leq \phi_r(t),$$

and

$$\liminf_{r\to+\infty}\frac{1}{r}\int_0^t \frac{\varphi_r(s)}{(t-s)^{1-\beta(1-\alpha)}}\mathrm{d}s=\sigma_1'<\infty,\qquad \liminf_{r\to+\infty}\frac{\|\phi_r\|_{L^\infty(0,T)}}{r}=\sigma_2<\infty.$$

(H_2') *$H : C([0,T];X_\alpha)\to X_\alpha$ is continuous and there exist $L_1,\ L_2>0$ such that*

$$\|H(u)\|_\alpha\le L_1|u|_\alpha+L_2.$$

(H_3) $ML_1+C_\alpha\sigma_1'+\frac{C_\alpha\widetilde{k}\sigma_2T^{(1-\alpha)\beta}}{(1-\alpha)\beta}<1.$

Then Cauchy problem (1.1) has at least one mild solution.

3. An example

Consider the fractional partial integro-differential equation with nonlocal initial condition

$$\begin{cases} {}^c\partial_t^{\frac{1}{2}}u(t,x)-\dfrac{\partial^2u(t,x)}{\partial x^2}=\dfrac{\sin u(t,x)}{t^{\frac{1}{3}}}+\displaystyle\int_0^t K(t-s)\left(\theta_1(s)u(s,x)+\theta_2(s)\frac{\partial u(s,x)}{\partial x}\right)\mathrm{d}s,\\ \quad 0\le t\le T,\ 0\le x\le\pi,\\ u(t,0)=u(t,\pi)=0,\quad 0\le t\le T,\\ u(0,x)=\displaystyle\sum_{i=1}^p\int_0^\pi K_0(x,y)\cos u(t_i,y)\mathrm{d}y,\quad 0\le x\le\pi, \end{cases}\tag{3.1}$$

where $0<t_1<\cdots<t_{p-1}<t_p<T$.

Let $X=L^2[0,\pi]$ and the operators $A=-\frac{\partial^2}{\partial x^2}:D(A)\subset X\mapsto X$ be defined by

$$D(A)=\{u\in X;\ u,u' \text{ are absolutely continuous, } u''\in X,\ \text{ and } u(0)=u(\pi)=0\}.$$

Then, A has a discrete spectrum and the eigenvalues are $n^2, n\in\mathbb{N}$, with the corresponding normalized eigenvectors $y_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$. Moreover, $-A$ generates a compact, analytic semigroup $\{T(t)\}_{t\ge0}$ on X and

$$T(t)u=\sum_{n=1}^\infty \mathrm{e}^{-n^2t}(u,y_n)y_n,\qquad \|T(t)\|_{\mathcal{L}(X)}\le\mathrm{e}^{-t}\quad\text{for all } t\ge0.$$

The following results are well known also:

(1) If $u\in D(A)$, then $Au=\sum_{n=1}^\infty n^2(u,y_n)y_n$.

(2) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}u=\sum_{n=1}^\infty n(u,y_n)y_n$$

for each $u\in D(A^{\frac{1}{2}})=\{v\in X;\ \sum_{n=1}^\infty n(v,y_n)y_n\in X\}$ and $\|A^{-\frac{1}{2}}\|_{\mathcal{L}(X)}=1$.

Denote by $E_{\rho,\beta}$ the generalized Mittag-Leffler special function (cf., e.g., [9]) defined by

$$E_{\rho,\beta}(t)=\sum_{k=0}^\infty\frac{t^k}{\Gamma(\rho k+\beta)}\quad \rho,\beta>0,\ t\in\mathbb{R}.$$

Therefore, we have

$$\mathcal{S}_\beta(t)u=\sum_{n=1}^\infty E_\beta(-n^2t^\beta)(u,y_n)y_n,\quad u\in X;\qquad \|\mathcal{S}_\beta(t)\|_{\mathcal{L}(X)}\le1\quad\text{for all } t\ge0,$$

$$\mathcal{P}_\beta(t)u=\sum_{n=1}^\infty e_\beta(-n^2t^\beta)(u,y_n)y_n,\quad u\in X;\qquad \|\mathcal{P}_\beta(t)\|_{\mathcal{L}(X)}\le\frac{\beta}{\Gamma(1+\beta)}\quad\text{for all } t\ge0,$$

where $E_\beta(t):=E_{\beta,1}(t)$ and $e_\beta(t):=E_{\beta,\beta}(t)$.

Assume that

(i) $K,\ \theta_1,\ \theta_2\in C([0,T];\mathbb{R}^+)$.

(ii) The function K_0 is measurable and

$$\int_0^\pi\int_0^\pi K_0^2(x,y)\mathrm{d}x\mathrm{d}y<\infty,\qquad c_0=\int_0^\pi\int_0^\pi\left(\frac{\partial K_0(x,y)}{\partial x}\right)^2\mathrm{d}x\mathrm{d}y<\infty.$$

Define

$$f(t,u(t))(x)=\frac{\sin u(t,x)}{t^{\frac{1}{3}}},$$
$$g(t,u(t))(x)=\theta_1(t)u(t,x)+\theta_2(t)\frac{\partial u(t,x)}{\partial x},$$
$$H(u)(x)=\sum_{i=1}^{p}\int_0^{\pi}K_0(x,y)\cos u(t_i,y)\mathrm{d}y.$$

Then it is easy to verify that $f,\ g : [0,T]\times X_{\frac{1}{2}}\to X$ and $H(u)\in X_{\frac{1}{2}}$ whenever $u\in C([0,T];X_{\frac{1}{2}})$. Moreover, we see that (H_1) and (H_2) hold with

$$\frac{1}{3}<\gamma<\frac{1}{2},\qquad \varphi_r(t)=\pi^{\frac{1}{2}}t^{-\frac{1}{3}},\qquad \phi_r(t)=r\max_{t\in[0,T]}\theta_1(t)+r\max_{t\in[0,T]}\theta_2(t),\qquad \Phi(r)=\sqrt{c_0}p\pi^{\frac{1}{2}},$$
$$\mu=\sigma_1=0,\qquad \sigma_2=\max_{t\in[0,T]}\theta_1(t)+\max_{t\in[0,T]}\theta_2(t).$$

Thus, when $C_{\frac{1}{2}}T^{\frac{1}{4}}\int_0^T K(t)\mathrm{d}t(\max_{t\in[0,T]}\theta_1(t)+\max_{t\in[0,T]}\theta_2(t))<\frac{1}{4}$ such that condition (2.1) is satisfied, (3.1) has at least one mild solution due to Theorem 2.2.

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