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AN OUTLINE OF THE LIFE AND WORK OF E. B. CHRISTOFFEL (1829-1900)

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SUMMARIES

This article begins with an account of the life of Christoffel and of the intellectual milieu in which he worked, namely, the Institutes of Technology in Zürich and Berlin, and the University of Strasbourg. A survey of Christoffel's work follows. His major contributions lie in diverse areas: (i) geometry, including the foundations of tensor analysis; (ii) function theory, including conformal mappings, abelian integrals, and theta functions; (iii) numerical analysis; (iv) orthogonal polynomials and continued fractions; (v) ordinary and partial differential equations, potential theory; (vi) theory of shock waves; and (vii) dispersion of light. Finally, an attempt is made to assess Christoffel's work and its subsequent influence upon the development of modern mathematics, physics, and mechanics.

Cet article commence par un rapport de la vie de Christoffel et du milieu intéllectuel dans lequel il travaillait, c'est à dire, des institutes polytechniques de Zürich et Berlin, ainsi que de l'université de Strasbourg. Ce rapport est suivi d'un résumé de l'œuvre de Christoffel. Ses contributions les plus importantes comprennent des articles dans des branches diverses: (i) géometrie, les fondements de l'analyse tensorielle y compris; (ii) théorie des fonctions, les applications conformes, les intégrales abéliennes et les fonctions théta y compris; (iii) analyse numérique; (iv) polynômes orthogonaux et fractions continues; (v) équations différentielles ordinaires et équations différentielles aux dérivées partielles, théorie du potentiel; (vi) théorie des ondes de choc; et (vii) dispersion de la lumière. Finalement, on essaie d'estimer l'importance de l'œuvre de Christoffel et de son influence succédant au développement de la mathématique moderne, la physique et la mécanique.

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Dieser Artikel beginnt mit einer Schilderung von Christoffels Leben und dem geistigen Milieu, in dem er wirkte, den Polytechnischen Instituten in Zürich und Berlin sowie der Universität Straßburg. An diesen Bericht schließt sich ein Überblick über Christoffels Werk an. Seine Hauptarbeiten liegen auf folgenden verschiedenen Gebieten: (i) Geometrie, einschließlich Grundlagen der Tensoranalysis; (ii) Funktionentheorie, einschließlich konforme Abbildungen, Abelsche Integrale und Thetafunktionen; (iii) numerische Analysis; (iv) orthogonale Polynome und Kettenbrüche; (v) gewöhnliche und partielle Differentialgleichungen, Potentialtheorie; (vi) Theorie der Stoßwellen; und (viii) Dispersion des Lichtes. Schließlich wird der Versuch unternommen, die Bedeutung von Christoffels Werk und dessen Einfluß auf die Entwicklung der modernen Mathematik, Physik und Mechanik zu bewerten.

INTRODUCTION

The purpose of this paper is to present a biography as well as an account of the scientific work of E. B. Christoffel on the occasion of the 150th anniversary of his birth. Since I have recently written a detailed article [Butzer 1978/1979] in German on this same theme, the outline of his life given here is fairly brief. The emphasis in this paper [1] is on Christoffel's early years as a student and instructor at the University of Berlin, and not so much on his professorial years at Zürich (1862-1869), Berlin (1869-1872), and Strasbourg (1872-1900).

The most important presentation of the life and work of Christoffel by his contemporaries was provided by two of his former students and colleagues, Carl Friedrich Geiser (1843-1934) and Ludwig Maurer (1859-1927) (see [Geiser & Maurer 1901] and in the obituary address of the philosopher Wilhelm Windelband (1848-1915) [Windelband 1901]). Whereas Geiser and Maurer provided a substantive account of Christoffel's life (enlarged upon in [Geiser 1910]), as well as a summary survey of the main highlights of his work, Windelband focused on his personality. Later biographies of Christoffel (including [Süss 1957; Struik 1933; Burau 1982]) are essentially based upon these two presentations. There is also the brief obituary [Anon. 1901]. The present biography (also [Butzer 1978/1979] as well as [Butzer & Fehér 1979, 1980]) makes use of additional sources and new archival material. In particular, two curicula vitae of Christoffel, dating back to 1859 and 1862, and letters of Christoffel written to Borchardt (1856/1857), F. E. Prym (1863-1865), F. Klein (1881), P. Groth (1874-1882), and the chancellor (Curator) of the University of

Strasbourg are utilized, as are letters of Minkowski addressed to Hilbert and of Weingarten written to Bianchi, as well as letters of recommendation on behalf of Christoffel. A further invaluable source is the report by Lorey [1916] concerning the mathematical institutes of the German-speaking universities and their teaching staffs.

The earlier account of Christoffel's work was published before the International Christoffel Symposium of 8-11 November 1979 [Butzer 1978/1979]. The present article has the advantage of the dozen festival lectures presented at this symposium, concentrating on various branches of Christoffel's work, as well as of many of the forty-five invited papers that dealt with particular aspects of Christoffel's contributions. Our survey of Christoffel's achievements complements the original appraisal of Geiser and Maurer [1901], in order to assess his work in the context of comtemporary mathematics, particularly in terms of the direct and indirect impact on the modern development of mathematics, physics, and mechanics.

1. A BIOGRAPHICAL OUTLINE

Elwin Bruno Christoffel was born on 10 November 1829 at Montjoie (Monschau since 1918), about 30 kilometers south of the city of Aachen in the westernmost part of Germany, near the Belgian border. He was the third son of Franz Karl Christoffel (1798-1832) and Maria Helena Christoffel (née Engels) (1795-1858). His father was a merchant and clothmaker; both parents came from families of smaller cloth merchants.

His grandfather, Charles Joseph Christoffel (Christophe) (1746-1822), was born in the Belgian town of Verviers (about 18 kilometers southwest of Aachen), which was also the birthplace of the grandfather of Johann Peter Gustav Lejeune Dirichlet (1805-1859). He left Verviers for Montjoie. His second wife there, Anna Maria van Wersch, born 1766 in Brandenberg, near Nideggen, was Christoffel's grandmother. His great-grandfather, Charles Joseph Christophe, as well as his forbears, came from the region Verviers-Liège; they have been partly traced back as far as the 11th century, in the 28th generation to a certain knight Michel d'Awir (a village south of Liège), a descendant of the Carolingian nobility. The detailed genealogical investigation of the Christoffel family tree and life in the town of Montjoie during the 19th century is to be found in [Jansen 1978/1979] and [Steinröx 1978/1979], respectively.

After attending the elementary school in Montjoie, Christoffel was privately educated at home, first in modern languages and mathematics, then also in classical languages. In 1844 he entered the Catholic Gymnasium at Cologne, also called the Jesuit or Marcellen-Gymnasium. One of his school fellows there was the German-American politician Carl Schurz (born 1829 in Liblar, died 1906 in New York City). Probably a year later Christoffel moved to the Friedrich-Wilhelms Gymnasium at Cologne, where he obtained his Maturitätszeugnis (final school certificate) with distinction in the fall of 1849 (see [Dieregsweiler 1978/1979] for details).

In 1850 Christoffel began a four-year course of study at the University of Berlin. His teachers in mathematics there included Carl Wilhelm Borchardt, Gotthold Eisenstein, Ferdinand Joachimsthal, and Jacob Steiner, as well as Dirichlet. He also attended many courses in physics and chemistry, particularly those given by Heinrich Wilhelm Dove and Franz Leopold Sonnenschein. The lectures by Dirichlet interested him the most, and one can probably regard Christoffel as first and foremost a student of Dirichlet.

Shortly after completion of his one-year military service (Guards Artillery Brigade), Christoffel received his doctoral degree with distinction on 12 March 1856. The referees of his thesis *De Motu permanenti electricitatis in corporibus homogeneis*, were Martin Ohm, Ernst Eduard Kummer, and the physicist Heinrich Gustav Magnus.

Christoffel subsequently returned to Montjoie, probably to take care of his ailing mother, and so spent three years away from structured academic activities. His character as a person and as a mathematician crystallized during this period of isolation, and it is not surprising that he became one of the truly free and independent mathematicians of his generation, but probably also a lonely man, as many great men are prone to be. He has also been described as shy, distrustful, unsociable, irritable, and brusque by his biographers. During this time he read and penetrated deeply the works of Bernhard Riemann, Dirichlet, and August Louis Cauchy. In his own biography of 1859 Christoffel [2] wrote that he continued especially his studies of mathematical physics, particularly elasticity of solid bodies, dispersion of light, and hydrodynamic and acoustic problems. These interests were related to his investigations of ordinary and partial differential equations. In fact, at this time he even spoke of himself as a mathematical physicist. In Montjoie Christoffel also wrote his first two papers published after his dissertation, namely, [1858a, b] of the list of his publications. (See appendix 2).

In the spring of 1859 Christoffel returned to Berlin, and, on May 12 he applied to begin the qualifying exam procedures to become a lecturer (Habilitationsverfahren) at the University of Berlin. The probationary lectures, one to the Faculty of Philosophy and the other in public, took place on or shortly before 2 August 1859, when he was appointed Privatdozent (lecturer).

Christoffel remained three years at Berlin in this capacity, teaching eight different courses, including numerical integration, theory of elasticity of solid bodies, and partial differential equations. (See Appendix 1.) He also wrote two additional papers, namely, [1861a, b].

In 1862 Christoffel accepted a professorship, as successor to Richard Dedekind, at the Swiss Federal Polytechnic School, at present the Eidgenössische Technische Hochschule (ETH) Zürich. (He was probably recommended by Kummer and Karl Weierstrass.) This school, founded only seven years earlier, had offered mainly mathematics courses for engineers, but Christoffel organized a new institute of mathematics and natural sciences. He was highly appreciated not only as a teacher but also as a researcher. Thirteen of his papers, [1864a] to [1869b], were written while he was in Zürich and they rank among his greatest scientific achievements. Indeed, on 2 April 1868 he was elected a Corresponding Member of the Berlin Academy of Sciences, and on 9 July 1868 also of the Reale Istituto Lombardo in Milan. (Concerning Christoffel's years at Zürich, see [Knus 1981].)

On 29 June 1868, Bismarck offered him a chair at the Gewerbeakademie (now the University of Technology) at Berlin. Franz Reuleaux (1829-1905), its director, had approached Christoffel earlier with a view to persuading him to accept a new position he was trying to establish. In the winter of 1868/1869 Christoffel declined an offer to become founding director of the Polytechnikum (the present Rheinisch-Westfälische Technische Hochschule) at Aachen, but he did resign at Zürich to join the faculty of the Gewerbeakademie Berlin on 1 April 1869. His successor in Zürich was Hermann Amandus Schwarz. One of Christoffel's two colleagues at Berlin was Siegfried Aronhold (1819-1884), whom he had probably come to know and respect in Berlin as early as 1850-1852. During his third period in Berlin, Christoffel continued his work on differential geometry and his work connected with the Christoffel-Schwarz formula begun in Zürich (see [1869c] to [1870c]). On 4 December 1869 he was again honored by election as a Corresponding Member of the Royal Society of Science (Königliche Gesellschaft der Wissenschaften) at Göttingen. However, Christoffel did not seem to have been happy in Berlin; neither he nor Aronhold was able to attract enough students to sustain higher-level courses in mathematics. The University of Berlin, Germany's best at the time, with Weierstrass, Kummer, Leopold Kronecker, Wilhelm Thomé, and Reinhold Hoppe, as well as the mathematical physicists G. Adolph Erman and Helmohltz, was too close by. (See [Knobloch 1978/1978, 1981] for details.)

The year 1872 marked another turning point in Christoffel's career, when he was appointed to a chair professorship at the University of Strasbourg. After the Prussian annexation of Alsace-Lorraine, this university, dating back to colleges founded in the 16th century, was reorganized. The plan was to convert it into a modern, exemplary institution of learning (see [v. Meyenn 1978/1979a, b; Wollmershäuser 1981]). Together with his colleague, Theodor Reye (1835-1919), Christoffel built a new mathematics department with a solid reputation, as he had done once before at Zürich. The new institute included one, and later two, associate professorships, occupied by Georg Roth (1845-1904?), Eugen Netto (1846-1927), Karl Schering (1854-1925), Adolf Krazer (1858-1926), and Emil Georg Cohn (1854-1944). Maurer became a lecturer there in 1888.

About two dozen candidates received their doctoral degrees in mathematics and astronomy at Strasbourg between 1872 and 1895. Some six of these were under the supervision of Christoffel. These included Otto Pauls (1857-1928; in 1882), Victor Doerr (born 1858; in 1883), Rikitaro Rigakuschi Fujisawa (1861-1933; in 1886), Maurer (in 1887), Joseph Wellstein (1869-1919; in 1894), and Paul Epstein (1871-1939; in 1895). The last four became university professors, at the Imperial University of Tokyo and at Tübingen, Strasbourg, and Frankfurt, respectively. Fujisawa and several of his famous students (T. Takagi, M. Fujiwara, T. Kubota, S. Kakeya, etc.) were responsible for raising the standard of mathematics in Japan to the European level [3].

Reye, who was a geometer in the tradition of Steiner and von Staudt, produced about a dozen doctorates (see [4], [Butzer & Stark 1976], [5]) during this time. These included B. Klein (1846-1891; in 1876), R. Krause (1879), G. Kilbinger (1880), St. Jolles (1857-1942; in 1883), Th. Meyer (1884), A. Pampusch (1886), C. Arnoldt (1887), C. Fink (1887), F. Schumacher (1889), and H. E. Timerding (1837-1945; in 1894). Of these, only Klein, Jolles, and Timerding became university professors--at Marburg, Berlin (Technological University), and Braunschweig, respectively. The remaining half-dozen were in the field of astronomy.

During his years at Strasbourg, Christoffel wrote thirteen additional papers, [1871] to [1902] that demonstrate the breadth of his research. At the age of 65 Christoffel chose to retire; one of the reasons was that he had broken his arm in an accident in 1892. His successor, appointed 1 January 1895, was Heinrich Weber. Four months after his 70th birthday, on 15 March 1900, Christoffel died in Strasbourg. Like Weierstrass, Christoffel never married, and thus left no family at the time of his death.

The Prussian state awarded two distinctions to Christoffel as a mathematician: the "Rote Adler Orden, III. Klasse mit der Schleife," bestowed on 24 January 1893, and the "Kronenorden II. Klasse" in March of 1895 [6]. Both orders were the second highest of all possible orders conferred at the time.

It should be mentioned that on 17 September 1875 the Faculty of Philosophy of the University of Vienna applied to the Austrian Minister of Education, Stremayr, for a third chair in mathematics. Their first choice was Christoffel, their second choice Friedrich Emil Prym (1841-1915) of Würzburg, equally ranked with

H. A. Schwarz. Stremayr, however, suggested to the Emperor Franz Josef that Emil Weyr, an Associate Professor at the Polytechnicum in Prague who was also under consideration, be offered the professorship. The emperor chose Weyr, an Austrian.

The recommendation was accompanied by the words, "According to their publications, Professors Christoffel, Prym and Schwarz rank among Germany's most proven scholars and can be evaluated through their former students, who have done independent scientific research...." The members of the appointment board included J. Petzval (1807-1891), L. Boltzmann (1844-1906), and the physicists J. Stefan (1835-1893), J. Loschmidt (1821-1895), and V. E. v. Lang (1838-1921) [7].

2. THE SCIENTIFIC WORK OF CHRISTOFFEL

Numerical Analysis; Orthogonal Polynomials

The work of Christoffel, in both its versatility and depth, has been of fundamental significance in the development of various branches of mathematics, physics, and mechanics.

Christoffel's first paper [1858a], concerned with approximate integration, has received and still receives wide recognition in the field of numerical analysis, as W. Gautschi emphasized in his paper [1981a] which surveys some 400 related articles. Indeed, with his quadrature formula [1858a] Christoffel generalized Gauss' method by allowing the quadrature sum to contain an arbitrary number of fixed preassigned nodes. He also succeeded in expressing the relevant polynomials in determinant form, now called Christoffel's theorem. In his second paper [1877a] on the subject Christoffel gave a synthesis of Gauss' method, this time for general nonnegative measures of integration. He also gave the connections with (what are now called) orthogonal polynomials and continued fractions. Later T. J. Stieltjes (1884) established the legitimacy of the Gauss-Christoffel quadrature formula by proving its convergence, while A. A. Markov (1885) endowed it with an error estimate. Among the more noteworthy later applications are those to Laplace transform inversion, conformal mapping, integral equations, approximation of iterated and multiple integrals, and, most recently, finite element computations. (Concerning the recognition of this work of Christoffel, see [Gautschi 1981b].)

The same two papers have also influenced the theory of special functions of mathematical physics, as has been pointed out by J. Meixner [1981]. Indeed, the earlier paper [Christoffel 1858a] also contains the Christoffel-Darboux summation formula in the particular case of Legendre polynomials. The other paper [Christoffel 1877a] gives the analogous formula for general orthogonal polynomials with an arbitrary positive weight function.

The formula in the latter instance has so far been credited to Gaston Darboux, who established it only a year (1878) after Christoffel. This formula, which is to be found in almost every book on the subject, is basic for summing up series of orthogonal polynomials. The so-called Christoffel functions and numbers, which have been studied in detail in a recent monograph [Nevai 1979], also have their origins in these two papers.

Function Theory; Automorphic Functions

Christoffel devoted a great part of his life's work to the theory of functions of a complex variable and, especially, applications in various branches of mathematics. One of his "lasting accomplishments" [Azzam & Kreyszig 1981] in this area is the Christoffel-Schwarz formula for the conformal mapping of the upper half-plane onto a polygon, a standard item in textbooks on function theory. This discovery, the subject of four papers by Christoffel [1868a, 1870a-c], is all the more remarkable since it is the first nontrivial constructive illustration of Riemann's mapping theorem of 1851. (Riemann's proof of his famous theorem was based on Dirichlet's principle; although this proof was generally accepted at the time, a correct proof was first supplied by D. Hilbert in 1900. For the history of Dirichlet's principle see [Monna 1975; Neuenschwander 1978, 23] and the references cited therein). H. A. Schwarz published the same formula with a different proof shortly afterward. However, it was only A. Weinstein who, in 1923, gave a complete and simplified proof of the mapping theorem for polygons using the method of continuity. (Concerning these questions, see especially [Pfluger 1981; Goodman 1981]; for the early history of the problem see [Lichtenstein 1918].)

The mapping formula, which has also been basic in the development of the theory of elliptic functions and remains a powerful tool in the study of boundary value problems for harmonic functions, has found many other applications, for example, in electrostatics, electron optics, fluid flow, and elasticity (see also [Azzam & Kreyszig 1981] and the references given therein).

One of the great achievements of mathematics in the 19th century was the theory of elliptic functions. Christoffel wrote two papers on Abelian integrals, which are generalizations of elliptic integrals [1879, 1881]. Riemann had indicated in 1857 that the number of linearly independent Abelian integrands (differentials) of the first kind is equal to the genus of the Riemann surface. Among Christoffel's results [1881] is to be found a proof of a form of this theorem using algebraic methods. This material is in the same circle of ideas with the Riemann-Roch theorem, which can also be proved in a purely algebraic context, and the famous dependence theorem (Abhängigkeitssatz), the first complete proof of which was only given around 1950 by C. L. Siegel and S. Bochner [Thimm 1966]. Concerning the above two papers, Brill and Noether [1892/1893], in the introduction to their survey article on the development of the theory of algebraic functions, state that "they report on ... the general progress that the theory of algebraic functions has made since 1876 ..., especially concerning an investigation of Christoffel that has so far received little appreciation..." This material is now part of the theory of automorphic functions (see, e.g., [Siegel 1971; Deuring 1973]). It is the feeling of the author that the influence of the two papers [Christoffel 1879, 1881] is a worthy subject for still further study. (See also [Pommerenke 1981], the literature cited in [Butzer 1978/1979, remark 8 to Sect. 6.3., p. 20], and [Stahl 1896].)

Christoffel contributed further to the theory of theta functions in [1896] and in the posthumous paper [1901]. Theta functions had been useful in representing elliptic functions. In order to treat surfaces of genus $p \geq 2$ it became necessary, as Riemann had noticed, to consider theta functions of p variables; the place of the independent variables is then taken by p linearly independent Abelian integrals (see e.g., [Siegel 1971, 1973]). Concerning this theory, which was decisively advanced by Frobenius and Poincaré, Siegel [1973, 163] writes: "The creation of this theory may be safely regarded as one of the most significant accomplishments of the nineteenth century." It was also one of the starting points of the modern theory of functions of several complex variables. The first chapter of Landfriedt's monograph [1904] is basically a reproduction of [Christoffel 1901]. The role played by Christoffel's work in this theory remains to be determined. Concerning this matter see also the literature cited in Audrey Terras' review [1980] of S. Lang's monograph *Introduction to Modular Forms* (1976).

Potential Theory; Partial Differential Equations; Shock Waves

Christoffel's contributions to potential theory, namely, the four papers [1868a, 1870a, 1871, 1865c], use the methods of function theory and are to be viewed within this frame. The first three of these basically deal with the Dirichlet problem for $\Delta u = 0$ or $\Delta u = \phi$ in R^2 and R^3, and the possible relation or equivalence with the existence of a Green's function or of a mapping into a disk or half-plane. This has since become classical. In this regard one may recall that an exact proof of Dirichlet's problem was only given in 1900. In his paper of 1871 Christoffel uses the method of proof now called "balayage." These three papers are three of the four that also contain the Christoffel-Schwarz formula. The fourth paper, [1865c], which is especially remarkable, deals with deep results on the continuation not only of singularities of harmonic functions, but also of potential functions, in R^3. The singularities or exceptional sets he allows are either isolated points or sufficiently regular bounded curves or surfaces. Marcel Brelot [1981], after his penetrating study of these four papers, was convinced that the theorems of continuation of [Christoffel 1865c] should lead to further generalizations and expressions of a harmonic function u in terms of the discontinuities of u and its derivatives at the exceptional sets. In fact, Brelot feels that Christoffel was until now almost unknown to the experts of potential theory. His results were rather important for his time, and even today they remain fresh and stimulating. Brelot [1982] has taken up the matter in the meantime.

As Müller [1981] has remarked, some of the questions dealt with by Christoffel were later considered by A. Liapounoff (1898), Poincaré [1899, 252-259], and Schmidt [1910], although these authors do not seem to have been aware of Christoffel's work. However, Schmidt cited Bruns [1876], who cited Stahl [1879] (Aachen), who in turn cited Christoffel [1865c]. (For the early history see [Bacharach 1883; Burkhardt & Meyer 1899-1916; Lichtenstein 1918, 197-217].) The matter was again taken up in the fifties by Müller [1951], Müller and Leis [1958], and Leis [1961].

The varied contributions of Christoffel also include two major papers on the propagation of discontinuities in solutions of partial differential equations. Whereas Riemann, who initiated the theory (1858), studied the propagation of shocks in one-dimensional isentropic gas flows Christoffel [1877b] studied the propagation of (first-order) discontinuity surfaces in three-dimensional flows of perfect fluids, and in [1877c] he applied his general theory to first-order waves in certain elastic materials. In his detailed investigation Hölder [1981] reports that the theory was studied independently by H. Hugoniot (1886/1887), then taken up and completed in Hadamard's celebrated monograph [1903], and later studied by P. Duhem (1903/1906), Truesdell [1969], and others. From the point of view of the engineering and physical sciences one must mention the important and practical results of L. Prandtl (1905), A. Busemann (1931), F. Kohlrausch (1897), M. v. Laue (1915), and others. The relevance of these achievements of Christoffel for the applied sciences is surely even greater today than could be appreciated in his time (beat of a drum, earthquakes, underwater explosions, aerodynamics, and the like). These papers indeed belong to the first contributions to the theory of shock waves and, from the pure mathematical side, they seem to anticipate the theory of characteristics and bicharacteristic rays (as Atkinson [1981] has phrased it so well), thus ranking Christoffel among the pioneers in the theory of hyperbolic equations, even though he may not often be remembered in this context. (See also [Lampariello 1957].)

Ordinary Differential Equations

Christoffel also wrote several papers, so far almost entirely unnoticed, related to ordinary differential equations. The paper [1864b] is devoted to small oscillations and wave-type movements in spatial lattices of particles subjected to attractive or repulsive forces. It improves earlier work by Lagrange, Fourier, Cauchy, and Weierstrass, as Mawhin [1981a] has remarked. The corresponding nonlinear theory has recently facilitated understanding of the so-called soliton-type solutions of some nonlinear partial differential equations, such as the Korteweg-De Vries equation. The paper [Christoffel 1866] treats qualitative properties (reality and continuity) of solutions of implicit first-order differential equations. According to Martinet [1981] and Mawhin [1981b], who arrived at their opinions independently, it can be regarded as a forerunner of the method of isoclines and of the qualitative theory of ordinary differential equations, which was later developed by Poincaré, Painlevé, G. Birkhoff, Lefschetz, and others.

Continued Fractions

Few mathematicians nowadays seem to know that Christoffel contributed to the theory of continued fractions, and that his paper [1877a] "marked the beginning of an epoch in the development of mathematics," as Peter Wynn [1981a] stated in his meticulous survey report on the matter. Christoffel's papers [1874, 1888] are concerned with the expression of a real number by means of a certain characteristic having a close connection with the regular continued fraction of this number. These ideas have unfortunately not been developed further. His paper [1861a] on the expression, in simple closed form, of the convergents of the continued-fraction expansion of quotients of Bessel functions of contiguous orders brought work of E. Heine (1846/1860) to an elegant completion. His paper [1877a], already mentioned in connection with quadrature formulas, and concerning which Chisholm and Common [1981] state that it "undoubtedly is of central importance in mathematics," deals with the continued-fraction expansion of functions defined as integral transforms of weight functions with respect to which the denominators and numerators of the convergents are orthogonal and associated orthogonal polynomials, respectively. As Wynn pointed out, this paper had an immediate impact, and one that will endure; the work was continued by A. Markoff (1895), T. J. Stieltjes (1894/1895), H. Hamburger (1920/1921), R. Nevanlinna (1922), M. Riesz (1921/1923), and others. None of these refer to Christoffel explicitly but to each other in sequence, as Wynn [1981b] notices. In any case, Christoffel's work has helped to give rise to the classical theory of moment problems (see [Shohat and Tamarkin 1963, Chap. 4], who cite [1877a]), as well as to more modern topics, such as the theory of Jacobi matrices (see [Stone 1932]), the theory of Hilbert spaces of entire functions of de Branges [1968] [8], and the theory of Padé approximation (see [Brezinski 1980; Saff & Varga 1977] and the literature cited therein), to name but a few.

Differential Geometry

Christoffel's name has been inextricably bound with differential geometry for more than a century. It is perhaps the famous Christoffel symbols that have helped most of all to make his name so well known to every mathematician. These symbols, $\Gamma_{ij,k}$ and Γ^k_{ij}, of the first and second kinds, respectively, which Christoffel actually wrote in the form

$$\begin{bmatrix} ij \\ k \end{bmatrix} \quad \text{and} \quad \begin{Bmatrix} ij \\ k \end{Bmatrix},$$

are nowadays used to describe a general linear connection in a vector bundle.

Before turning to the papers dealing with them, let us consider the four papers dedicated more or less to geometry proper, namely, [1865a, 1868a, 1867, 1868d]. The results of the first paper which, according to Klingenberg [1981], undoubtedly belong to his best-known achievements, are to be found in most textbooks on the subject (e.g., [Blaschke 1945, 204-206; Bonnesen & Fenchel 1934; Busemann 1958; Spivak 1975, V, 229]). It deals with the unique determination (apart from a translation) of a convex surface in Euclidean 3-space from a specification of its mean radius of curvature as a function of the outer normal direction. For the many recent generalizations and variations to this theorem of Christoffel see the reports by Firey [1968, 1981].

From the point of view of geodetical science Grafarend [1981] judges Christoffel, provided one interprets his basic theorem of [1865a] in a proper "physical" sense, with the following statement: "Without doubt one can consider Christoffel as the founder of modern three-dimensional geodesy."

Of the second and third papers--both have connections with the theory of functions--the third [1867] is devoted to isothermic surfaces. Christoffel's results here were rediscovered by S. Lie and G. Scheffers (1891). (See especially [Leichtweiss 1981].) The fourth paper [1868d], in which the general theory of geodetic triangles is constructed and the free mobility property of infinitesimal triangles is solved completely, is a forerunner of the von Helmholtz (1868) result that a surface in R^3 can be mapped isometrically and locally into itself if and only if it is of constant Gaussian curvature (see [Laugwitz 1965] and also the survey papers of Sager [1903], Struik [1971], and Reich [1979].) Incidentally, [1868d] already includes a particular case of the Christoffel symbols.

Invariant Theory

Christoffel's six papers on the theory of invariants, namely, [1868b, c, 1869a-c, 1882], have been and remain of fundamental importance for the evolution of physics and mechanics. In [1868b] he solved the problem of equivalence for two general algebraic forms (and for systems of forms) using only algebraic eliminations. In his "extraordinarily impressive" [Spivak 1975, V, 636] paper [1869b], which appeared just one year after the publication of Riemann's *Habilitationsvortrag* of 1854 and which introduces the Christoffel symbols in their general classical form, Christoffel gave the basic ideas to the solution of the local equivalence problem, even for two quadratic differential forms (termed the Christoffel reduction theorem by Klein [1927, 196]), by using covariant differentiations to adjoin to the given forms other forms of higher degrees, so that the question is reduced to the equivalence problem for systems of algebraic forms (treated in [1869b]). Thus he avoided the use of integration and the calculus of variations, which was the approach of E. Beltrami (1835-1900) and R. Lipschitz (1832-1903). (For an account of the work of Beltrami, Christoffel, and Lipschitz in its historical setting see especially [Kline 1972; Peschl 1970].) The procedure Christoffel employed in his solution of the equivalence problem is what Gregorio Ricci-Curbastro (1853-1925) later called covariant differentiation; Christoffel also used the latter concept to define the basic Riemann-Christoffel curvature tensor. (For an excellent review of Christoffel's approach the reader is referred to [Ehlers 1981]; see also [Burau 1981; Pinl 1981]).

The importance of this approach and the two concepts Christoffel introduced, at least implicitly, can only be judged when one considers the influence it has had. Indeed, with the help of the curvature tensor and the notion of the covariant derivative, Ricci was able to develop (especially in the years 1887-1896, and after 1900 jointly with his student Tullio Levi-Cività (1873-1941)) the so-called absolute differential calculus, designated as Ricci calculus or, since the days of Einstein, as tensor analysis. Here the word "absolute" means that the calculus does not depend on a particular coordinate system.

It must first be emphasized that Christoffel's paper [1869b] had a lasting effect upon Riemannian and differential geometry. Doubts about the geometry or physical space had led N. I. Lobachevsky (1792-1856), J. Bólyai (1802-1860), and Gauss to the exploration of non-Euclidean geometry. This was the work taken up by Riemann. He allowed non-Euclidean spaces of nonconstant curvature; the curvature of these Riemann spaces cannot be described by a scalar but by a tensor, namely, the Riemann-Christoffel curvature tensor. Already Levi-Cività (and J. A. Schouten) recognized (1917/1918) that the Christoffel symbols can be regarded geometrically to determine a natural parallel transport of vectors and tensors on Riemannian manifolds. Subsequently, Weyl [1918] showed how this notion can be introduced intrinsically, and it led Ehresmann [1950] to the concept of connections in principal bundles (see, e.g., [Spivak 1970, II, 8-1 to 8-60]). Concerning the notion of connection, Barthel [1981] writes:

> *Christoffel's great achievement for differential geometry is, in my opinion, that he discovered the most important differential-geometrical structure, namely that of connection in a manifold, even if he himself could hardly have been aware of the significance of this discovery.*

Christoffel's local equivalence problem was only put into a definitive form in 1933/1946 by the work of Cartan [1946] (and Sternberg [1964]; see also [Kobayashi & Nomizu 1963/1969]; for discussion of the influence of Christoffel's work in this direction, see [Ehlers 1981]; for early books on tensor analysis see [9]).

Applications to General Relativity and other Physical Sciences

One of the first and at the same time farthest-reaching applications of the tensor analysis initiated by Christoffel was the application to the general theory of relativity. A consistent utilization of tensor analysis in the four-dimensional geometry of curved space-time made it possible for Albert Einstein (1879-1955) to phrase the physical concepts of his general theory in mathematical terms and to express the gravitational field equations in covariant form. Indeed, Einstein interpreted the Christoffel symbols as the components of the gravitational-inertial field, and the curvature tensor as the field gradient, or tidal field, as Ehlers [1981] has pointed out. Christoffel's methods and results, in contrast to many later investigations, do apply to general relativity since they are not restricted to positive-definite metrics. Einstein himself frequently emphasized the significance of Christoffel's and Ricci's results in this regard. In one of his famous papers on general relativity Einstein [1915] stated literally that

> *the fascination of this theory would hardly leave anybody who has really grasped it. It represents a real triumph of the method of the general differential calculus founded by Gauss, Riemann, Christoffel, Ricci and Levi-Cività.*

Einstein's letter of 28 November 1915 to A. Sommerfeld (1868-1951) is also pertinent here (see [A. Hermann 1968, 32]).

Einstein was greatly aided in this research by his friend and former student peer at Zürich, the differential geometer Marcel Grossmann (1878-1936). Indeed, their three joint papers of 1913/1914 must be regarded as critical precursors of Einstein's cited papers of 1915. Recently the question whether Einstein can be considered the sole author of general relativity has again been raised (see, e.g., [Mehra 1973; Earman & Glymour 1978]). David Hilbert [10] had sent his results on the field equations to the Göttingen Academy of Science on 20 November 1915, whereas Einstein presented his paper on the field equations with the decisive additional term involving the trace to the Royal Prussian Academy on 25 November 1915. (This was the fourth of his papers presented on 8, 11, 18, and 25 November.) Hilbert's paper of 1912/1913, inspired by earlier results not only of Einstein but also of Gustav Mie (1868-1957), naturally also employed tensor analysis; it was more elegant and far reaching according to [Mehra, 1973]. (For books on general relativity the reader may consult [Misner, et al. 1973; Hawking & Ellis 1973], and especially [Sachs & Wu 1977].)

Tensor analysis or, more generally, global geometric techniques have again come into the limelight during the past few years. One of the great goals of modern theoretical physics is a unified description of the basic physical laws and structures or, more specifically, a unified (at least partly) geometrical theory encompassing not only gravitational and electromagnetic "forces," but also all of the other fundamental interactions between subatomic particles. A very promising approach for reaching this goal is based on recent gauge field theories; these in turn are based on modern tensor analysis, in particular on the notion of connection in principal bundles (i.e., a set of generalized Christoffel symbols related to a Lie group) as mediators of interactions between matter fields (combined with quantum-theoretical structures) (see [Ehlers 1981]). (For details and references on gauge theories attached to the names of F. London (1972), H. Weyl (1929), C. N. Yang and R. L. Mills (1954), the 1979 Nobel laureates S. Glashow (1961), S. Weinberg (1967), and A. Salam (1968), and others, see the proceedings volumes [Bleuler, et al. 1978; Nelkowski et al. 1979; R. Hermann 1973].)

Finally, it may be noted that differential geometry provides the natural language for classical mechanics (and the calculus of variations) (see, e.g., [Bishop & Goldberg 1968; Abraham 1967; Corbin & Stehle 1960; Whittaker 1959; Rund 1966]) and also for continuum dynamics (see, e.g., Truesdell & Tourpin 1960]) and geometrical optics (see [Kline 1981]). Furthermore, with regard to stellar physics, physicists have recently applied global geometric techniques to the study of the singularities of gravitational collapse (i.e., "big band" results) as well as the nature of black holes (see, e.g., the literature cited in Thorpe's review [1978] of [Sachs & Wu 1977].)

The following statement by Bleuler [1981] is a fitting conclusion to this section:

> *One can declare with full right that Christoffel's idea finally led, via many intermediate stages (decisive points since Einstein were contributed by Weyl, Yang and Mills, Gell-Mann, t'Hooft, and others) to the setting up of the basic laws of present-day physics.*

Optics; Dispersion of Light

Of less lasting significance than Christoffel's mathematical achievements were his early contributions to physics, in particular to optics. At that time the foundation of (mechanical) optics was the light ether, as carrier of elastic waves. The problem of the constitution of this ether--whether continuous or discrete--occupied both Christoffel and his contemporaries a great deal, just as did the old dispute concerning the atomism of matter. With the discovery of anomalous dispersion and its relation to absorption phenomena, the problem of the coupling of matter and ether later moved into the realm of experimental and theoretical research.

In continuation of work begun by Augustin Fresnel (1788-1827) and Cauchy, Christoffel obtained some interesting conclusions which met with the approval of experimenters, as Karl von Meyenn pointed out in his paper [1981] on the subject. Mathematically this effort concerned the dymanics of elastic point systems, i.e., with a lattice of atoms that are surounded by a point-like, discrete ether cloud.

But because of the rapid development of the field, which finally led to the abandonment of the mechanical conception in favor of an electromagnetic theory, Christoffel's results, despite their initial success [11], were soon superseded. However, Christoffel's optical investigations may have contributed to the development of his geometrical sense of intuition, so basic to much of his later work.

3. A TENTATIVE APPRAISAL OF CHRISTOFFEL'S WORK

The Nature of His Teaching and Work

The name Christoffel has long been known to most mathematicians, mainly because of the "elusive" symbols in differential geometry named after him. Moreover, most theoretical physicists knew of the applications of tensor analysis (basically under the terms "covariant differentiation" and "Riemann-Christoffel curvature tensor") to the theory of general relativity. But, for the most part, only specialists were familiar with Christoffel's contributions to the theory of conformal mappings (Christoffel-Schwarz formula), to numerical analysis (Gauss-Christoffel quadrature formula, Christoffel functions), to the theory of special functions of mathematical physics (Christoffel-Darboux summation formula), and to geometry (Christoffel's theorem on determination of convex bodies). Yet even the experts were probably not always fully aware of the broad scope of his work.

It may surprise many mathematicians to learn that among Christoffel's achievements are also results on ordinary and partial differential equations (he was a pioneer in the theory of hyperbolic equations), potential theory, shock waves, and continued fractions. The relevance of a good deal of this work is far greater today than it was in his own time. Christoffel was often far ahead of his generation, so that parts of his work were not fully understood and therefore were not appreciated. Consequently, some of it was not taken up by his contemporaries and has been forgotten.

On the other hand, it is inexplicable that some mathematicians who did follow up his work directly did not always bother to cite him. Since only a few journals were available for the publication of mathematical topics during the second part of the 19th century, one might expect that authors were aware of all relevant papers; also, the *Jahrbuch über die Fortschritte der Mathematik* had begun to appear in 1868.

The bibliography of Christoffel lists 35 published papers. On the other hand, as his biographer Geiser [1910] emphasized, perhaps the greater part of Christoffel's work was not published but was propagated through his lectures. This was not unusual for the time. Similar remarks would apply to Gauss, Dirichlet, Riemann, and Weierstrass, if one also were to take their correspondence into consideration. In Christoffel's case, this suggests that his students were sufficiently good to be able to understand and appreciate the new material presented to them; otherwise either such material would have disappeared or the origins would have been permanently lost. There were never more than 60 students enrolled in the mathematics department at Strasbourg between 1872 and 1900, so that the selection of good students was small, particularly since most top-level students seem to have gone to Berlin or Göttingen. In addition, Strasbourg was not a well-established university when Christoffel joined the department.

Nonetheless, Christoffel seems to have made the best of the opportunities at Strasbourg. He never gave a first-year course during his 22 years at Strasbourg, lecturing only in the second-, third-, and fourth-year university courses, which were comparable to our graduate courses. These were mainly concerned with fields just developing at the time. Indeed, he brought his students to the very frontiers of research, very often the research he himself was working on.

Concerning Christoffel's teaching, Heinrich Emil Timerding (1873-1945) writes that

> *Christoffel was one of the most polished teachers ever to occupy a chair. His lectures were meticulously prepared, to the smallest detail, so that at any moment he had the material at his presence from beginning to end. His delivery was lucid and of the greatest esthetic perfection,--in his manner of speaking, in the executions of computations, in the writing of the formulae, even in his movements. The core of the lectures was the course on complex function theory, distinguished by the inspirational name of Riemann. Christoffel had developed Riemann's function theory independently, particularly in the area of ultraelliptic functions, but did not publish his research, presenting them only in his lectures, after the model of Weierstrass. I have often regretted that this immense intellecutal effort and artful delivery were expended on such a small group of students. The later teaching of anyone who had taken Christoffel's courses was bound to be affected. One can almost recognize Christoffel's students by the way they pick up the chalk, since even in such externals do they show the influence of their master. The positive aspects were also complemented by negative ones. Christoffel had no interest to dilute the esthetic purity of his presentation with the addition of new research, and he did not have the slightest inclination to establish personal relations with his disciples. He could not tolerate the imperfect delivery of students and therefore quickly abandoned his efforts to set up a mathematical seminar.* [Timerding 1916]

But, in an article on the life of Reye, written as late as 1920, Geiser had the following to say about Christoffel:

I can still see the image of Christoffel, unfaded despite the more than half century that has elapsed, as the incomparable lecturer who drew future engineers to his lectures at the Zürich Polytechnic School and kindled their permanent interest and willing efforts.
[Geiser 1921]

The first impression of Christoffel's work is that he "picked up many different threads in the fabric of science," as Baker and Gubernatis [1981] have put it. What is characteristic of him, according to Meyer-König [12], is that "he explored so many fields at depth. The 'many' can be documented in his bibliography; the 'in depth' is verified by the many standard formulae and techniques that bear his name." Indeed, Christoffel did research in some eight different branches and directions of pure and applied mathematics and mathematical physics, and to each branch he contributed on the average 3 to 5 papers. This accounts for the 35 papers attached to his name.

In these papers he primarily worked out the basic concepts underlying the problems he tackled, seldom carrying out a problem to its final resolution. Thus he always left abundant material for others to take up. He confined his published work to what he personally felt was original, creative, and pioneering. He seems to have thrived as a lone worker. This is in sharp contrast to the teamwork of today. The fact that none of his publications were coauthored is not a criterion, since this was usual in his time. He did not have many students who were his direct concern as apprentices, assistants, and disciples. He did not have a school as, for example, did Weierstrass.

Christoffel strived to emulate his own teachers and role models, Dirichlet and Riemann. Minkowski [1905] once characterized Dirichlet's philosophy of work roughly as "to solve a problem with a minimum of blind effort and a maximum of enlightened perception." Geiser wrote:

The main appeal and value of all his lectures lay in the fact that in conscious imitation of Dirichlet, he treated mathematics as a science of abstract ideas. The general fundamentals and methods were always explained with the greatest precision, the special problems clearly and sharply defined, their solutions comprehensively discussed in terms of the necessary and sufficient conditions for their validity.
[Geiser 1910]

This is exactly how Christoffel perceived the nature and goals of mathematics. As Klingenberg [1981] has phrased it:

If we look back again on Christoffel's work, it gives us a sense of satisfaction that he always resisted the temptation to engage in an empty formalism. His papers were motivated by a sense of geometry and executed in a spirit that should serve all mathematicians as a guideline. [Klingenberg 1981]

Christoffel in the Eyes of his Contemporaries

Turning to the esteem which Christoffel's contemporaries had for him as a mathematician, it is symptomatic that he was elected a corresponding member of three learned societies as early as 1868/1869, and that his collected works [13] were published (in 1910). After Minkowski visited Christoffel at Strasbourg he wrote in a letter [Rüdenberg & Zassenhaus 1973, 34] of 19 June 1889 to D. Hilbert: "I was particularly anxious to meet Christoffel, because I had just studied a very excellent essay of his 'on the movement of a periodically organized system' (in Crelle's journal [probably [Christoffel 1846b]]) with great interest." In their joint letter of 17 February 1868 recommending Christoffel as a Corresponding Member of the Berlin Academy of Science, C. W. Borchardt, A. v. Auwers, L. Kronecker, E. E. Kummer, and Weierstrass wrote:

Shortly upon this important paper [Christoffel 1864b] *a new study followed, one in which Christoffel's achievements maintained their first place in terms of originality. This one* [Christoffel 1865a] *solves the problem to determine a surface, when the sum of the curvatures is given for every normal direction.... The problem posed in this paper is by itself new and its solution, which promises to be of importance for geodesy, is of eminent scientific interest.* [Knobloch 1978/1979, 62]

From a letter of F. Klein of 12 December 1881 [14] addressed to Christoffel, one learns that he found Christoffel's paper [1871] on conformal mappings so important that he wished to republish it in the *Mathematische Annalen*. In addition, a number of sections of Klein's *Vorlesungen uber die Entwicklung der Mathematik im 19. Jahrhundert* [1927] are dedicated to Christoffel's work on differential geometry. A number of Italian geometers also followed in his footsteps. Klein, with whom Ricci spent the year 1877/1878 in Munich, writes [1927, 189]: "His influence reached from Zürich to Italy where Bianchi attached himself to him and his ideas found a sympathetic development through Ricci." To this, that historian of mathematics, Pierre Speziali [1975], adds: "Ricci greatly admired Klein, and his esteem was soon reciprocated; nevertheless, Ricci does not seem to have been decisively influenced by Klein's teaching. It was, rather, Riemann, Christoffel and Lipschitz, who inspired his future research. Indeed, their influence on him was even greater than that of his Italian teachers." Levi-Cività, in the foreword to his book *The Absolute Differential Calculus* [1926], writes: "Riemann's general metric and a formula of Christoffel constitute the premises of the absolute differential calculus." In a letter dated 18 January 1886, Weingarten [Bianchi 1952-1959] wrote to Bianchi at Pisa: "Two years ago Christoffel wrote me that he planned to publish his research and his lectures on infinitesimal geometry. But I haven't heard from him again. I believe his book should also prove a good one." Thus Weingarten also thought highly of Christoffel. However, the relationship between Christoffel and Weierstrass seems to have been tense. Although Weierstrass and Kummer probably recommended Christoffel for the professorship in Zürich in 1862, they did not offer him such a position in Berlin in 1868 when L. Fuchs left Berlin for Greifswald, even though Dove was very active on Christoffel's behalf. In a letter of 24 March 1885, addressed to Sonja Kowalewsky [Biermann 1973, 74], Weierstrass named Christoffel a "wunderlichen Kauz," perhaps translatable as a "strange guy." On the other hand, in his letter of 26 February 1872 to von Roggenbach, Christoffel [v. Meyenn 1978/1979a, 73] wrote: "... the lectures of Messrs. Kummer and Weierstrass, however excellent they may otherwise be, offer no opportunity of a reasonable course of study...." Perhaps one basic cause of this mutual dislike was the antagonism between the Riemann and Weierstrass schools of mathematics. Christoffel belonged to the Riemann school in the wider sense, as did Clebsch, K. F. W. Hattendorf, Klein, Sophus Lie, Carl Neumann, Prym, G. Rost, H. Stahl, H. Weber, and H. Weyl.

Christoffel's Rank as a Mathematician

Any attempt to rank Christoffel, or for that matter any other mathematician, among the mathematicians of his time and in the context of all fields is difficult if not impossible. However, Gautschi [15] would place Christoffel, as a contributor to the field of numerical integration, as follows: "Gauss stands above anybody else. Christoffel's work is about equally significant as Jacobi's, not quite as deep as Stieltjes' and more important than that of Lobatto, Mehler, Posse, Radau and Markov." Klingenberg [1981] ranks him among differential geometers as follows: "After the actual founders of differential geometry, namely Gauss and Riemann, Christoffel was the equal of Beltrami, Weingarten, Bonnet and Darboux." In this respect Leichtweiss [16] writes: "In regard to geometry, I would indeed rank Christoffel behind Gauss, Riemann and Hilbert, but still ahead of Levi-Cività and Ricci." Concerning Christoffel's role as a function theorist, Pfluger [17] rates him behind Cauchy, Riemann, Weierstrass, and H. A. Schwarz, but refrains from further differentiation.

From the point of view of his work in the field of continuum mechanics, in particular on shock waves, Hölder [18] argues that Christoffel "stands on the same level as Riemann, Hugoniot [Pierre Henri; 1851-1887] and Hadamard."

If one were to rate Christoffel among mathematicians of all fields, the following thesis could be offered. It is widely recognized that, at least in the German-speaking countries of Europe, Riemann was one of the best mathematicians of the 19th century, behind Gauss and ahead of Weierstrass. In this author's opinion, Christoffel's teacher, Dirichlet, belongs to the next group of major mathematicians, which includes Jacobi, Kummer, Kronecker, Dedekind, Cantor, and Klein. Christoffel himself should be placed immediately behind this group.

In concluding a paper on Christoffel's contributions to potential theory, Brelot [1982] writes: "These are in part still new today, they were and remain stimulating, ... and that, perhaps, is the best one can expect of a mathematician."

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APPENDIX 1: COURSES TAUGHT BY CHRISTOFFEL

The first part of the list contains the courses offered fairly regularly in at least one of the four institutes where Christoffel taught, or those given before his Strasbourg years. The second part lists those courses taught only at Strasbourg, the majority being more advanced courses. The date indicates the first time a course was presented. Courses with similar titles are listed only once.

Of the advanced courses given fairly regularly between 1873 and 1894, one can note "Theorie der Abelschen Functionen" and "Theorie der binären Formen."

ETH stands for the Polytechnic School at Zürich, TUB for the Gewerbeakademie at Berlin, UB for the University of Berlin, and USTR for the University of Strasbourg.

The information concerning the courses given at the University of Berlin, at Zürich, and at the Gewerbeakademie at Berlin is taken from [2], [Knus 1981], and [Knobloch 1981], respectively. The list of the courses offered at Strasbourg from 1872 to 1895 was kindly supplied by K. v. Meyenn [6].

Differential- und Integralrechnung I und II (ETH); Ausgewählte Kapitel der (Differential- und) Integralrechnung (ETH, TUB, USTR); Analytische Geometrie des Raumes (ETH, TUB); Theorie der gewöhnlichen Differentialgleichungen (UB, ETH, USTR); Theorie der partiellen Differentialgleichungen und Anwendungen (UB, ETH, TUB, USTR); Theorie der Functionen einer komplexen veränderlichen Größe (ETH, TUB, USTR); Theorie der Elastizität fester Körper, nebst Anwendungen (auf Akustik u. Optik) (UB, TUB); Theorie der Dispersion des Lichtes (UB); Über Methoden zur angenäherten Berechnung bestimmter Integrale (Gauss'sche Quadratur und drei ähnliche Methoden) (UB); Methode der kleinsten Quadrate (ETH, TUB); Theorie der bestimmten Integrale (und unendlichen Reihen) (UB, TUB, USTR); Theorie der Anziehungskräfte, welche nach dem umgekehrten Quadrat der Entfernung wirken (ETH, USTR); Analytische Mechanik (TUB); Theorie und Anwendung der Fourier'schen Reihen (TUB, USTR);

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Theorie der Abel'schen Functionen (1873); Anwendungen der Abel'schen Functionen (auf die ultraelliptischen Functionen) (1875, 1878/1879); Wahrscheinlichkeitsrechnung und ihre Anwendungen (1874); Anwendung der Integralrechnung auf die Wahrscheinlichkeitsrechnung (1885/1886); Theorie der homogenen Formen und der invarianten Substitutionen (1873-1874); Theorie der quadratischen und der bilinearen Formen mit Anwendungen (1874-1875); Invariantentheorie (1875); Theorie der binären Formen (1876-1877); Einleitung in die Functionentheorie und elliptische Functionen (1876); Einleitung in die Theorie der Functionen einer complexen veränderlichen Größe und Anwendung auf die doppelt-periodischen Functionen (1878); Theorie der ein- und der zweiwerthigen Functionen einer einzigen complexen Variable (1882); Anwendung der complexen Integration auf elliptische Functionen (1884); Theorie der Determinanten (1877/1878); Infinitesimalgeometrie (1878); Einleitung in die Theorie der algebraischen Functionen (1882); Ultraelliptische Functionen (1891).

APPENDIX 2: PUBLICATIONS OF E. B. CHRISTOFFEL

1856 *De motu permanenti electricitatis in corporibus homogeneis*. Inaugural Dissertation, University of Berlin.

1858a Über die Gaussische Quadratur und eine Verallgemeinerung derselben. *Journal für die reine und angewandte Mathematik* 55, 61-82.

1858b Über die lineare Abhängigkeit von Functionen einer einzigen Veränderlichen. *Journal für die reine und angewandte Mathematik* 55, 281-299.

1861a Zur Abhandlung von Heine: "Über Zähler und Nenner der Näherungswerthe von Kettenbrüchen" im 57. Band des Journals für die Reine und Angewandte Mathematik. *Journal für die reine und angewandte Mathematik* 58, 90-92.

1861b Über die Dispersion des Lichts. *Monatsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin,* 906-921, 997-999. [*Annalen der Physik und Chemie* 117 (1862), 27-45.]

1864a Verallgemeinerung einiger Theoreme des Herrn Weierstrass. *Journal für die reine und angewandte Mathematik* 63, 255-272.

1864b Über die kleinen Schwingungen eines periodisch eingerichteten Systems materieller Punkte. *Journal für die reine und angewandte Mathematik* 63, 273-288.

1865a Über die Bestimmung der Gestalt einer krummen Oberfläche durch lokale Messungen auf derselben. *Journal für die reine und angewandte Mathematik* 64, 193-209.

1865b Über die Dispersion des Lichtes. *Annalen der Physik und Chemie* 124, 53-60.

1865c Zur Theorie der einwerthigen Potentiale. *Journal für die reine und angewandte Mathematik* 64, 321-368.

1886 Über den Einfluss von Realitäts- und Stetigkeits-Bedingungen auf die Lösung gewöhnlicher Differentialgleichungen. *Journal für die reine und angewandte Mathematik* 66, 1-14.

1868a Sul problema delle temperature stazionarie e la rappresentazione di una data superficie. *Annali di Matematica pura ed applicata, Serie II* 1, 89-103.

1867 Über einige allgemeine Eigenschaften der Minimumsflächen. *Journal für die reine und angewandte Mathematik* 67, 218-228.

1868b Beweis des Fundamentalsatzes der Invariantentheorie. *Journal für die reine und angewandte Mathematik* 68, 246-252.

1868c Theorie der bilinearen Funktionen. *Journal für die reine und angewandte Mathematik* 68, 253-272.

1868d Allgemeine Theorie der geodätischen Dreiecke. *Mathematisch-physikalische Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin,* 119-176.

1869a Über die Transformation ganzer homogener Differentialausdrücke. *Monatsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin,* 1-6.

1869b Über die Transformation der homogenen Differentialausdrücke zweiten Grades. *Journal für die reine und angewandte Mathematik* 70, 46-70.

1869c Über ein die Transformation homogener Differentialausdrücke zweiten Grades betreffendes Theorem. *Journal für die reine und angewandte Mathematik* 70, 241-245.

1870a Sopra un problema proposto da Dirichlet. *Annali di Matematica pura ed applicata, Serie II* 4, 1-9.

1870b Über die Abbildung einer einblättrigen, einfach zusammenhängenden, ebenen Fläche auf einem Kreise. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-August-Universität zu Göttingen,* 283-298.

1870c Über die Abbildung einer *n*-blättrigen, einfach zusammenhängenden, ebenen Fläche auf einem Kreise. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-August-Universität zu Göttingen,* 359-369.

1871 Über die Integration von zwei partiellen Differentialgleichungen. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-August-Universität zu Göttingen,* 435-453.

1874 Observatio Arithmetica. *Annali di Matematica pura ed applicata, Serie II* 6, 148-152.

1877a Sur une classe particulière de fonctions entières et de fractions continues. *Annali di Matematica pura ed applicata, Serie II* 8, 1-10.

1877b Untersuchungen über die mit dem fortbestehen linearer, partieller Differentialgleichungen verträglichen Unstetigkeiten. *Annali di Matematica pura ed applicata, Serie II* 8, 81-112.

1877c Über die Fortpflanzung von Stössen durch elastische feste Körper. *Annali di Matematica pura ed applicata, Serie II* 8, 193-243.

1879 Über die canonische Form der Riemann'schen Integrale erster Gattung. *Annali di Matematica pura ed applicata, Serie II* 9, 240-301.

1881 Algebraischer Beweis des Satzes von der Anzahl der linear unabhängigen Integrale erster Gattung. *Annali di Matematica pura ed applicata, Serie II* 10, 81-100.

1882 Bemerkungen zur Invariantentheorie. *Mathematische Annalen* 19, 280-290.

1888 Lehrsätze über arithmetische Eignschaften der Irrationalzahlen. *Annali di Matematica pura ed applicata, Serie II* 15, 253-276.

1896 Die Convergenz der Jacobi'schen θ-Reihe mit den Moduln Riemann's. *Vierteljahresschrift der Naturforschenden Gesellschaft in Zürich* 41, 2, 3-6.

1900 Über die Vollwerthigkeit und die Stetigkeit analytischer Ausdrücke. *Mathematische Annalen* 53, 465-492.

1901 Vollständige Theorie der Riemann'schen θ-Function. *Mathematische Annalen* 54, 347-399.

1902 Querschnittstheorie. *Mathematische Annalen* 55, 497-515.

NOTES

1. This paper is essentially the same as the one appearing in [Butzer & Fehér 1981].

2. Curriculum vitae by Christoffel of 1859 and 1862, respectively. In *Heimatblätter des Kreises Aachen* 34/35, No. 3/4 (1978); No. 1 (1979), 44, 52-53.

3. The Book of Memoirs for Dr. Fujisawa Tokyo 1938, pp. 271-272 (in Japanese). The author would like to thank Professor G. Sunouchi (Sendai) for this information.

4. Verzeichnis der seit 1850 an den Deutschen Universitäten erschienen Doktor-Dissertationen und Habilitationsschriften. Edited for Catalogue of the German University Exhibition in Chicago. Munich 1893. Separate edition by the Deutsche Mathematiker-Vereinigung.

5. Burau, W. Written communications.

6. v. Meyenn, K. Written communications.

7. Archivalien des österreichischen Staatsarchives im Allgemeinen Verwaltungsarchiv, Aktenzeichen 15. 190/1875 6832. This information was kindly communicated by Hofrätin Dr. Auguste Dick (Vienna) in April 1980.

8. Rovnyak, J. Oral communication, dated 10 March 1980.

9. [Levi-Cività 1926; Schouten 1924; Eisenhart 1926, 1927; Cartan 1946; Blaschke 1923]. For literature on invariant theory, which "has already been pronounced dead several times" [1971, 1], see especially [Wright 1908; Weitzenböck 1923; Schur (and Grunsky) 1968; Dieudonné & Carrell 1971; Springer 1977]. For the history of tensor analysis and invariant theory see [Meyer 1890/1891; Pauli 1920; Fisher 1966; Reich 1979].

10. According to Earman and Glymour [1978, 307], "it seems probable that Hilbert had derived his field equations and communicated at least an outline of his results to Einstein sometime before November 18...." However, Einstein makes no mention of Hilbert's work. For Einstein in general, see [French 1979].

11. A measure of this success is the fact that the formulas Christoffel applied were cited in many texts and journals at the time; see, e.g., Wüllner, H., *Lehrbuch der Experimentalphysik,* Vol. 2, 3rd ed., 1875, pp. 105 f.; Gercken, W., 1877, *Über die mathematische Theorie der Dispersion des Lichtes*. Göttingen: Universitätsdruck von E. A. Muth (reviewed in *Blättern zu den Annalen der Physik und Chemie,* Vol. II, 1878, pp. 407 ff.); Chwolson, O. D., 1902. *Lehrbuch der Physik,* Vol. II, p. 416. For additional literature see [v. Meyenn 1981].--Communicated by K. v. Meyenn.

12. Meyer-König, W. Written communication, dated 18 January 1980.

13. Of the mathematicians at Berlin during the last century, the collected works of the following have appeared: Steiner, J., 1881/1891; Jacobi, G. G. J., 1881/1891; Borchardt, 1888; Dirichlet, 1889/1897; [Schwarz, H. A., 1890]; Weierstrass 1894/1927 (incomplete); Kronecker, L., 1895/1930; Fuchs, L., 1904/1909; Eisenstein, G., 1975; Kummer, E. E., 1975.

14. Klein, F. Letter of Klein located in the University Library at Göttingen. A copy was kindly communicated to the author by v. Meyenn, K. (Stuttgart).

15. Gautschi, W. Written communication, dated 31 March 1980.

16. Leichtweiss, K. Written communication, dated 1 April 1980.

17. Pfluger, A. Written communication, dated 19 February 1980.

18. Hölder, E. Written communication, dated 28 February 1980.

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