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A NOTE ON THE ETA FUNCTION FOR QUOTIENTS OF $\mathrm{PSL}_2(\mathbf{R})$ BY CO-COMPACT FUCHSIAN GROUPS

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IN THIS PAPER we compute the value at 0 of the eta function [4] associated to the Dirac operator on $\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})$, where Γ is a co-compact Fuchsian group. This is done by considering a family of metrics parameterized by $t \in (0, \infty)$. We thus have $\eta_t^\Gamma(0)$ defined for each t and we calculate $\lim_{t\to 0} \eta_t^\Gamma(0)$. (It turns out that if one makes the analogous constructions for SU(2), obtaining $\eta_t^c(0)$, then $\lim_{t\to 0} \eta_t^\Gamma(0)$ and $\lim_{t\to 0} \eta_t^c(0)$ are related by the Hirzebruch proportionality factor [7] provided Γ has no elliptic elements.) Our calculation uses little of the geometry of $\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})$ but requires substantial information about the representations of $SL_2(\mathbf{R})$.

The group $\mathrm{PSL}_2(\mathbf{R})$ acts transitively and freely on $T_1\mathscr{H}$, the space of unit tangent vectors to the upper half-plane $\mathscr{H}$, and may be identified with the orbit of $(i, 1)$. If we give $\mathscr{H}$ the standard Poincaré metric $(dx^2+dy^2)/y^2$ and give $T_1\mathscr{H}$ the induced metric; this metric is invariant under $\mathrm{PSL}_2(\mathbf{R})$ and the basis vectors

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\mathrm{SL}_2(\mathbf{R})$ have length 2 and are mutually perpendicular. We give $\mathrm{PSL}_2(\mathbf{R})$ the spin structure corresponding to the left invariant trivialization of its tangent bundle and for which K, A, H are unit perpendicular vectors. We may now form the bundle of spinors S and the Dirac operator $P: \Gamma(S) \to \Gamma(S)$. If Γ is any co-compact discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ we have similarly a space of spinors and a Dirac operator too. The Dirac operator is an elliptic operator and since $\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})$ is compact its eigenspaces are finite dimensional and we can form its eta function. A basic theorem of [4, 5, 6] is that $\eta(s)$ can be defined as a meromorphic function on the whole of C and is finite at 0. We prove the following results concerning P and $\eta(0)$ on $\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})$.

THEOREM 1. *$\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})$ has no harmonic spinors for any co-compact Fuchsian group Γ: that is,* $\ker P = 0$.

THEOREM 2. *If $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ is a co-compact Fuchsian group of signature $\{g; \alpha_1, \ldots, \alpha_n\}$ then*

$$\eta(0) = \frac{1}{48\pi} \ \mathrm{vol}\ \ F_\Gamma + \frac{1}{6}\left(\sum_{i=1}^{n}\left(\alpha_i - \frac{1}{\alpha_i}\right)\right)$$
$$= \frac{1}{24}\left\{2g-2+n+\sum_{i=1}^{n}(4\alpha_i - 5/\alpha_i)\right\},$$

where F_Γ is a fundamental domain for the action of Γ on $\mathscr{H}$.

THEOREM 3. *Let $\Sigma \subset \mathrm{PSL}_2(\mathbf{R})$ be a co-compact Fuchsian group without elliptic elements (so that Σ operates freely on $\mathscr{H}$ and its signature is $\{g\}$, where g is the genus of $\Sigma\backslash\mathscr{H}$). Then $\Sigma\backslash\mathrm{PSL}_2(\mathbf{R}) = \partial DT(\Sigma\backslash\mathscr{H})$, where $DT(\Sigma\backslash\mathscr{H}) = X_\Sigma$ denotes the unit tangent disc bundle of $\Sigma\backslash\mathscr{H}$ and the spin structure on $\Sigma\backslash\mathrm{PSL}_2(\mathbf{R})$ extends uniquely to X_Σ. If P_Σ denotes the corresponding Dirac operator on X_Σ, then index $P_\Sigma = 0$.*

In as much as the scalar curvature of $\Gamma\backslash PSL_2(\mathbf{R})$ is negative (proposition 3.9), neither Theorem 1 nor Theorem 3 is a consequence of Lichnerowicz's theorem, though they are no doubt special cases of more general theorems.

Theorem 1 is proved by direct computation in §1, and in §2 we obtain sufficient information on the eigenvalues of the Dirac operator to be able to prove Theorem 2 for a group Σ without elliptic elements in §3. This section is probably the most interesting part of the paper, for it defines an eta function $\eta_t(s)$ for a varying family of metrics ρ_t (where the metric we are interested in is ρ_1) and computes $\lim_{t\to 0} \eta_t(0)$. The result for $t = 1$ is then deduced by using a formula for the variation of $\eta_t(0)$ with t. (When the authors described this to Professor Atiyah he pointed out that it was very similar to the idea of E. Witten in [19].) In §4 we prove Theorem 3 by using the general index formula [4] and finally, in §5, we establish Theorem 2 in general using Theorem 3 and the special case of Theorem 2 for a group without elliptic elements. Naturally this paper is much indebted to [12], where the compact case is considered.

§1

Here we shall write Γ for a general co-compact subgroup of $\mathrm{PSL}_2(\mathbf{R})$ and Σ for one without elliptic elements.

Take as a basis for $sl_2(\mathbf{R})$ the vectors

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The commutation laws are:

$$[H, K] = 2A, \; [H, A] = 2K, \; [A, K] = -2H.$$

If $t \in \mathbf{R}$ and $t > 0$, set $E_1 = \frac{1}{t}K$, $E_2 = A$ and $E_3 = H$. Take these as unit perpendicular vectors and endow $\mathrm{PSL}_2(\mathbf{R})$—and $\mathrm{SL}_2(\mathbf{R})$—with the metric ρ_t which is obtained from this by left translation. The corresponding Levi–Civita connexion is then determined by the formula

$$2X.\nabla_Z Y = Z[X, Y] + Y.[X, Z] - X.[Y, Z],$$

where the dot denotes scalar product, and results in the following formulae:

$$\begin{array}{lll} \nabla_H H = 0 & \nabla_H A = K = tE_1 & \nabla_H K = -t^2 A \\ \nabla_A H = -K & \nabla_A = 0 & \nabla_A K = t^2 H \\ \nabla_{E_1} H = -\left(\dfrac{2+t^2}{t}\right)A & \nabla_{E_1} A = \left(\dfrac{2+t^2}{t}\right)\mathscr{H} & \nabla_{E_1} K = 0. \end{array} \tag{1.1}$$

In as much as $\mathrm{PSL}_2(\mathbf{R})$ is not simply connected we must specify a spin structure: we take that one determined by the left invariant trivialization. With any orthonormal basis of tangent vectors we have a corresponding basis of spinors and a lifted connexion. Calculating exactly as in [12] we find that, if ψ is a basic spinor, the lifted connexion is

$$\begin{aligned} \nabla_H \psi &= \frac{1}{4}(AK - KA)\psi = \frac{1}{2}AK\psi, \\ \nabla_A \psi &= \frac{1}{4}(-HK + KH)\psi = \frac{1}{2}KH\psi, \\ \nabla_{E_1} \psi &= \left(\frac{2+t^2}{4t}\right)(-HA + AH)\psi = \frac{1}{2t}(2+t^2)AH\psi. \end{aligned} \tag{1.2}$$

Consequently, if P_t is the Dirac operator for the given spin structure and for the metric ρ_t, we see that for a basic spinor ψ

$$\begin{aligned} P_t \psi &= \frac{1}{2}E_1 AH\left\{tHAE_1 + tAE_1H + \left(\frac{2+t^2}{t}\right)E_1AH\right\}\psi \\ &= \left(\frac{2-t^2}{2t}\right)\psi. \end{aligned}$$

We take the spin representation to be given by

$$\begin{aligned} \omega H &\mapsto \mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \omega E_1 &\mapsto -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \omega A &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{1.3}$$

where $\mathrm{i} = \sqrt{-1}$ and $\omega = E_1AH = \frac{1}{t}KAH$. We may write any spinor, ϕ, as $\alpha\psi_1 + \beta\psi_2$ where ψ_1 and ψ_2 are the basic spinors and α, β are smooth functions. Let us denote this ϕ by the vector $\binom{\alpha}{\beta}$. Then

$$P_t\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\frac{2-t^2}{2t}\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} -\mathrm{i}E_1 & A + \mathrm{i}H \\ -A + \mathrm{i}H & \mathrm{i}E_1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

It is convenient to write

$$Z = \mathrm{i}K,\ 2X_- = A - \mathrm{i}H,\ 2X_+ = A + \mathrm{i}H$$

so as to have a basis for $sl_2(\mathbf{C})$ in standard form. In terms of this basis (and with the above notation) we find that

$$P_t = \left(\frac{2-t^2}{2t}\right)I + \begin{pmatrix} -Z/t, & 2X_+ \\ 2X_-, & Z/t \end{pmatrix}. \tag{1.4}$$

Comparison with [12] shows that the first order terms are the same, as indeed they must be since they are determined by the symbol.

Now that we have an expression for P_t, we may prove Theorem 1. In fact we prove the following theorem which is a little stronger.

THEOREM 1.5. *Let $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ be co-compact, and endow $\Gamma\backslash\mathrm{PSL}_2(\mathbf{R})$ with the metric ρ_t. Then, for the invariant spin structure, there are no non-zero harmonic spinors on $\Gamma\backslash\mathrm{PSL}_2(\mathbf{R})$ if $0 < t < \sqrt{2}$: that is,* $\ker P_t = 0$ *if* $0 < t < \sqrt{2}$.

Let S denote the space spinors. As we have tacitly noted,

$$L^2(\Gamma\backslash \mathrm{PSL}_2(\mathbf{R}); S) \cong L^2(\Gamma\backslash \mathrm{PSL}_2(\mathbf{R}))\oplus L^2(\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})).$$

Since $\Gamma\backslash \mathrm{PSL}_2(\mathbf{R})$ is compact this space decomposes into the (completed) sum of irreducible representations of $\mathrm{PSL}_2(\mathbf{R})$ and the operator P_t respects each isotypic component. We investigate the behaviour of P_t on each. The representation theory of $\mathrm{SL}_2(\mathbf{R})$, due to Bargmann [7], is well known and there are several accounts [1, 7, 11]; our notation will be closest to [1]. Let π be an irreducible unitary representation of $\mathrm{SL}_2(\mathbf{R})$. Then π is determined by a parameter $s\in\mathbf{C}$ and by a sign. If V_π is the Hilbert space for π then we may decompose V_π with respect to the action of the compact subgroup $K=\left\{\begin{pmatrix}\cos\theta, & \sin\theta\\ -\sin\theta, & \cos\theta\end{pmatrix}=a_\theta;\ 0\le\theta<2\pi\right\}\cong S^1$ of $\mathrm{SL}_2(\mathbf{R})$. So

$$V_\pi=\hat{\bigoplus_{n\in\mathbf{Z}}} D_n^\pi$$

where a_θ acts on D_n^π by multiplication by $e^{ni\theta}$. Each D_n^π has dimension at most 1 and, moreover, X_+ carries D_n^π into D_{n+2}^π whilst X_- carries it into D_{n-2}^π. Consequently X_+X_- is an endomorphism of D_n^π, and it is entirely determined by the parameter $s\in\mathbf{C}$: on D_n^π, $X_+X_-=\frac{1}{4}(s^2-(n-1)^2)$.

Let us fix an irreducible representation $\pi(\pi\neq 0)$ corresponding to a parameter $s\in\mathbf{C}$ and let us set $l=(2-t^2)/t$. If ϕ is a spinor given by the vector $\binom{\alpha}{\beta}$, where $\alpha\in D_n^\pi$, and if $P_t=0$ then formula 1.4 gives us the following equations.

$$0=-\frac{Z\alpha}{t}+\frac{l\alpha}{2}+2X_+\beta$$

$$0=\frac{Z\beta}{t}+\frac{l\beta}{2}-2X_-\alpha$$

Consequently $\beta\in D_{n-2}^\pi$ and

$$\begin{aligned}(l-2Z/t)\alpha &= -4X_+\beta\\ &= -16X_+(l+2Z/t)^{-1}X_-\alpha\end{aligned}$$

Remembering that Z acts on D_n^π by multiplication by n and that X_+X_- acts by multiplication by $\frac{1}{4}(s^2-(n-1)^2)$ we find the equation.

$$\left(l-\frac{2n}{t}\right)\alpha=-16\left(l+\frac{2n-4}{t}\right)^{-1}\left(\frac{s^2-(n-1)^2}{4}\right)\alpha.$$

We may have equality with $\alpha\neq 0$ iff

$$\left(l-\frac{2n}{4}\right)\left(l+\frac{2n-4}{t}\right)=4((n-1)^2-s^2); \tag{1.6}$$

that is, iff n and s satisfy

$$-(n-1)^2+\left(\frac{tl}{2}-1\right)^2=t^2((n-1)^2-s^2)$$

or equivalently,

$$(t^2+1)(n-1)^2=t^2s^2+\frac{t^4}{4}. \tag{1.7}$$

Since we are working over $\Gamma \backslash PSL_2(\mathbf{R})$, the only permissible representations of $SL_2(\mathbf{R})$ are those which descend to $PSL_2(\mathbf{R})$, namely those for which $-I$ acts as the identity. So n must always be even and hence the left-hand side of 1.6 is never zero. If we fix n we may solve for the possible parameters $s: s = \pm\frac{1}{2}(4(1+t^{-2})(n-1)^2 - t^2)^{1/2}$. Now, the principal series representations are parametrized by $s \in i\mathbf{R}$. For an element in a principal series representation to be in $\ker P_t$ we must then have $t^2 \geq 4(t^2+1)(n-1)^2$: at the very least, $t > 2$. Considering next the cases of representations in the discrete series and in the complementary series we see that there can be no non-trivial solutions of $P_t(\frac{\alpha}{\beta}) = 0$ if $t < \sqrt{2}$, as asserted in Theorem 1.5. (Notice that if $t = \sqrt{2}$ then the basic spinors are decidedly in $\ker P_t$.)

§2

The formula 1.4 for P_t also gives information about the eigenvalues. We extract some of that information in this section. If a given representation π associated to $s \in i\mathbf{R} \cup (-1, 1)$—so a representation in the principal or complementary series—appears in $L^2(\Gamma \backslash PSL_2(\mathbf{R}))$ with multiplicity $N_\pi \neq 0$ then from 1.4 we see that for each $n \in \mathbf{Z}$ there are two eigenvalues

$$\lambda = -\frac{t}{2} \pm \left((2n-1)^2\left(1+\frac{1}{t^2}\right) - s^2\right)^{\frac{1}{2}}$$

each occurring with multiplicity N_π.

On the other hand, for $s \in \mathbf{N} - 0$ we have the rather different case of a discrete series representation. To the parameter s correspond two discrete series representations, often written π_{s+1}^+ and π_{s+1}^- [1]. When decomposed with respect to the action of the vector Z in the one case we only have components in which Z acts by multiplication by positive integers, in the other only ones where Z acts by multiplication by negative integers. Naturally, the sign tells us which. The subscript gives us the additional information that the first non-zero component (counting up or counting down as the case may be) is where Z acts by multiplication by $s+1$ (positive case) or by $s-1$ (negative case). We shall refer to a vector in this component as an *extreme* vector. If v is in a component where Z acts by multiplication by l we shall call v a vector of type l. Recall that since all representations in question must descend to $PSL_2(\mathbf{R})$ only even types may occur. Hence $s+1$ is even and we shall write $s+1 = 2k$, $k \in \mathbf{N} \backslash 0$. There are now two distinct cases. If we have an extreme vector u of type $2k$ for π_{2k}^+ then $(u, 0)$ is an eigenvector for P_t with eigenvalue $-\frac{t}{2}+\frac{1}{t}-\frac{2k}{t}$. Similarly if u is an extreme vector of type $-2k$ for π_{2k}^-, then $(0, u)$ is an eigenvector for P_t with eigenvalue $-\frac{t}{2}+\frac{1}{t}-\frac{2k}{t}$. On the other hand, if we are not at the extreme point then, just as in the case of the principal and complementary series, we get two eigenvalues

$$\lambda = -\frac{t}{2} \pm ((2n-1)^2(1+t^2) - (2k-1)^2)^{1/2}$$

for each $\pi_{2k}^\pm$ and for each $n > k$. Both appear with multiplicity that of the representation $\pi_{2k}^\pm$ in $L^2(\Gamma \backslash PSL_2(\mathbf{R}))$. These multiplicities are known [11, 13, 17]. If $N_{2k}^\pm$ denotes the multiplicity of $\pi_{2k}^\pm$ in $L^2(\Gamma \backslash PSL_2(\mathbf{R}))$, where Γ is a co-compact Fuchsian group of signature $\{g; \alpha_1, \ldots, \alpha_n\}$, then

$$N_{2k}^\pm = \frac{\text{vol } F_\Gamma}{4\pi}(2k-1) \pm \sum_{j=1}^{n} \sum_{r=1}^{\alpha_j-1} \frac{i}{2\alpha_j} e^{\pm i\pi(2k-1)/\alpha_j} \left(\sin\frac{\pi r}{\alpha_j}\right)^{-1}, \text{ if } \quad k > 1;$$

$$N_2^\pm = \frac{\text{vol } F_\Gamma}{4\pi} + 1 \pm \sum_{j=1}^{n} \sum_{r=1}^{\alpha_j-1} \frac{i}{2\alpha_j} e^{\pm i\pi/\alpha_j} \left(\sin\frac{\pi r}{\alpha_j}\right)^{-1}; \tag{2.1}$$

where F_Γ denotes a fundamental domain for the action of Γ on $\mathscr{H}$ and $\mathscr{H}$ has the standard Poincaré metric. In the case of a discrete subgroup Σ without elliptic points (so of signature $\{g_\Sigma\}$, where g_Σ denotes the genus of $\Sigma\backslash\mathscr{H}$) the formulae simplify:

$$N^{\pm}_{2k} = (2k-1)(g_\Sigma - 1) \qquad \text{if} \quad k > 1$$

$$N^{\pm}_{2} = g_\Sigma. \tag{2.2}$$

We collect the information about the eigenvalues needed in the next section together in the following proposition.

Proposition 2.3 *Let $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ be a co-compact Fuchsian group of signature $\{g; \alpha_1, \ldots, \alpha_n\}$ and let P_t denote the Dirac operator on $\Gamma\backslash\mathrm{PSL}_2(\mathbf{R})$ corresponding to the metric ρ_t and the trivial spin structure. Then the eigenvalues of P_t are:*

(i) $-\frac{t}{2}-\left(\frac{2k-1}{t}\right)$ *for $k \geq 1$ with multiplicity $N^+_{2k}+N^-_{2k}$ where $N^{\pm}_{2k}$ are as in* 2.1;

(ii) $-\frac{t}{2}-((2n-1)^2(1+t^{-2})-(2k-1)^2)^{1/2}$ *with multiplicity $N^+_{2k}+N^-_{2k}$ for each $k \geq 1$ and $n > k$;*

(iii) $-\frac{t}{2}-((2n-1)^2(1+t^{-2})-s^2)^{1/2}$ *for $n \in \mathbf{Z}$ and $s \in \Lambda$, where Λ is some countable subset of $(-1, 1) \cup i\mathbf{R}$;*

(iv) $-\frac{t}{2}+\frac{1}{t}$ *with multiplicity* 2;

(v) $-\frac{t}{2}+((2n-1)^2(1+t^{-2})-(2k-1)^2)^{1/2}$ *with multiplicity $N^+_{2k}+N^-_{2k}$ for each $k \geq 1$ and $n > k$;*

(vi) $-\frac{t}{2}+((2n-1)^2(1+t^{-2})-s^2)^{1/2}$ *for $n \in \mathbf{Z}$ and $s \in \Lambda$ and with the same multiplicity as the corresponding eigenvalue in* (iii).

To calculate the value of the η-function $\eta_t(s)$ at 0 we do not, fortunately, need to know Λ explicitly.

§3

In this section we calculate $\eta_t^\Sigma(0)$, $0 < t < \sqrt{2}$, for a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ without elliptic points. The value of $\lim_{t\to 0} \eta_t^\Gamma(0)$ for a co-compact subgroup Γ of signature $\{g; \alpha_1, \ldots, \alpha_n\}$ may be calculated directly from (2.1) and (2.3). Here we compute first for a subgroup without elliptic points and then deduce the general case using theorems from [5, 6] and [9]. Let Σ be a co-compact Fuchsian group without elliptic points and let us write simply $\eta_t(s)$ for $\eta_t^\Sigma(s)$ and g for g_Σ, the genus of the Riemann surface $\Sigma\backslash\mathscr{H}$. For $\mathrm{Re}(s)$ large we may divide $\eta_t(s)$ into two pieces,

$$\eta_t(s) = \eta_t^d(s) + \eta_t^c(s),$$

where $\eta_t^d(s)$ is the part of the η-function coming from the discrete series and $\eta_t^c(s)$ is the remainder.

PROPOSITION 3.1. *Both* $\eta_t^d(s)$ *and* $\eta_t^c(s)$ *may be analytically continued to functions meromorphic in the whole complex plane and finite at* 0. *Moreover,*

(i) $\lim_{t\to 0} \eta_t^d(0) = (1-g)/6,$

(ii) $\lim_{t\to 0} \eta_t^c(0) = 0.$

From this proposition and the smooth variation of $\eta_t(0)$ as t varies, $0 < t < \sqrt{2}$, we can deduce Theorem 2 for a subgroup Σ without elliptic elements.

We begin the proof of Proposition 3.1 by considering $\eta_t^d(s)$ and we shall suppose, although it is not necessary here, that $0 < t < \sqrt{2}$. Divide $\eta_t^d(s)$ into two pieces, as well:

$$\eta_t^d(s) = \eta_t^1(s) + \eta_t^2(s),$$

where $\eta_t^1(s)$ is the contribution to $\eta_t^d(s)$ by the eigenvalues of (i) and (iv) of Proposition 2.3 (so from the trivial representation and the extreme vectors of the discrete series representations) and $\eta_t^2(s)$ is the contribution by the eigenvectors of (ii) and (v) of Proposition 2.3. The functions are thus explicitly known and, in particular,

$$\eta_t^1(s) = t^s(g-1)\left\{-\sum_{k=1}^{\infty} \frac{2(2k-1)}{\left(2k-1+\frac{t^2}{2}\right)^s}\right\} - \frac{2t^s}{\left(1+\frac{t^2}{2}\right)^s} + \frac{2t^s}{\left(1-\frac{t^2}{2}\right)^s}.$$

Comparison with the Riemann zeta function

$$\zeta(a,s) = \sum_{n=0}^{\infty} (n+a)^{-s}$$

tells us that $\eta_t^1(s)$ has a meromorphic extension to the whole of C, holomorphic if $\text{Re}(s) > 1$ and with only simple poles. If we set

$$\zeta_0(a,s) = \sum_{n=1}^{\infty} (2n-1+a)^{-s}$$

then

$$\zeta_0(s,a) = \zeta(a,s) - 2^{-s}\zeta\left(\frac{a}{2}, s\right) \text{ and}$$

$$\eta_t^1(s) = 2(1-g)\left\{\zeta_0\left(\frac{t^2}{2}, s-1\right) - \frac{t^2}{2}\zeta_0\left(\frac{t^2}{2}, s\right)\right\} - \frac{2t^s}{\left(1+\frac{t^2}{2}\right)^s} + \frac{2t^s}{\left(1-\frac{t^2}{2}\right)^s}. \tag{3.2}$$

It is known [18] that $\zeta(a,s)$ is holomorphic at 0 and -1 with $\zeta(a,0) = -\frac{1}{2} - a$ and $\zeta(a,-1) = -\frac{1}{6}\phi_3'(a)$, where ϕ_3' is the derived polynomial of the third Bernoulli polynomial $\phi_3(z) = z^3 - (3/2)z^2 + (1/2)z$. As a result,

$$\eta_t^1(0) = (1-g)\left\{\frac{1}{6} + \frac{t^4}{8}\right\},$$

so that $\lim_{t\to 0} \eta_t^1(0) = (1-g)/6$.

To handle $\eta_t^2(s)$, introduce as in [12] the auxiliary function

$$f_t(s) = (g-1) \sum_{n > k \geq 1} (2k-1)\, q_t(n,k)^{-s},$$

where $q_t(n,k) = \{(2n-1)^2(1+t^2) - t^2(2k-1)^2\}^{1/2}$. A similar comparison with the Riemann zeta function shows that $f_t(s)$ is holomorphic for $\mathrm{Re}(s) > 3$ and may be analytically continued to the whole of $\mathbf{C}$ as a meromorphic function with simple poles, independent of t for t small and with the residues continuous functions of t. But

$$\eta_t^2(s) = \frac{4}{t^s} \sum_{n > k \geq 1} (g-1)(2k-1) \left\{ \left(q_t(n,k) - \frac{t^2}{2}\right)^{-s} - \left(q_t(n,k) + \frac{t^2}{2}\right)^{-s} \right\}$$

so that the same is true of $\eta_t^2(s)$. Not only this, but for small t we have an expansion of $\eta_t^2(s)$ in terms of $f_t(s)$:

$$\eta_t^2(s) = \frac{4}{t^s} \left\{ t^2 s f_t(s+1) + t^6 \frac{s(s+1)(s+2)}{24} f_t(s+3) \right\} + t^{8-3}\theta(t,s),$$

where $\theta(t,s)$ is holomorphic at 0 for small t. Since $f_t(s)$ has only simple poles and there the residues are continuous in t, we see that $\eta_t^2(0) = t^2\phi(t)$ where $\phi(t)$ is continuous at 0. So $\lim_{t\to 0} \eta_t^2(0) = 0$ and we have established (i).

A similar argument establishes that $\lim_{t \to 0} \eta_t^c(0) = 0$. The good behaviour of $\eta_t^c(s) = \eta_t(s) - \eta_t^d(s)$ is clear because we have established it for $\eta_t^d(s)$ and that for $\eta_t(s)$ is proved as a general fact in [6; 4.5].

Let us write $\eta_0(0)$ for $\lim_{t\to 0} \eta_t(0)$. We shall complete the proof of Theorem 2 for a group without elliptic elements by calculating the difference $\eta_t(0) - \eta_0(0)$ when $0 < t < \sqrt{2}$. Since $\ker P_t = 0$ when $0 < t < \sqrt{2}$, we see from [4; 3.10] and [6; 2.10] and a comparison of their conventions and ours (see §4) that

$$\eta_t(0) - \eta_0(0) = -2 \int_{M_\Sigma \times (0,t]} \omega_{\hat{A}}, \tag{3.3}$$

where $M_\Sigma = \Sigma \backslash \mathrm{PSL}_2(\mathbf{R})$ and $\omega_{\hat{A}}$ is the $\hat{A}$ polynomial in the Pontrjagin forms associated to the connexion which on $M_\Sigma + \{u\}$ is the Levi–Civita connexion given by the metric σ_u and which is standard along $(0, t]$. Comparing definitions we realize that

$$\int_{M_\Sigma \times (0,t]} \omega_{\hat{A}} = -\frac{1}{24} \left(\int_{M_\Sigma \times \{t\}} TP_1(\rho_t) - \lim_{s\to 0} \int_{M_\Sigma \times \{s\}} TP_1(\rho_s) \right), \tag{3.4}$$

where $TP_1(\rho)$ denotes the Chern–Simons form [8] associated to the first Pontrjagin form. (It is a horizontal form because M_Σ is parallelized.)

An orthonormal base for the tangent space to $M_\Sigma \times \{t\}$ consists of the left translates of the vectors

$$E_1 = \frac{1}{t} K,\ E_2 = A,\ E_3 = H.$$

With respect to these fields the Levi–Civita connexion has connexion matrix θ_{ij} (determined according to the rule $\nabla_X(E_i) = \sum_{j=1}^{3} \theta_{ji}(X) E_j$), where

$$\theta_{12} = tE_3^* = tH^*, \quad \theta_{13} = -tE_2^* = -tA^*,$$

$$\theta_{23} = -\frac{(2+t^2)}{t} E_1^* = -(2+t^2)K^*; \tag{3.5}$$

and the star denotes the dual vectors. So the curvature matrix $\Omega_{ij}\left(\Omega_{ij} = d\theta_{ij} - \sum_{k=1}^{3} \theta_{ik} \wedge \theta_{jk}\right)$ is as follows:

$$\begin{aligned}\Omega_{12} &= t^3 K^* \wedge A^*,\\ \Omega_{13} &= t^3 K^* \wedge H^*,\\ \Omega_{23} &= (4+3t^2)H^* \wedge A^*.\end{aligned} \tag{3.6}$$

Consequently [8],

$$\begin{aligned} TP_1(\rho_t) &= \frac{1}{4\pi^2}\{\theta_{12}\wedge\theta_{13}\wedge\theta_{23}+\theta_{12}\wedge\Omega_{12}+\theta_{13}\wedge\Omega_{13}+\theta_{23}\wedge\Omega_{23}\}\\ &= \frac{1}{\pi^2}\{2+t^2(2+t^2)\}\, K^*\wedge A^*\wedge H^*.\end{aligned}$$

Now under the identification $\mathrm{PSL}_2(\mathbf{R}) \cong T_1\mathscr{H}$ (defined by mapping $\mathrm{PSL}_2(\mathbf{R})$ to the orbit of $(i, 1)$) the vectors K, A, H each map into vectors of length 2. Hence $K^*\wedge A^*\wedge H^*$ corresponds to $\omega/8$ where ω is the volume form on $T_1\mathscr{H}$. This implies that

$$\begin{aligned}\int_{M_\Sigma\times\{t\}} TP_1(\rho_t) &= \frac{1}{4\pi}(2+t^2(2+t^2))\,\mathrm{vol}\,(\Sigma\backslash\mathscr{H})\\ &= 2(g-1)+t^2(2+t^2)\,(g-1).\end{aligned} \tag{3.7}$$

This (3.3, 3.4 and 3.7) proves the following theorem which is an extension of Theorem 2 in the case of a subgroup without elliptic elements.

THEOREM 3.8. *If* $\Sigma \subset \mathrm{PSL}_2(\mathbf{R})$ *is a co-compact subgroup without elliptic elements and* g *is the genus of* $\Sigma\backslash\mathscr{H}$ *then*

$$\eta_t(0) = -\frac{(g-1)}{12}(2-t^2(2+t^2))$$

if $0 < t < \sqrt{2}$.

From (3.6) we can compute immediately the scalar curvature of M_Σ.

PROPOSITION 3.9. *The scalar curvature of* $M_\Sigma = \Sigma\backslash\mathrm{PSL}_2(\mathbf{R})$ *endowed with the metric* ρ_t *is* $-8-2t^2$.

§4

Here we verify Theorem 3. We know [16] (it is easily shown) that $X_\Sigma = \Sigma\backslash D\,(T\mathscr{H})$ has a unique spin structure extending the one on its boundary given by the left invariant trivialization $\mathscr{L}$. Let P denote the Dirac operator on X_Σ. Our orientation convention, that of the boundary followed by the *outward-pointing* normal [16], unfortunately is not that of [4] where the *inward-pointing* normal is used (see theorem 10, p. 57). So the appropriate boundary value problem is the adjoint of the one discussed in [4] and the index theorem for manifolds with boundary thus tells us that

$$\mathrm{index}\, P = -\int_{X_\Sigma}\hat{A} - \frac{1}{2}(h+\eta(0)),$$

where $\hat{A} = -\frac{1}{24} P_1$ and P_1 is the first Pontrjagin form in the curvature of X_Σ. From Theorem 1 and Theorem 3.8 above we know that

$$\tfrac{1}{2}(h+\eta(0)) = \frac{g-1}{24}, \text{ where } g = g_\Sigma \text{ denotes the genus of } \Sigma\backslash\mathscr{H}.$$

The integral of P_1 over X_Σ may be calculated directly, but it is easier here to use the work of §3 and the relative Pontrjagin class P'_1 defined with respect to $\mathscr{L}$. In [4] it is noted that

$$P'_1[X_\Sigma] = \int_{M_\Sigma} P_1 + \int_{M_\Sigma \times [0,1]} P_1(c),$$

where c is a connexion joining the Riemannian connexion ρ_1 (at 0) to the flat connexion defined by $\mathscr{L}$ (at 1). In this case the second integral is just

$$-\int_{M_\Sigma} TP_1 = -\frac{5}{4\pi}\operatorname{vol}(\Sigma\backslash\mathscr{H})$$
$$= -5(g-1);$$

by (3.7). The class $P'_1 \in H^4(X_\Sigma, M_\Sigma; \mathbf{Z})$ is easily calculated.

LEMMA 4.1. $P'_1[X_\Sigma] = 4(1-g)$.

Proof. Let c'_i denote the relative Chern classes of X_Σ with respect to $\mathscr{L}$. Then

$$P'_1 = (c'_1 - 2c'_2).$$

But $(c'_1)^2[X_\Sigma] = 4(2-2g)$, since the canonical class $K = -2[\Sigma\backslash\mathscr{H}]$, and $c'_2[X_\Sigma] = 2-2g$ by obstruction theory and the Poincaré-Hopf theorem.

From this we see that $\int_{X_\Sigma} P_1 = (g-1)$

so that $\int_{X_\Sigma} \hat{A} = \frac{1-g}{24} = -2\eta(0)$.

Consequently index $P = 0$. (Of course, it is perfectly feasible to compute the index of P directly and so verify the index theorem in this case.)

(At this point it may be convenient to compare the compact and non-compact cases. If we identify $\mathrm{PSL}_2(\mathbf{R})$ with $T_1\mathscr{H}$, the unit sphere bundle to the upper half plane, and decide that $T\mathscr{H}$ shall receive its canonical orientation coming from the complex structure *then* (K, A, H, N), where N denotes the *outward* pointing normal, is a compatible frame. On the other hand, if we identify $SO(3)$ $T_1^* CP^1$, the unit cotangent vectors to the Riemann sphere, then $(K, A, H, -N)$ is a compatible frame of unit vectors, where K, A, H satisfy

$$[K, A] = 2H,\ [A, H] = 2K,\ [H, K] = 2A$$

and so are the vectors e_1, e_2, e_3 of [12].)

§5

Finally, we complete in this section the calculation of $\eta_t(0)$, $0 < t < \sqrt{2}$, for a general co-compact Fuchsian group Γ of signature $\{g; \alpha_1, \ldots, \alpha_n\}$. Let $\Sigma \subset \Gamma$ be a normal subgroup of finite index and with no elliptic elements. Such exists by the theorem of Fox [10]. Let g_Σ

denote the genus of $\Sigma\backslash\mathscr{H}$. The finite group $G = \Gamma/\Sigma$ acts on $X_\Sigma = \Sigma\backslash D(T\mathscr{H})$, where $D(T\mathscr{H})$ denotes the unit disc bundle of the bundle of tangents to $\mathscr{H}$. The action is free on the boundary $M_\Sigma \cong \Sigma\backslash \mathrm{PSL}_2(\mathbf{R})$ and there we have a finite Galois covering

$$M_\Sigma \to M_\Gamma,$$

with group G. As noted in [16] the action of G on X_Σ lifts to an action on the principal spin-bundle and so is an action on the spin-manifold X_Σ. The spinor fixed point index $f(g) = \mathrm{spin}(g, X_\Sigma)$ is thus defined for each $g \in G\backslash 1$, and may be used (since the index of the Dirac operator on X_Σ is zero) to express $\eta_t^\Gamma(0)$ in terms of $\eta_t^\Sigma(0)$.

Proposition 5.1. *For* $0 < t < \sqrt{2}$,

$$\eta_t^\Gamma(0) = \frac{1}{|G|}\eta_t^\Sigma(0) - \frac{2}{|G|}\sum_{g \in G\backslash 1} f(g);$$

where $f(g)$ *is the spinor index of* $g \in G\backslash 1$ *acting on* X_Σ *and* $G = \Gamma/\Sigma$.

The last term on the right hand side may be computed using [2] and [3]. We find that

$$\frac{1}{|G|}\sum_{g \in G\backslash 1} f(g) = -\frac{1}{12}\left\{\sum_{i=1}^{n}\left(\alpha_i - \frac{1}{\alpha_i}\right)\right\}.$$

So once Proposition 5.1 is proved we have established Theorem 2 and, in fact, found $\eta_t(0)$ for $0 < t < \sqrt{2}$. We see in particular that

$$\eta_0^\Gamma(0) = \lim_{t \to 0}\eta_t^\Gamma(0) = -\frac{1}{24\pi}\operatorname{vol} F_\Gamma + \frac{1}{12}\sum_{i=1}^{n}\left(\alpha_i - \frac{1}{\alpha_i}\right); \tag{5.2}$$

something we could have established otherwise.

Proposition 5.1 itself is a consequence of the theory of [4, 5, 6]. We shall now fix $t = 1$ (although this is not necessary) so that under the identification of $\mathrm{PSL}_2(\mathbf{R})$ with the orbit of $(i, 1) \in T\mathscr{H}$ the unit vectors map into vectors of length 2. For each $\alpha \in \hat{G}$, where $\hat{G}$ denotes the set of equivalence classes of irreducible representations of G, we have a flat bundle V_α defined over M_Γ and hence a Dirac operator P_α with coefficients in V_α. Set

$$\xi_\alpha^\Gamma = \frac{1}{2}(\dim\ker P_\alpha + \eta_\alpha^\Gamma(0)),$$

$$\tilde{\xi}_\alpha^\Gamma = \xi_\alpha^\Gamma - \dim\alpha\,\xi_0^\Gamma,$$

as in [5], where $\eta_\alpha^\Gamma(0)$ denotes the value at 0 of the η-function for P_α. (Since the metric is fixed we do not reduce $\tilde{\xi}_\alpha^\Gamma$ mod $\mathbf{Z}$.) On M_Σ we have an action of G on the eigenspaces of P and so we may define

$$\eta^\Sigma(g, s) = \sum_\lambda \operatorname{sign}(\lambda)\frac{\operatorname{trace}(g|V_\lambda)}{|\lambda|^s}, \qquad \text{where } V_\lambda \text{ denotes } \lambda\text{-eigenspace};$$

$$\xi^\Sigma(g) = \frac{1}{2}(\operatorname{tr}(g|\ker p) + \eta^\Sigma(g, 0));\ g \in G. \tag{5.3}$$

Considering P as the lift of the Dirac operator on M_Γ we see that

$$\xi^\Sigma(g) = \sum_{\alpha \in \hat{G}} \bar{\chi}_\alpha(g)\, \xi^\Gamma_\alpha,\ g \in G,$$

$$= \begin{cases} \sum_{\alpha \in \hat{G}} \bar{\chi}_\alpha(g)\, \tilde{\xi}^\Gamma_\alpha & \text{if } g \neq 1 \\ \sum_{\alpha \in \hat{G}} \dim \alpha\, \tilde{\xi}^\Gamma_\alpha + |G|\, \xi^\Gamma_0 & \text{if } g = 1. \end{cases} \tag{5.4}$$

Since the index of P on X_Σ is zero by Theorem 3, the G-index theorem for manifolds with boundary [5] tells us that

$$\xi^\Sigma(g) = -\operatorname{spin}(g, X_\Sigma) = -f(g) \quad \text{if } g \neq 1$$

$$\xi^\Sigma(1) = -\int_{X_\Sigma} \hat{A},$$

where $f(g) = \operatorname{spin}(g, X_\Sigma)$ is the spinor fixed point index [2]. From (5.4) and (5.5) we see that

$$\tilde{\xi}^\Gamma(\alpha) = -\sum_{g \in G \setminus 1} \chi_\alpha(g) f(g),$$

so that, in particular,

$$\tilde{\xi}^\Gamma(0) = -\sum_{g \in G \setminus 1} f(g).$$

Proposition 5.1 now follows immediately from this and 5.4.

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