Linear Algebra and its Applications 325 (2001) 177–189

LINEAR ALGEBRA AND ITS APPLICATIONS

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On the characterization of 2×2 ρ-contraction matrices

Kazuyoshi Okubo [a,1], Ilya Spitkovsky [b,*,2]

[a] *Mathematics Laboratory, Hokkaido University of Education, Sapporo 002-8502, Japan*
[b] *Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA*

Received 1 October 1999; accepted 7 September 2000

Submitted by R.A. Brualdi

Abstract

We give an explicit description of all ρ-contractive (in Nagy–Foiaş sense) 2×2 matrices. This description leads to the formulas for ρ-numerical radii when the eigenvalues of such matrices either have equal absolute values or equal (mod π) arguments. We also discuss (natural) generalizations to the case of decomposable operators with at most two-dimensional blocks covering, in particular, the quadratic operators. © 2001 Elsevier Science Inc. All rights reserved.

AMS classification: Primary 47A12; Secondary 15A60; 47A20

Keywords: ρ-Numerical radii; ρ-Contractions; Decomposable operators

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$, and let ρ be a positive parameter, $\rho \in (0, \infty)$. A bounded linear operator A acting on $\mathfrak{H}$ (notation: $A \in \mathscr{L}[\mathfrak{H}]$) is called a *$\rho$-contraction* if it admits a unitary ρ-dilation, that is, if there exists a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and a unitary operator U on $\mathfrak{K}$ such that

$$A^k = \rho P U^k | \mathfrak{H}, \quad k = 1, 2, \ldots,$$

* Corresponding author.
E-mail addresses: okubo@atson.sap.hokkyodai.ac.jp (K. Okubo), ilya@math.wm.edu (I.M. Spitkovsky).

[1] Supported in part by a Grant-in AID for Scientific Research.
[2] Supported in part by NSF Grant DMS-98-00704.

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where P is the orthoprojection of $\mathfrak{K}$ onto $\mathfrak{H}$.

This concept was introduced by Nagy and Foiaş [6], see also [7]. As was shown in [6,7], the operator A is a ρ-contraction if and only if

$$(\rho-2)\|(I-zA)h\|^2+2\,\mathrm{Re}\,\langle(I-zA)h,h\rangle\geqslant 0 \tag{1.1}$$

for all $h\in\mathfrak{H}$ and for all z in the unit disk $\mathbb{D}=\{z\colon |z|<1\}$. According to Davis [2], condition (1.1) is equivalent to

$$\|zA(\rho-(\rho-1)zA)^{-1}\|\leqslant 1,\quad z\in\mathbb{D}. \tag{1.2}$$

Directly from the definition it follows that the set $\mathscr{C}_\rho$ of all ρ-contractions is invariant under unitary similarities and multiplications by unimodular constants. It is also closed under taking orthogonal sums. More accurately, let $\mathfrak{H}$ be a direct integral of Hilbert spaces, and let A be a *decomposable* operator on $\mathfrak{H}$:

$$\mathfrak{H}=\int_X^{\oplus}\mathfrak{H}_x\,\mathrm{d}\mu(x),\quad A=\int_X^{\oplus}A_x\,\mathrm{d}\mu(x),\quad A_x\in\mathscr{L}[\mathfrak{H}_x] \tag{1.3}$$

(see [9, Chapter IV] for the detailed definition and properties of direct integrals of Hilbert spaces and decomposable operators associated with them). Then, due to (1.1), $A\in\mathscr{C}_\rho$ if and only if $A_x\in\mathscr{C}_\rho$ for each $x\in X$.

Condition (1.1) also implies that the spectrum of any ρ-contraction lies in the closed unit disk $\overline{\mathbb{D}}=\{z\colon |z|\leqslant 1\}$:

$$A\in\mathscr{C}_\rho\quad\Rightarrow\quad\sigma(A)\subset\overline{\mathbb{D}}. \tag{1.4}$$

According to Holbrook [4], the *ρ-numerical radius* $w_\rho(A)$ is defined as

$$w_\rho(A)=\inf\left\{r>0\colon \frac{1}{r}A\in\mathscr{C}_\rho\right\}. \tag{1.5}$$

Hence, A is a ρ-contraction if and only if $w_\rho(A)\leqslant 1$.

From the aforementioned properties of $\mathscr{C}_\rho$ it follows that

$$w_\rho(U^*AU)=w_\rho(A)\quad\text{for any unitary } U, \tag{1.6}$$

$$w_\rho(\xi A)=|\xi|\,w_\rho(A),\quad \xi\in\mathbb{C}, \tag{1.7}$$

and

$$w_\rho(A)=\sup\{w_\rho(A_x)\colon x\in X\} \tag{1.8}$$

if A is given by (1.3).

It was shown in [1] that

$$w_\rho(A)\text{ is a non-increasing function of }\rho\text{ on }(0,\infty) \tag{1.9}$$

and

$$\rho w_\rho(A)=(2-\rho)w_{2-\rho}(A),\quad 0<\rho<2. \tag{1.10}$$

It is well known [4] that $w_1(A)=\|A\|$, $w_2(A)$ equals the so-called *numerical radius* of A (the maximum absolute value of $\langle Ah,h\rangle$, where $h\in\mathfrak{H}$ and $\|h\|=1$), and

$$w_\infty(A) = \lim_{\rho\to\infty} w_\rho(A)$$

is the spectral radius $r(A)$ $(= \max\{|\lambda| : \lambda \in \sigma(A)\})$.

Recall that the operator A is called *normaloid* if $\|A\| = r(A)$. For such operators, $w_\rho(A)$ is constant on $[1, \infty)$ due to (1.9)—the result originally established in [3].

In particular, when A is normal, $w_\rho(A) = \|A\| = r(A)$ for $1 \leqslant \rho \leqslant \infty$. The latter can be represented in the form (1.3) with one-dimensional blocks A_x. Hence, it seems natural to consider the class of operators next in order of complexity, namely, operators (1.3) with at most two-dimensional blocks A_x. This is the subject of our paper.

In Section 2, we derive the ρ-contraction criterion for 2×2 "building blocks" of A. The particular cases of matrices with the spectrum lying on a line passing through the origin or on a circle centered there are considered in Sections 3 and 4, respectively. Section 5 deals with the general case covering, in particular, the so-called quadratic operators.

2. ρ-Contractive 2 × 2 matrices

In this section, we consider 2×2 matrices A. Due to (1.6), we may without loss of generality suppose that A is upper triangular

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}. \tag{2.1}$$

Theorem 2.1. *Let A be of the form* (2.1). *Then $A \in \mathscr{C}_\rho$ if and only if*

$$|a|, |b| \leqslant \begin{cases} 1 & \text{for } \rho \geqslant 1, \\ \rho/(2-\rho) & \text{for } \rho \leqslant 1, \end{cases} \tag{2.2}$$

and

$$|c|^2 + |a-b|^2 \leqslant \min_{-\pi\leqslant\theta\leqslant\pi} \left|z(\theta) + \rho + \overline{a}b(\rho-2)\right|^2, \tag{2.3}$$

where $z(\theta) = (1-\rho)(b\mathrm{e}^{\mathrm{i}\theta} + \overline{a}\mathrm{e}^{-\mathrm{i}\theta})$.

Proof. Without loss of generality, we may suppose that $a, b \in \overline{\mathbb{D}}$: for necessity, these conditions follow from (1.4); for sufficiency, they are contained in (2.2). Then the matrix function $f(z) = zA(\rho - (\rho-1)zA)^{-1}$ is analytic in $\mathbb{D}$. Combining criterion (1.2) with the maximum modulus principle, we conclude that $A \in \mathscr{C}_\rho$ if and only if

$$\max_{-\pi\leqslant\theta\leqslant\pi} \|\Phi_A(\theta)\| \leqslant 1,$$

where $\Phi_A(\theta) = A(\rho - (\rho-1)\mathrm{e}^{\mathrm{i}\theta}A)^{-1}$. From (2.1)

$$\Phi_A(\theta) = \begin{bmatrix} u & w \\ 0 & v \end{bmatrix},$$

where

$$u = a(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}a)^{-1}, \quad v = b(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b)^{-1},$$

$$w = c\rho(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}a)^{-1}(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b)^{-1}.$$

Condition $\|\Phi_A(\theta)\| \leqslant 1$ can be rewritten as

$$|u| \leqslant 1, \quad |v| \leqslant 1, \qquad |u - v|^2 + |w|^2 \leqslant |1 - \overline{u}v|^2$$

or, equivalently,

$$|a| \leqslant \left|\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}a\right|, \quad |b| \leqslant \left|\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b\right| \tag{2.4}$$

and

$$\begin{aligned} &\left|a(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b) - b(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}a)\right|^2 + |c\rho|^2 \\ &\quad \leqslant \left|(\rho - (\rho - 1)\mathrm{e}^{-\mathrm{i}\theta}\overline{a})(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b) - \overline{a}b\right|^2. \end{aligned} \tag{2.5}$$

Inequalities (2.4) are satisfied for all $\theta \in [-\pi, \pi]$ if and only if

$$|a| \leqslant |\rho - |\rho - 1| \cdot |a||, \quad |b| \leqslant |\rho - |\rho - 1| \cdot |b||.$$

These conditions are satisfied automatically (for $a, b \in \overline{\mathbb{D}}$) if $\rho \geqslant 1$, and are equivalent to $|a|, |b| \leqslant \rho/(2 - \rho)$ if $\rho \leqslant 1$. In other words, (2.4) is equivalent to (2.2). It remains to observe that

$$a(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b) - b(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}a) = (a - b)\rho$$

and

$$\begin{aligned} &(\rho - (\rho - 1)\mathrm{e}^{-\mathrm{i}\theta}\overline{a})(\rho - (\rho - 1)\mathrm{e}^{\mathrm{i}\theta}b) - \overline{a}b \\ &\quad = \rho^2 - \rho(\rho - 1)(\mathrm{e}^{-\mathrm{i}\theta}\overline{a} + \mathrm{e}^{\mathrm{i}\theta}b) + (\rho^2 - 2\rho)\overline{a}b \\ &\quad = \rho(\rho + z(\theta) + (\rho - 2)\overline{a}b), \end{aligned}$$

so that (2.5) is equivalent to (2.3). □

Remark 1. Formula (1.2) was used in [8] to prove the following result.

Theorem 2.2. *Let A be of the form* (2.1). *Then* $A \in \mathscr{C}_\rho$ *if and only if* $|a|, |b| \leqslant 1$ *and*

$$|c|^2 + |a - b|^2 \leqslant \inf_{\zeta \in \mathbb{D}} |F(\zeta)|^2, \tag{2.6}$$

where $F(\zeta) = z(\theta) + \rho x^{-1} + \overline{a}b(\rho - 2)x$, $\zeta = x\mathrm{e}^{\mathrm{i}\theta}$.

Theorem 2.1 shows that, for $\rho \geqslant 1$, inf in the right-hand side of (2.6) can be changed to the min along the boundary $\mathbb{T} = \{z\colon |z| = 1\}$ of $\mathbb{D}$. This result does not follow from the maximum modulus principle directly, because the expression under the inf sign in (2.6) is not analytic in $\mathbb{D}$. However, a straightforward elementary (though somewhat cumbersome) derivation of Theorem 2.1 from Theorem 2.2 also can be given. [3]

Remark 2. Geometrically, the set $E = \{z(\theta): -\pi \leqslant \theta \leqslant \pi\}$ is an ellipse (degenerating into an interval if $|a| = |b|$) centered at the origin. Therefore, the set $F = \{F(x\mathrm{e}^{\mathrm{i}\theta}): 0 < x \leqslant 1,\ \theta \in [-\pi, \pi]\}$ can be thought of as the union $\bigcup_{x\in(0,1]} E_x$ of ellipses E_x obtained from E by shifting and with their centers located along the portion of the hyperbola (degenerating into a ray if $\overline{a}b$ is real) $H = \{\rho x^{-1} + \overline{a}b(\rho - 2)x: x \in (0, 1]\}$. The right-hand side of (2.6) is then the distance from the closest point of F to the origin. Theorem 2.1 claims that, for $\rho \geqslant 1$, this closest point actually lies on the "last" ellipse E_1.

Remark 3. Condition $\rho \geqslant 1$ in Remark 2 is essential. Consider, for example, an arbitrary $\rho \in (0, 1)$ and $-b = a \in (\sqrt{\rho/(2-\rho)}, 1)$. Then E_x is the line segment joining the points $\rho x^{-1} + a^2(2-\rho)x + 2\mathrm{i}a(1-\rho)$ and $\rho x^{-1} + a^2(2-\rho)x - 2\mathrm{i}a(1-\rho)$. The distance from E_x to the origin is $\rho x^{-1} + a^2(2-\rho)x$, and its minimal value on the interval $x \in (0, 1]$ is assumed at $x = a^{-1}\sqrt{\rho/(2-\rho)}$, not at $x = 1$.

3. Spectrum on the line

For an arbitrary matrix (2.1), an attempt to find the right-hand side of (2.3) explicitly leads to a fourth degree algebraic equation. Remark 2 shows, however, that the particular cases $|a| = |b|$ and $\overline{a}b \in \mathbb{R}$ deserve a special attention. In this section, we deal with the latter.

Theorem 3.1. *Let A be unitarily similar to the matrix* (2.1) *with $\overline{a}b \in \mathbb{R}$. Then $A \in \mathscr{C}_\rho$ if and only if* (2.2) *holds and*

$$|c|^2 + |a-b|^2 \leqslant (\rho + (\rho-2)\overline{a}b - |\rho - 1| \cdot |a+b|)^2 . \tag{3.1}$$

Proof. From (1.10) and (1.7) it follows that for $0 < \rho < 1$, $A \in \mathscr{C}_\rho$ if and only if $((2-\rho)/\rho)A \in \mathscr{C}_{2-\rho}$. Since conditions (2.2) and (3.1) are invariant under the transformation $\rho \mapsto 2-\rho$, $A \mapsto ((2-\rho)/\rho)A$ $(0 < \rho < 2)$, we may without loss of generality suppose that $\rho \geqslant 1$. Both the property $A \in \mathscr{C}_\rho$ and condition (3.1) are also invariant under multiplication by any complex number with absolute value 1. Hence, we may even suppose that

$$\rho \geqslant 1, \quad a, b \in \mathbb{R}, \quad a + b \geqslant 0. \tag{3.2}$$

[3] In fact, that was our original proof. The new, shorter and more self-contained version was suggested to us by the referee.

Due to Theorem 2.1, it remains to show that in situation (3.2)

$$\min_{-\pi\leqslant\theta\leqslant\pi}\left|(\rho-1)(b\mathrm{e}^{\mathrm{i}\theta}+a\mathrm{e}^{-\mathrm{i}\theta})+\rho+ab(\rho-2)\right|$$

is assumed at $\theta=\pm\pi$. To this end, observe that

$$\begin{aligned}&\left|(\rho-1)(b\mathrm{e}^{\mathrm{i}\theta}+a\mathrm{e}^{-\mathrm{i}\theta})+\rho+ab(\rho-2)\right|\\&\quad=|\rho+ab(\rho-2)+(\rho-1)(a+b)\cos\theta+\mathrm{i}(\rho-1)(b-a)\sin\theta|\\&\quad\geqslant|\rho+ab(\rho-2)+(\rho-1)(a+b)\cos\theta|\\&\quad=(\rho-1)(1-a)(1-b)+(1-ab)+(\rho-1)(a+b)(1+\cos\theta)\\&\quad\geqslant(\rho-1)(1-a)(1-b)+(1-ab),\end{aligned}\tag{3.3}$$

and that for $\theta=\pm\pi$ all the inequalities in (3.3) turn into the equalities. □

Several particular cases of (3.1) ($a, b\geqslant 0$; $a=-b$, etc.) were earlier formulated in [8].

Corollary 3.1. *In the setting of Theorem* 3.1, *the* ρ*-numerical radius of the matrix* A *is given by the formula*

$$w_\rho(A)=\frac{P+\sqrt{P^2-4\rho(\rho-2)\overline{a}b}}{2\rho},\tag{3.4}$$

where $P=|\rho-1|\cdot|a+b|+\sqrt{|c|^2+|a-b|^2}$.

Proof. As in Theorem 3.1, it suffices to consider the case (3.2). Applying this theorem to matrices $\alpha^{-1}A$ in place of A, we see from (1.5) that $\alpha\geqslant w_\rho(A)$ if and only if

$$\alpha\geqslant\max\{|a|,|b|\}\tag{3.5}$$

and

$$|c|^2+(a-b)^2\leqslant(\rho\alpha+(\rho-2)ab\alpha^{-1}-(\rho-1)(a+b))^2.\tag{3.6}$$

Since

$$\begin{aligned}&\rho\alpha+(\rho-2)ab\alpha^{-1}-(\rho-1)(a+b)\\&\qquad=(\rho-1)(\alpha-(a+b)+ab\alpha^{-1})+\alpha-ab\alpha^{-1}\\&\qquad=\alpha\left((\rho-1)\left(1-\frac{a}{\alpha}\right)\left(1-\frac{b}{\alpha}\right)+\left(1-\frac{a}{\alpha}\cdot\frac{b}{\alpha}\right)\right)\geqslant 0\end{aligned}$$

due to (3.5), (3.6) can be rewritten as

$$\sqrt{|c|^2+(a-b)^2}\leqslant\rho\alpha+(\rho-2)ab\alpha^{-1}-(\rho-1)(a+b).\tag{3.7}$$

Solving the quadratic inequality (3.7), we conclude that $\alpha\geqslant w_\rho(A)$ if and only if (3.5) holds and

$$\alpha \geqslant \frac{P + \sqrt{P^2 - 4\rho(\rho-2)\overline{a}b}}{2\rho} \tag{3.8}$$

or

$$\alpha \leqslant \frac{P - \sqrt{P^2 - 4\rho(\rho-2)\overline{a}b}}{2\rho}. \tag{3.9}$$

Observe now that, for fixed a, b ($\in \mathbb{R}$) the right-hand sides of (3.8) and (3.9) are monotonic functions of $|c|$ (increasing and decreasing, respectively). If $c = 0$, then $P = (\rho - 1)(a + b) + |a - b|$, and these right-hand sides are respectively equal to the maximum and the minimum of

$$\left\{a,\ \frac{\rho-2}{\rho}b\right\} \text{ if } a \geqslant b, \text{ and } \left\{\frac{\rho-2}{\rho}a,\ b\right\} \text{ if } a \leqslant b.$$

Therefore, condition (3.5) follows from (3.8) automatically but contradicts (3.9). In other words, $\{(3.5) \wedge ((3.8) \vee (3.9))\} \Longleftrightarrow (3.8)$. $\square$

For $\rho = 2$, formula (3.4) implies that

$$w_2(A) = \frac{|a+b| + \sqrt{|c|^2 + |a-b|^2}}{2},$$

the result stated in [5].

4. Spectrum on the circle

Another particular case in which the right-hand side of (2.3) can be computed explicitly is that of $|a| = |b|$ ($\overset{\text{def}}{=} R$). If $R = 0$, then Theorem 3.1 implies that $A \in \mathscr{C}_\rho$ if and only if $|c| \leqslant \rho$. Equivalently, $w_\rho(A) = |c|/\rho$ for

$$A = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$$

and $\rho \geqslant 1$. It remains therefore to consider $R > 0$.

Theorem 4.1. *Let A be unitarily similar to the matrix* (2.1) *with* $|a| = |b|$ $(= R) > 0$. *Then* $A \in \mathscr{C}_\rho$ *if and only if* (2.2) *holds and*

$$\begin{aligned} |c|^2 + |a-b|^2 \leqslant{} & \frac{1}{4}(\rho R^{-1} - (\rho-2)R)^2\, |a-b|^2 \\ & + \left(\max\left\{0, \frac{1}{2}(\rho R^{-1} + (\rho-2)R)\, |a+b| - 2\,|\rho-1| \cdot R\right\}\right)^2. \end{aligned} \tag{4.1}$$

Proof. Similarly to the proof of Theorem 3.1, we may suppose that $\rho \geqslant 1$ and multiply A by any unimodular complex number. Therefore, it suffices to consider the case

$$\rho \geqslant 1, \quad \overline{b} = a = R\mathrm{e}^{\mathrm{i}\phi} \text{ with } \phi \in [-\pi/2, \pi/2]. \tag{4.2}$$

The right-hand side of (4.1) can be then rewritten as

$$(\rho - (\rho - 2)R^2)\sin^2\phi \\ +(\max\{0, (\rho + (\rho - 2)R^2)\cos\phi - 2(\rho - 1)R\})^2. \tag{4.3}$$

According to Theorem 2.1, it remains to show that the right-hand side of (2.3) coincides with (4.3) for a, b given by (4.2). To this end, observe that

$$\begin{aligned}
&(1-\rho)(b\mathrm{e}^{\mathrm{i}\theta} + \overline{a}\mathrm{e}^{-\mathrm{i}\theta}) + \rho + \overline{a}b(\rho - 2) \\
&\quad = 2(1-\rho)R\mathrm{e}^{-\mathrm{i}\phi}\cos\theta + \rho + R^2\mathrm{e}^{-2i\phi}(\rho - 2) \\
&\quad = \mathrm{e}^{-\mathrm{i}\phi}\left(\rho\mathrm{e}^{\mathrm{i}\phi} + R^2\mathrm{e}^{-\mathrm{i}\phi}(\rho-2) - 2(\rho-1)R\cos\theta\right) \\
&\quad = \mathrm{e}^{-\mathrm{i}\phi}\Big((\rho + (\rho-2)R^2)\cos\phi - 2(\rho-1)R\cos\theta \\
&\qquad + \mathrm{i}(\rho - (\rho-2)R^2)\sin\phi\Big).
\end{aligned}$$

So, the right-hand side of (2.3) coincides with the square of the distance d from the origin to the horizontal segment Ω with the length $4(\rho - 1)R$ and centered at $(\rho + (\rho - 2)R^2)\cos\phi + i(\rho - (\rho - 2)R^2)\sin\phi$.

Due to (1.4), we may without loss of generality suppose that $R \leqslant 1$. Then, obviously, the center of Ω is located in the right half plane Π_+. If $2(\rho - 1)R \leqslant (\rho + (\rho - 2)R^2)\cos\phi$, the whole segment Ω is located in Π_+. Then the distance from Ω to the origin is assumed at the leftmost point of Ω, that is, at $(\rho + (\rho - 2)R^2)\cos\phi - 2(\rho - 1)R + \mathrm{i}(\rho - (\rho - 2)R^2)\sin\phi$.

If $2(\rho - 1)R \geqslant (\rho + (\rho - 2)R^2)\cos\phi$, the segment Ω intersects the imaginary axis. The distance from Ω to the origin is then assumed at this intersection, that is, at $\mathrm{i}(\rho - (\rho - 2)R^2)\sin\phi$. Either way, d^2 coincides with (4.3). □

Remark 4. The cases $a = b$ and $a = -b$ are covered both by Theorems 3.1 and 4.1. Of course, inequalities (3.1) and (4.1) are in these cases equivalent. They can be simplified as follows:

$$|c| \leqslant \rho + (\rho - 2)R^2 - 2\,|\rho - 1| \cdot R$$

if $a = b$, and

$$|c|^2 + 4R^2 \leqslant (\rho - (\rho - 2)R^2)^2$$

if $a = -b$.

Theorem 4.1 allows us to compute the ρ-numerical radius for matrices (2.1) with $|a| = |b|$.

Corollary 4.1. *In the setting of Theorem* 4.1, *the ρ-numerical radius of the matrix A is given by the formula*

$$w_\rho(A) = \frac{R}{\rho}\left(\sqrt{1+\frac{|c|^2}{|a-b|^2}}+\sqrt{(\rho-1)^2+\frac{|c|^2}{|a-b|^2}}\right) \tag{4.4}$$

if

$$|c|\cdot|a+b| < |\rho-1|\cdot|a-b|^2, \tag{4.5}$$

and

$$w_\rho(A) = \frac{Q+\sqrt{Q^2-4\rho(\rho-2)R^2}}{2\rho}, \tag{4.6}$$

where

$$Q = |c| + |\rho-1|\cdot|a+b| \tag{4.7}$$

otherwise.

Proof. It suffices to consider the case (4.2). Applying Theorem 4.1 to $\alpha^{-1}A$ in place of A, we see that $\alpha \geqslant w_\rho(A)$ if and only if $\alpha \geqslant R$ and

$$|c|^2+|a-b|^2 \leqslant \left(\rho\alpha-\frac{(\rho-2)R^2}{\alpha}\right)^2\sin^2\phi + \left(\max\{0, \left(\rho\alpha+\frac{(\rho-2)R^2}{\alpha}\right)\cos\phi - 2(\rho-1)R\}\right)^2. \tag{4.8}$$

Let us introduce a new variable,

$$z = \rho\alpha + \frac{(\rho-2)R^2}{\alpha}. \tag{4.9}$$

There is a monotonic (and therefore one-to-one) correspondence between $\alpha \in [R,\infty)$ and $z \in [2(\rho-1)R,\infty)$. Since

$$\left(\rho\alpha-\frac{(\rho-2)R^2}{\alpha}\right)^2 = \left(\rho\alpha+\frac{(\rho-2)R^2}{\alpha}\right)^2 - 4\rho(\rho-2)R^2$$

and $|a-b|^2 = 4R^2\sin^2\phi$, inequality (4.8) can be rewritten as

$$|c|^2 \leqslant \left(z^2-4R^2(\rho-1)^2\right)\sin^2\phi + \left(\max\{0, z\cos\phi-2(\rho-1)R\}\right)^2. \tag{4.10}$$

We now consider two cases separately, according to whether or not (4.5) holds.

Due to (4.2), the latter condition can be rewritten as follows:

$$|c| < 2(\rho-1)R\frac{\sin^2\phi}{\cos\phi} \tag{4.11}$$

(with the understanding that for $\cos\phi = 0$ the right-hand side of (4.11) equals $+\infty$).

If (4.11) holds, then, in particular, $\sin\phi \neq 0$, and

$$\begin{aligned}&\sqrt{\frac{|c|^2}{\sin^2\phi}+4(\rho-1)^2R^2}\\&\quad<\sqrt{(2(\rho-1)R\tan\phi)^2+4(\rho-1)^2R^2}=\sqrt{4(\rho-1)^2R^2(1+\tan^2\phi)}\\&\quad=\frac{2(\rho-1)R}{\cos\phi}.\end{aligned}$$

Inequality (4.10) for $(0<)z \leqslant (2(\rho-1)R)/\cos\phi$ can be simplified to

$$\sqrt{\frac{|c|^2}{\sin^2\phi}+4(\rho-1)^2R^2} \leqslant z.$$

Hence, the smallest solution of (4.10) in case (4.11) is given by

$$z=\sqrt{\frac{|c|^2}{\sin^2\phi}+4(\rho-1)^2R^2}=2R\sqrt{(\rho-1)^2+\frac{|c|^2}{|a-b|^2}}.$$

The corresponding smallest value of $\alpha \in [R,\infty)$ satisfying (4.8) is then equal to (4.4).

If (4.11) fails, then

$$\sqrt{\frac{|c|^2}{\sin^2\phi}+4(\rho-1)^2R^2} \geqslant \frac{2(\rho-1)R}{\cos\phi},$$

and the inequality (4.10) has no solutions to the left of $(2(\rho-1)R)/\cos\phi$. For $z \geqslant (2(\rho-1)R)\cos\phi$ the right-hand side of (4.10) can be rewritten as

$$\begin{aligned}&(z^2-4(\rho-1)^2R^2)\sin^2\phi+(z\cos\phi-2(\rho-1)R)^2\\&\quad=z^2-4(\rho-1)^2R^2\sin^2\phi+4(\rho-1)^2R^2-4(\rho-1)Rz\cos\phi\\&\quad=(z-2(\rho-1)R\cos\phi)^2.\end{aligned}$$

Therefore, the set of all solutions of (4.10) is in this case

$$[|c|+2(\rho-1)R\cos\phi,+\infty)$$

(note that $|c|+2(\rho-1)R\cos\phi \geqslant 2(\rho-1)R((\sin^2\phi/\cos\phi)+\cos\phi)=(2(\rho-1)R)/\cos\phi)$. The smallest solution of (4.10) is therefore equal to Q, as defined by formula (4.7). The corresponding smallest value of α is then given by (4.6). □

As in Section 3, the particular case of formulas (4.4) and (4.6) for $\rho=2$ was stated in [5]. In our notation, it reads

$$w_2(A)=\begin{cases}(R/|a-b|)\sqrt{|c|^2+|a-b|^2} & \text{if } |c|\cdot|a+b|<|a-b|^2,\\ (|c|+|a+b|)/2 & \text{otherwise.}\end{cases}$$

5. Variations and generalizations

The results of Sections 2–4 can be reformulated in a unitarily invariant form. To this end, observe that

$$|a|^2 + |b|^2 + |c|^2 = \|A\|^2 + \frac{|ab|^2}{\|A\|^2} \tag{5.1}$$

for any non-zero matrix A of the form (2.1). Hence, (2.3) can be rewritten as

$$\|A\|^2 + \frac{|ab|^2}{\|A\|^2} - 2\,\mathrm{Re}\,\overline{a}b \leqslant \min_{-\pi\leqslant\theta\leqslant\pi} |z(\theta) + \rho + \overline{a}b(\rho-2)|^2 . \tag{5.2}$$

If $\overline{a}b \in \mathbb{R}$, the left-hand side of (5.2) can be further rewritten as $(\|A\| - (\overline{a}b/\|A\|))^2$, so that after taking the square roots, (3.1) becomes

$$\|A\| - \frac{\overline{a}b}{\|A\|} \leqslant \rho + (\rho-2)\overline{a}b - |\rho-1|\cdot|a+b|$$

(note that both sides are non-negative). Respectively, P in (3.4) can be substituted by $|\rho-1|\cdot|a+b| + \|A\| - (\overline{a}b/\|A\|)$.

If $|a| = |b| = R$, then (5.1) implies that $|c| = \|A\| - (R^2/\|A\|)$, and formulas (4.4) and (4.6) can be modified accordingly.

Let now A be an operator defined by (1.3), with $\dim \mathfrak{H}_x \leqslant 2$ for all $x \in X$. Applying the above-mentioned results and then using (1.8), it is easy to formulate a criterion for $A \in \mathscr{C}_\rho$, as well as to compute $w_\rho(A)$.

We will not formulate these statements in such a generality. Observe, instead, that A has form (1.3) with at most two-dimensional $\mathfrak{H}_x$ if and only if A belongs to a W^*-subalgebra of $\mathscr{L}[\mathfrak{H}]$ generated by two orthoprojections. This class includes, in particular, all *quadratic* operators, that is, operators A satisfying the equation

$$A^2 + pA + qI = 0 \tag{5.3}$$

for some $p, q \in \mathbb{C}$.

Theorem 5.1. *Let A $(\neq 0)$ be a quadratic operator, and let a, b be the roots of its minimal polynomial $z^2 + pz + q$. Then A is a ρ-contraction if and only if a, b satisfy* (2.2) *and*

$$\|A\|^2 + \frac{|q|^2}{\|A\|^2} \leqslant 2\,\mathrm{Re}\,\overline{a}b + \min_{-\pi\leqslant\theta\leqslant\pi} \left|\rho + \overline{a}b(\rho-2) - (\rho-1)(b\mathrm{e}^{\mathrm{i}\theta} + \overline{a}\mathrm{e}^{-\mathrm{i}\theta})\right|^2 .$$

Proof. There exists an orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ relative to which A has the form

$$A = \begin{bmatrix} aI & C \\ 0 & bI \end{bmatrix}.$$

Using the polar decomposition of the block $C\colon \mathfrak{H}_2 \to \mathfrak{H}_1$ and the spectral representation of $(C^*C)^{1/2}$, we can further decompose A into a direct integral with two-dimensional blocks (2.1), $x \in \sigma((C^*C)^{1/2})$, and possibly one-dimensional blocks $\{a\}$ and $\{b\}$. We now apply (5.2) to each of the blocks A_x and then use the fact that

$$\sup\{\|A_x\|\colon x \in \sigma((C^*C)^{1/2})\} = \|A\|. \qquad \square$$

If the coefficients p, q in (5.3) are real, the roots a, b are either real or complex conjugate. Applying the results of Section 3 (for $a, b \in \mathbb{R}$) or Section 4 (for $b = \overline{a}$) to the blocks A_x in the direct integral representation (1.3) of A, we come to the following formula for its ρ-numerical radius.

Theorem 5.2. *Let A ($\neq 0$) be a quadratic operator such that the coefficients p, q in* (5.3) *are real. Then*

$$w_\rho(A) = \frac{S_\rho + \sqrt{S_\rho^2 - 4\rho(\rho-2)q}}{2\rho}, \tag{5.4}$$

where

$$S_\rho = |\rho - 1| \cdot |p| + \|A\| - \frac{q}{\|A\|}$$

if

$$p^2 \geqslant 4q \quad \text{or} \quad p^2 \leqslant \left(\|A\| + \frac{q}{\|A\|}\right)^2 - (\rho-1)^2\left(\frac{4q}{p} - p\right)^2, \tag{5.5}$$

and

$$w_\rho(A) = \frac{\sqrt{q}}{\rho}\left(\sqrt{1 + T_\rho} + \sqrt{(\rho-1)^2 + T_\rho}\right) \tag{5.6}$$

with

$$T_\rho = \frac{\left(\|A\| - \frac{q}{\|A\|}\right)^2}{4q - p^2}$$

otherwise.

Observe that $w_2(A) = \frac{1}{2}S_2$ and $S_\rho = 2\sigma\,|p| + S_2$, where

$$\sigma = \begin{cases} (\rho-2)/2 & \text{if } \rho \geqslant 1, \\ -\rho & \text{if } \rho \leqslant 1. \end{cases}$$

Hence, formula (5.4) for p, q satisfying

$$p^2 \geqslant 4q \quad \text{or} \quad p^2 \leqslant \left(\|A\| + \frac{q}{\|A\|}\right)^2 - \max\{1, (\rho-1)^2\}\left(\frac{4q}{p} - p\right)^2$$

can be rewritten as

$$w_\rho(A) = \frac{w_2(A) + \sigma\,|p| + \sqrt{(w_2(A) + \sigma\,|p|)^2 - \rho(\rho-2)q}}{\rho}.$$

In particular,

$$w_\rho(A) = \frac{w_2(A) + \sqrt{w_2(A)^2 + \rho(\rho-2)s}}{\rho} \quad \text{if} \quad A^2 = sI,\ s \geqslant 0,$$

and

$$w_\rho(A) = \frac{2w_2(A) + 2\sigma\,|p|}{\rho} = \frac{\|A\| + |\rho-1|\cdot|p|}{\rho} \quad \text{if} \quad A^2 = pA, \quad p \in \mathbb{R}.$$

The last two formulas (with $s = 1$ and $p = 1$, respectively) were obtained in [1].

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