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The Busemann–Petty problem in hyperbolic and spherical spaces

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Abstract

The Busemann–Petty problem asks whether origin-symmetric convex bodies in $\mathbb{R}^n$ with smaller central hyperplane sections necessarily have smaller n-dimensional volume. It is known that the answer to this problem is affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$. We study this problem in hyperbolic and spherical spaces.

MSC: 52A55; 52A20

Keywords: Busemann–Petty problem; Hyperbolic and spherical spaces; Fourier transform

1. Introduction

The Busemann–Petty problem asks the following question. Given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) \leqslant \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbb{R}^n$, does it follow that

$$\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(L)?$$

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The answer to this problem in $\mathbb{R}^n$ is known to be affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$. The solution appeared as the result of work of many mathematicians (see [6] or [23] for historical details).

In this paper, we consider the Busemann–Petty problem in hyperbolic and spherical spaces in place of the Euclidean space. We prove

Theorem 1.1. *Let K and L be centrally symmetric convex bodies in the spherical space* $\mathbb{S}^n$, $n \leqslant 4$ (*more precisely in a hemisphere*) *such that*

$$\mathrm{vol}_{n-1}(K \cap H) \leqslant \mathrm{vol}_{n-1}(L \cap H) \tag{1}$$

for every central totally-geodesic hyperplane H in $\mathbb{S}^n$. *Then*

$$\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(L).$$

On the other hand, if $n \geqslant 5$ *there are convex symmetric bodies* $K, L \subset \mathbb{S}^n$ *that satisfy* (1) *but* $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

So, the answer to the Busemann–Petty in $\mathbb{S}^n$ is exactly the same as in the Euclidean space. However, the situation in the hyperbolic space is different. Trivially, the answer is affirmative if $n = 2$, since the condition (1) in this case is equivalent to $K \subseteq L$, but for higher dimensions we have the following:

Theorem 1.2. *There are convex centrally symmetric bodies* $K, L \subset \mathbb{H}^n$, $n \geqslant 3$ *that satisfy the condition*

$$\mathrm{vol}_{n-1}(K \cap H) \leqslant \mathrm{vol}_{n-1}(L \cap H)$$

for every central totally-geodesic hyperplane H in $\mathbb{H}^n$, *but* $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

The idea to find analogs of known results in non-Euclidean spaces is not new. For example in [4] the authors study intrinsic volumes in hyperbolic and spherical spaces. The Brunn–Minkowski inequality in different spaces is discussed in [5]. Also a number of papers is concerned with other generalizations of the Busemann–Petty problem. In our proof we will be using results from [24], where Zvavitch studied the Busemann–Petty problem for arbitrary measures. For other generalizations of the Busemann–Petty problem see [2,11–14,21,16].

2. Preliminaries

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$. Using the stereographic projection (from the north pole onto the hyperplane containing the equator) we can think of it as $\mathbb{R}^n$ equipped with the metric of constant curvature $+1$:

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 + (x_1^2 + \cdots + x_n^2))^2},$$

where $x_1, \ldots, x_n$ are the standard Euclidean coordinates in $\mathbb{R}^n$. (See [3, §9, §10, 20, §4.5] for details about the spherical and hyperbolic spaces). It is well-known that geodesic lines on the sphere are great circles. Later on, in order to define convexity, we will need the uniqueness property of geodesics joining given 2 points. But this is not the case on the sphere. However if we restrict ourselves to an open hemisphere, then for any two points there exists a unique geodesic segment connecting them. Under the stereographic projection the open south hemisphere gets mapped onto the open unit ball B^n in $\mathbb{R}^n$. This is the model we will be working in. The geodesics in this model are arcs of the circles intersecting the boundary of the ball B^n in antipodal points and straight lines through the origin.

Also it is well-known that the hyperbolic space $\mathbb{H}^n$ can be identified with the interior of the unit ball in $\mathbb{R}^n$ with the metric:

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - (x_1^2 + \cdots + x_n^2))^2}.$$

This is the Poincaré model of the hyperbolic space in the ball. Note that it can be also obtained from the pseudosphere in the Lorentzian space via the stereographic projection. The geodesic lines in this model are arcs of the circles orthogonal to the boundary of the ball B^n and straight lines through the origin.

Since both geometries are defined in the unit ball in $\mathbb{R}^n$, we will treat them simultaneously, considering the open ball $B^n \subset \mathbb{R}^n$ with the metric

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 + \delta\ (x_1^2 + \cdots + x_n^2))^2}, \tag{2}$$

where $\delta = -1$ for the hyperbolic case, $+1$ for the spherical space. In addition if we consider $\delta = 0$ we get the original case of the Euclidean space.

The definition of convexity in hyperbolic and spherical spaces (recall that we work in an open hemisphere) is analogous to that in the Euclidean space (see [19, Chapter I, §12]). A body K (compact set with non-empty interior) is called *convex* if for every pair of points in K the geodesic segment joining them also belongs to the body K. For our definition of convexity in $\mathbb{S}^n$ it is crucial that we work in an open hemisphere, since in this case we have a unique geodesic segment through any two points.

Let K be a body in the open unit ball B^n. In order to distinguish between different types of convexity we will adopt the following system of notations. The body K is called s-convex (or $+1$-convex), if it is convex in the spherical metric defined in the ball B^n. Similarly it is called h-convex (or -1-convex) if it is convex with respect to

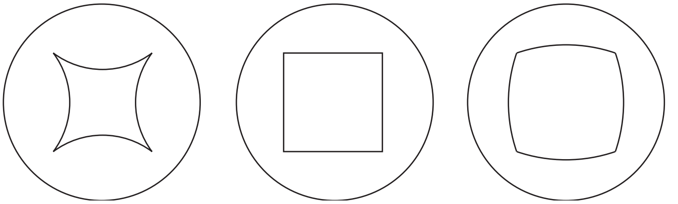

Fig. 1. Three types of convexity.

the hyperbolic metric. e-convex bodies (or 0-convex) are the bodies convex in the usual Euclidean sense. Analogously s-(h-,e-)geodesics are the straight lines of the spherical (hyperbolic, Euclidean) metric. (In this terminology we follow [18]. Note that in the literature there are other definitions of h-convexity or δ-convexity which have absolutely different meaning).

Some examples of convex hulls of four points with respect to hyperbolic, Euclidean and spherical metrics correspondingly are shown in Fig. 1.

Clearly, any s-convex body containing the origin is also e-convex and any e-convex body containing the origin is h-convex. (See for example [18]).

A submanifold $\mathcal{F}$ in a Riemannian space $\mathcal{R}$ is called *totally geodesic* if every geodesic in $\mathcal{F}$ is also a geodesic in the space $\mathcal{R}$. In the Euclidean space the totally geodesic submanifolds are Euclidean planes, on the sphere they are great subspheres. In the Poincaré model of the hyperbolic space described above the totally geodesic submanifolds are represented by the spheres orthogonal to the boundary of the unit ball B^n. In a sense, totally geodesic submanifolds are analogs of Euclidean planes in Riemannian spaces. For elementary properties of totally geodesic submanifolds see [1, Chap. 5, §5].

The *Minkowski functional* of a star-shaped origin-symmetric body $K \subset \mathbb{R}^n$ is defined as

$$\|x\|_K = \min\{a \geqslant 0 : x \in aK\}.$$

The *radial function* of K is given by $\rho_K(x) = \|x\|_K^{-1}$. If $x \in S^{n-1}$ then the radial function $\rho_K(x)$ is the Euclidean distance from the origin to the boundary of K in the direction of x.

For a centrally-symmetric δ-convex body $K \in B^n$ ($\delta = 0, 1, -1$) consider the section of K by the hypersurface $\xi^\perp = \{\langle x, \xi\rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. Clearly such a hypersurface is a totally geodesic hyperplane in the metric (2) for any $\delta = 0, 1, -1$. This hyperplane passes through the origin with the normal vector ξ.

The volume element of the metric (2) equals

$$d\mu_n = 2^n \frac{dx_1 \cdots dx_n}{(1 + \delta\ (x_1^2 + \cdots + x_n^2))^n} = 2^n \frac{dx}{(1 + \delta|x|^2)^n}.$$

Therefore the volume of a body K is given by the formula

$$\mathrm{vol}_n(K) = \int_K d\mu_n = 2^n \int_K \frac{dx}{(1+\delta\ |x|^2)^n}.$$

Note that in polar coordinates the latter formula looks as follows:

$$\mathrm{vol}_n(K) = 2^n \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-1}}{(1+\delta r^2)^n} dr\, d\theta. \tag{3}$$

Similarly the volume element of the hypersurface $\xi^{\perp}$ is

$$d\mu_{n-1} = 2^{n-1} \frac{dx}{(1+\delta\ |x|^2)^{n-1}},$$

therefore the $(n-1)$-volume of the section of K by the hyperplane $\xi^{\perp}$ is given by the formula

$$S_K(\xi) = \int_{K\cap\langle x,\xi\rangle=0} d\mu_{n-1} = 2^{n-1} \int_{K\cap\langle x,\xi\rangle=0} \frac{dx}{(1+\delta\ |x|^2)^{n-1}}.$$

One of the tools of this paper is the Fourier transform of distributions. Let ϕ be a function from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. We define the Fourier transform of ϕ by

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-i\langle x,\xi\rangle}\, dx, \quad \xi \in \mathbb{R}^n.$$

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ from the space $\mathcal{S}$.

We say that a distribution f is *positive definite*, if its Fourier transform is a positive distribution, in the sense that $\langle \hat{f}, \phi\rangle \geqslant 0$ for every non-negative test function ϕ.

If a distribution f acts on test functions in the same way as a continuous function g then we write that $f(x) = g(x)$ pointwise. This is just notation meaning that f and g coincide on all test functions. In particular if $\hat{f} = g$ on test functions we write $\hat{f}(x) = g(x)$ pointwise, where in the left-hand side we do not mean the convergent Fourier integral, but understand this as equality of distributions. We write $(\|x\|^p)^\wedge(\xi)$ meaning the values of the continuous function to which this is equal as a distribution.

The spherical Radon transform $R : C(S^{n-1}) \to C(S^{n-1})$ is defined by

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^{\perp}} f(x)\, dx.$$

The following Lemma gives a relation between the spherical Radon transform and the Fourier transform. (See [8, Lemma 2.5], or [22] for more general results.)

Lemma 2.1. *Let $g(x)$ be an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$, so that $g(x)|_{S^{n-1}} \in C(S^{n-1})$ then*

$$Rg(\xi) = \frac{1}{\pi}\hat{g}(\xi), \qquad \forall \xi \in S^{n-1}.$$

The latter equality means that $\hat{g}$ is a homogeneous function of degree -1 on $\mathbb{R}^n$, whose values on S^{n-1} are equal to Rg.

Now we derive a formula for the function $S_K(\xi)$ using the Fourier transform, similar to [24]. For $\delta = 0$ this is the formula from [9].

Lemma 2.2. *Let K be an origin-symmetric δ-convex body in B^n with Minkowski functional $\|\cdot\|_K$. Let $\xi \in S^{n-1}$ and $\xi^\perp$ be the hyperplane through the origin orthogonal to ξ. Then the volume of the section of the body K by the hyperplane$\xi^\perp$ in the metric* (2) *equals*

$$S_K(\xi) = \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}\,dr\right)^\wedge(\xi).$$

Proof. Passing to spherical coordinates we get

$$\begin{aligned} S_K(\xi) &= 2^{n-1}\int_{\xi^\perp}\chi(\|x\|_K)\frac{dx}{(1+\delta\ |x|^2)^{n-1}} \\ &= 2^{n-1}\int_{S^{n-1}\cap\xi^\perp}\int_0^{\|\theta\|_K^{-1}}\frac{r^{n-2}dr}{(1+\delta\ r^2)^{n-1}}\ d\theta. \end{aligned}$$

We can rewrite the integral above as follows (note that $|x| = 1$, since $x \in S^{n-1}$):

$$S_K(\xi) = 2^{n-1}\int_{S^{n-1}\cap\xi^\perp}|x|^{-n+1}\int_0^{|x|/\|x\|_K}\frac{r^{n-2}dr}{(1+\delta\ r^2)^{n-1}}\ dx.$$

The function under the spherical integral is a homogeneous function of x of degree $-n+1$ and therefore by Lemma 2.1

$$S_K(\xi) = \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}\ dr\right)^\wedge(\xi). \qquad \square$$

3. Proofs of main results

First we construct counterexamples to the Busemann–Petty problem in $\mathbb{H}^n$ and $\mathbb{S}^n$ for $n \geqslant 5$.

Theorem 3.1. *There exist convex origin-symmetric bodies K and L in $\mathbb{S}^n$ (or $\mathbb{H}^n$), $n \geqslant 5$ such that*

$$\mathrm{vol}_{n-1}(K \cap H) \leqslant \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane, but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

Proof. We will show the proof only for the case of the spherical space, the hyperbolic case is similar. The idea here is to use the property that any Riemannian space locally looks as "almost" Euclidean.

Let K and L be convex origin-symmetric bodies in $\mathbb{R}^n$ that give a counterexample to the original Busemann–Petty problem (see for example [15, Section 5.1]). That is

$$\mathrm{EVol}_{n-1}(K \cap H) \leqslant \mathrm{EVol}_{n-1}(L \cap H) \tag{4}$$

for every central hyperplane H, but

$$\mathrm{EVol}_n(L) < \mathrm{EVol}_n(K). \tag{5}$$

(Here we denote the usual Euclidean volume by EVol to avoid confusion with the spherical volume.)

In fact, since the inequality (5) is strict, we can dilate one of the bodies a little to make the inequality (4) strict. Recall also, that in the original counterexample the body L was strictly convex, and the body K was obtained from the body L by small perturbations. Note that K can also be made strictly convex.

In view of the latter remarks, we will assume that K and L are strictly convex origin-symmetric bodies that satisfy the strict version of (4). Moreover, the function $\mathrm{EVol}_{n-1}(K \cap H)/\mathrm{EVol}_{n-1}(L \cap H)$ is a continuous function of $\xi \in S^{n-1}$, where ξ is the normal vector to the hyperplane H. Since this function is strictly less than 1, there exists an $\varepsilon > 0$ such that

$$\mathrm{EVol}_{n-1}(K \cap H) < (1-\varepsilon)\mathrm{EVol}_{n-1}(L \cap H)$$

for all H and

$$\mathrm{EVol}_n(L) < (1-\varepsilon)\mathrm{EVol}_n(K).$$

Clearly, any dilations αK and αL also provide a counterexample. We can take α so small that both bodies K and L lie in a ball of radius r that satisfies the inequality:

$$1-\varepsilon \leqslant \frac{1}{(1+r^2)^n} < 1.$$

Now the volumes of the bodies K and L in the spherical metric are related by the inequality:

$$\begin{aligned}\mathrm{vol}_n(L) &= 2^n \int_L \frac{dx}{(1+|x|^2)^n} \leqslant 2^n \int_L dx = 2^n \mathrm{EVol}_n(L) < (1-\varepsilon) 2^n \mathrm{EVol}_n(K)\\ &= (1-\varepsilon)\ 2^n \int_K dx \leqslant 2^n \int_K \frac{dx}{(1+|x|^2)^n} = \mathrm{vol}_n(K).\end{aligned}$$

Analogously, for the volumes of sections we have

$$\begin{aligned}\mathrm{vol}_{n-1}(K\cap\xi^\perp) &= 2^{n-1}\int_{K\cap\langle x,\xi\rangle=0} \frac{dx}{(1+|x|^2)^{n-1}} \leqslant 2^{n-1}\int_{K\cap\langle x,\xi\rangle=0} dx\\ &< (1-\varepsilon)2^{n-1}\int_{L\cap\langle x,\xi\rangle=0} dx \leqslant 2^{n-1}\int_{L\cap\langle x,\xi\rangle=0} \frac{dx}{(1+|x|^2)^{n-1}}\\ &= \mathrm{vol}_{n-1}(L\cap\xi^\perp).\end{aligned}$$

To finish the proof we only need to show that if K is a strictly e-convex body, then αK is s-convex for sufficiently small α. Consider the boundary of the body K. Define

$$k = \min\{k_i(x) : x \in \partial K,\ \ i = 1, \dots, n-1\},$$

where $k_i(x)$, $i = 1, \dots, n-1$, are the principal curvatures at the point x on the boundary of K. Since K is strictly e-convex the quantity defined above is strictly positive: $k > 0$. For the body αK it is equal to k/α. On the other hand in a small neighborhood of the origin the totally geodesic s-planes are the spheres with almost zero curvature (from the Euclidean point of view). Consider all the spheres, which are totally geodesic in the spherical metric and tangent to the body αK, and let R be the smallest radius of all such spheres. We can choose an α so small that

$$k/\alpha > 1/R$$

and therefore the body αK lies on one side with respect to any tangent totally geodesic s-hyperplane. Hence αK is s-convex.

The situation in the hyperbolic space is even easier since every e-convex body containing the origin is also h-convex. $\square$

In 1988 Lutwak [17] introduced the concept of intersection body and proved that the Busemann–Petty problem has affirmative answer if the body with smaller sections is an intersection body. Later, in [10] Koldobsky proved that a body K is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution. Then in [11] Koldobsky generalized Lutwak's connection using the following Parseval's formula on the sphere:

Lemma 3.2. *If K and L are origin symmetric infinitely smooth bodies in $\mathbb{R}^n$ and $0 < p < n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and*

$$\int_{S^{n-1}} \left(\|x\|_K^{-p}\right)^\wedge (\xi) \left(\|x\|_L^{-n+p}\right)^\wedge (\xi)\, d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_L^{-n+p}\, dx.$$

In fact we will be using the following version of this Lemma, see [11, Corollary 1].

Corollary 3.3. *Let f and g be functions on $\mathbb{R}^n$, continuous on S^{n-1} and homogeneous of degree -1 and $-n+1$, respectively. Suppose that f represents a positive definite distribution. Then there exists a measure γ_0 on S^{n-1} such that*

$$\int_{S^{n-1}} \widehat{g}(\theta)\, d\gamma_0(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta)\, g(\theta)\, d\theta.$$

Here we do not assume that f is an infinitely differentiable function, so its Fourier transform is not necessarily a function, but merely a measure.

Later, Zvavitch [24] solved the Busemann–Petty problem for arbitrary measures. Namely, let $f_n(x)$ be a locally integrable function on $\mathbb{R}^n$, and $f_{n-1}(x)$ a function on $\mathbb{R}^n$, locally integrable on central hyperplanes. Then let μ_n be the measure on $\mathbb{R}^n$ with density $f_n(x)$ and μ_{n-1} be the $(n-1)$-dimensional measure on central hyperplanes with density $f_{n-1}(x)$ such that $t\frac{f_n(tx)}{f_{n-1}(tx)}$ is an increasing function of t for any fixed x. Then if

$$\|x\|_K^{-1} \frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}$$

is a positive definite distribution on $\mathbb{R}^n$ then the Busemann–Petty problem for these measures has affirmative answer, i.e. $\mu_{n-1}(K \cap \xi^\perp) \leqslant \mu_{n-1}(L \cap \xi^\perp)$ implies $\mu_n(K) \leqslant \mu_n(L)$. Our next result is a particular case of Zvavitch's theorem, but for the sake of completeness we include a proof.

Theorem 3.4. *Let K and L be δ-convex origin-symmetric bodies in B^n such that $\dfrac{\|x\|_K^{-1}}{1+\delta\,(\frac{|x|}{\|x\|_K})^2}$ is a positive definite distribution. If*

$$\mathrm{vol}_{n-1}(K \cap H) \leqslant \mathrm{vol}_{n-1}(L \cap H)$$

for every totally geodesic hyperplane through the origin, then

$$\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(L).$$

Proof. Let us first prove the following elementary inequality (cf. [24]). For any $a, b \in (0, 1)$

$$\frac{a}{1+\delta\ a^2}\int_a^b \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr \leqslant \int_a^b \frac{r^{n-1}}{(1+\delta\ r^2)^n}\,dr.$$

Indeed, since the function $\dfrac{r}{1+\delta\ r^2}$ is increasing on the interval $(0, 1)$ we have the following

$$\begin{aligned}\frac{a}{1+\delta\ a^2}\int_a^b \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr &= \int_a^b \frac{r^{n-1}}{(1+\delta\ r^2)^n}\frac{a}{1+\delta\ a^2}\left(\frac{r}{1+\delta r^2}\right)^{-1} dr \\ &\leqslant \int_a^b \frac{r^{n-1}}{(1+\delta\ r^2)^n}dr.\end{aligned}$$

Note that latter inequality does not require that $a \leqslant b$.

Using the previous inequality with $a = \|x\|_K^{-1}$ and $b = \|x\|_L^{-1}$ we get

$$\int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}}\int_{\|x\|_K^{-1}}^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr\ dx \leqslant \int_{S^{n-1}}\int_{\|x\|_K^{-1}}^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1+\delta\ r^2)^n}dr\ dx.$$

Suppose we can show that the left-hand side is non-negative, then it will follow that

$$\int_{S^{n-1}}\int_0^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1+\delta\ r^2)^n}dr\ dx \leqslant \int_{S^{n-1}}\int_0^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1+\delta\ r^2)^n}dr\ dx,$$

that is $\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(L)$, see the polar formula (3).

So we only need to show that

$$\begin{aligned}&\int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}}\int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr\ dx \\ &\quad\leqslant \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}}\int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr\ dx.\end{aligned}$$

But this follows from the assumption of the theorem, Parseval's formula on the sphere (Corollary 3.3) and formula for the volume of central sections (Lemma 2.2). Indeed, let γ_0 be the measure from Corollary 3.3 corresponding to the Fourier transform of the positive definite distribution $\frac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2}$, then

$$
\begin{aligned}
&(2\pi)^n \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr\, dx \\
&= \int_{S^{n-1}} \left(\frac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2} \right) \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr \right) dx \\
&= \int_{S^{n-1}} \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr \right)^{\wedge} (\theta)\, d\gamma_0(\theta) \\
&= \int_{S^{n-1}} \frac{\pi}{2^{n-1}} S_K(\theta)\, d\gamma_0(\theta) \leqslant \int_{S^{n-1}} \frac{\pi}{2^{n-1}} S_L(\theta)\, d\gamma_0(\theta) \\
&= \int_{S^{n-1}} \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_L}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}\, dr \right)^{\wedge} (\theta)\, d\gamma_0(\theta) \\
&= (2\pi)^n \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr. \qquad \square
\end{aligned}
$$

Remark 3.5. Since $\|x\|_K^{-1}$ is positive definite for any convex origin-symmetric body in $\mathbb{R}^n$, $n \leqslant 4$ (see [6]), the previous theorem implies the affirmative part of the original Busemann–Petty problem in $\mathbb{R}^n$.

Now we investigate for which classes of bodies $\dfrac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2}$ is a positive definite distribution.

Proposition 3.6. *Let K be an origin-symmetric body in B^n, $n \leqslant 4$.*

(i) *If K is h-convex then $\dfrac{\|x\|_K^{-1}}{1+(\frac{|x|}{\|x\|_K})^2}$ is positive definite.*

(ii) *If K is s-convex then $\dfrac{\|x\|_K^{-1}}{1-(\frac{|x|}{\|x\|_K})^2}$ is positive definite.*

Proof. (i) Consider a h-convex origin-symmetric body $K \subset B^n$, $n \leqslant 4$. Define a body M by the formula:

$$\|x\|_M^{-1} = \frac{\|x\|_K^{-1}}{1+(\frac{|x|}{\|x\|_K})^2}.$$

It is enough to show that M is e-convex. If we pass to polar coordinates then the map

$$(r, \theta) \mapsto \left(\frac{r}{1+r^2}, \theta\right)$$

transforms the body K into the body M.

Take two points in K and connect them by a hyperbolic segment. This segment belongs to K since K is h-convex. Consider the 2-dimensional plane through the origin and these 2 points. The section of the body K by this plane is a 2-dimensional h-convex body. Introduce polar coordinates on this plane and (without loss of generality) assume that the h-geodesic segment has the equation $r^2 - a\ r \cos\phi + 1 = 0$. Applying the above transformation one can see that this h-segment gets mapped into an e-segment given by the equation $r = \dfrac{1}{a\cos\phi}$. Therefore the body M is e-convex and $(\|x\|_M^{-1})^\wedge$ is positive in dimensions $n \leqslant 4$ (see [6]).

(ii) Similar to (i). Take a s-geodesic given by the equation $r^2 + a\ r \cos\phi - 1 = 0$. The image of this geodesic under the map

$$(r, \theta) \mapsto \left(\frac{r}{1-r^2}, \theta\right) \tag{6}$$

is an e-geodesic $r = \dfrac{1}{a\cos\phi}$. $\square$

Since every s-convex body containing the origin is h-convex, we have the following

Corollary 3.7. $\dfrac{\|x\|_K^{-1}}{1+(\frac{|x|}{\|x\|_K})^2}$ *is positive-definite for every origin-symmetric s-convex body K in dimension $n \leqslant 4$.*

This fact combined with Theorem 3.4 *implies the affirmative answer to the spherical Busemann–Petty problem for $n \leqslant 4$.*

However not every h-convex body is s-convex and this idea will be used in constructing counterexamples to the hyperbolic Busemann–Petty problem.

First we remind the following fact:

Theorem 3.8 (*Gardner et al. [6, Theorem 1]*). *Let K be an origin-symmetric star body in $\mathbb{R}^n$ with C^∞ boundary, and let $k \in \mathbb{N} \setminus \{0\}$, $k \neq n-1$. Suppose that $\xi \in S^{n-1}$, and let A_ξ be the corresponding parallel section function of K: $A_\xi(z) = \int_{K \cap \langle x,\xi\rangle = z} dx$. (We also assume that $K \cap \{\langle x, \xi\rangle = z\}$ is star-shaped for small z).*

(a) *If k is even, then*

$$(\|x\|^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_\xi^{(k)}(0).$$

(b) *If k is odd, then*

$$(\|x\|^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2} 2(n-1-k)k! \times \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \cdots - A_\xi^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}} dz,$$

where $A_\xi^{(k)}$ stands for the derivative of the order k and the Fourier transform is considered in the sense of distributions.

In particular, it follows that for infinitely smooth bodies the Fourier transform of $\|x\|^{-n+k+1}$ restricted to the unit sphere is a continuous function (see also [15, Section 3.3]).

Now we can prove the following

Proposition 3.9. *There exist h-convex origin-symmetric bodies in B^n, $n \geqslant 3$ that give a counterexample to the hyperbolic Busemann–Petty problem.*

Proof. In view of Theorem 3.1 we are interested only in the cases $n = 3$ and 4. First we construct a body L for which $\dfrac{\|x\|_L^{-1}}{1 - (\frac{|x|}{\|x\|_L})^2}$ is not positive definite. Let L be a circular cylinder of radius $\sqrt{2}/2$ with x_1 being its axis of revolution. (See Fig. 2) To the top and bottom of the cylinder attach spherical caps, that are totally geodesic in the spherical metric. Clearly the body L constructed this way is e-convex and therefore

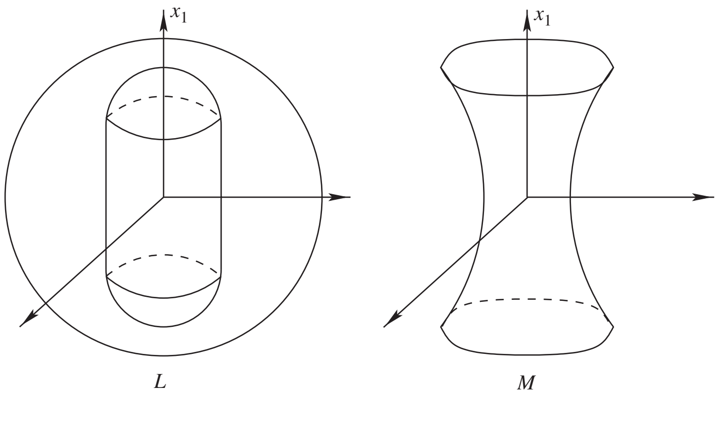

Fig. 2. The bodies L and M from Proposition 3.9.

h-convex. Using the formula

$$\|x\|_M^{-1} = \frac{\|x\|_L^{-1}}{1-(\frac{|x|}{\|x\|_L})^2} \tag{7}$$

we define a body M.

Clearly the body M is the image of L under the map (6). It can be checked directly that the cylinder is mapped into the surface of revolution obtained by rotating the hyperbola $x_2 = \frac{1}{2}\left(\sqrt{2}+\sqrt{2+4x_1^2}\right)$ about the x_1-axis, and the top and bottom spherical caps are mapped into flat disks.

In fact the body L constructed above is not smooth. But we can approximate it by infinitely smooth e-convex bodies that differ from L only in a small neighborhood of the edges. Since the body M is obtained from L by (7), and the denominator in (7) is never equal to zero, the body M is also infinitely smooth. (Now that the bodies L and M are smooth, Fig. 2 might be confusing, but we wanted to make it as simple as possible, just to emphasize the idea).

Now that we defined the body M, we can explicitly compute its parallel section function $A_{M,\xi}$ in the direction of the x_1-axis.

$$A_{M,\xi}(t) = \begin{cases} \pi\left(\dfrac{\sqrt{2}+\sqrt{2+4t^2}}{2}\right)^2, & \text{in dimension } n=3, \\ \dfrac{4\pi}{3}\left(\dfrac{\sqrt{2}+\sqrt{2+4t^2}}{2}\right)^3, & \text{in dimension } n=4. \end{cases}$$

Since M is an infinitely smooth body, $(\|x\|_M^{-1})^\wedge$ is a function. Applying Theorem 3.8 with $n=3$ and $q=1$ we get

$$(\|x\|_M^{-1})^\wedge(\xi) = -2\int_0^\infty \frac{A_{M,\xi}(t)-A_{M,\xi}(0)}{t^2}\,dt.$$

Let the height of the cylindrical part of L be equal to $\sqrt{2}-2\varepsilon$ and the height of its image under (6) equal to N. If ε tends to zero, the top and bottom parts of the body L get closer to the sphere $x_1^2+\cdots+x_n^2=1$. Recalling the definition of the radial function of M:

$$\rho_M(x) = \frac{\rho_L(x)}{1-\rho_L(x)^2}, \quad \forall x \in S^{n-1},$$

one can see that the body M becomes larger in the direction of x_1 as $\varepsilon \to 0$, and therefore its height N approaches infinity.

Since in dimension $n = 3$ the section function can be written as $A_{M,\xi}(t) = \pi\left(1 + t^2 + \sqrt{1 + 2t^2}\right)$ for $-N \leqslant t \leqslant N$, we get

$$
\begin{aligned}
(\|x\|_M^{-1})^\wedge(\xi) &= -2\pi \int_0^N \frac{1 + t^2 + \sqrt{1 + 2t^2} - 2}{t^2} dt - 2\pi \int_N^\infty \frac{(-2)}{t^2} dt \\
&\leqslant -2\pi \int_0^N dt + 4\pi \int_N^\infty \frac{1}{t^2} dt = -2\pi N + \frac{4\pi}{N} < 0
\end{aligned}
$$

for N large enough.

If $n = 4$ and $q = 2$ Theorem 3.8 implies

$$
(\|x\|_M^{-1})^\wedge(\xi) = -\pi A''_{M,\xi}(0) < 0,
$$

since the second derivative of the function $A_{M,\xi}$ in dimension $n = 4$ equals: $A''_{M,\xi}(0) = 8\sqrt{2}\pi$.

Thus we have proved that $\left(\dfrac{\|x\|_L^{-1}}{1 - (\frac{|x|}{\|x\|_L})^2}\right)^\wedge(\xi) = (\|x\|_M^{-1})^\wedge(\xi)$ is negative for some direction ξ.

Now apply a standard argument to construct another body K which along with the body K provides a counterexample to the hyperbolic Busemann–Petty problem (cf. [11, Theorem 2] or [24, Theorem 2]). By continuity of $(\|x\|_M^{-1})^\wedge$ there is a neighborhood of ξ where this function is negative. Let

$$
\Omega = \{\theta \in S^{n-1} : (\|x\|_M^{-1})^\wedge(\theta) < 0\}.
$$

Choose a non-positive infinitely-smooth even function v supported on Ω. Extend v to a homogeneous function $r^{-1}v(\theta)$ of degree -1 on $\mathbb{R}^n$. By Lemma 5 from [11] we know that the Fourier transform of $r^{-1}v(\theta)$ is equal to $r^{-n+1}g(\theta)$ for some infinitely smooth function g on S^{n-1}.

To construct a counterexample to the Busemann–Petty problem, define another body K as follows:

$$
\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-2}}{(1 - r^2)^{n-1}} dr = \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-2}}{(1 - r^2)^{n-1}} dr + \varepsilon g(\theta)
$$

for some $\varepsilon > 0$ small enough (to guarantee that K is still convex in hyperbolic sense). Indeed, define a function $\alpha_\varepsilon(\theta)$ such that

$$
\int_0^{\|\theta\|_L^{-1}} \frac{r^{n-2}}{(1 - r^2)^{n-1}} dr + \varepsilon v(\theta) = \int_0^{\|\theta\|_L^{-1} + \alpha_\varepsilon(\theta)} \frac{r^{n-2}}{(1 - r^2)^{n-1}} dr,
$$

then

$$\|\theta\|_K^{-1} = \|\theta\|_L^{-1} + \alpha_\varepsilon(\theta).$$

Note that in our construction L is e-convex, but we can perturb it a little (by adding $\alpha|\theta|_2$ to the norm $\|\theta\|_L$ with $\alpha > 0$ small enough), so we can assume that L is strictly e-convex. Therefore one can choose ε small enough such that K is also e-convex (for details see [24, Proposition 2]). Hence we can assume that both L and K are h-convex. Using Lemma 2.2 we get

$$\begin{aligned}
\mathrm{vol}_{n-1}(K \cap \xi^\perp) &= \frac{2^{n-1}}{\pi}\left(|x|^{-n+1}\int_0^{|x|/\|x\|_K} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi)\\
&= \frac{2^{n-1}}{\pi}\left(|x|^{-n+1}\int_0^{|x|/\|x\|_L} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi) + \varepsilon v(\xi)\\
&\leqslant \frac{2^{n-1}}{\pi}\left(|x|^{-n+1}\int_0^{|x|/\|x\|_L} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi)\\
&= \mathrm{vol}_{n-1}(L \cap \xi^\perp).
\end{aligned}$$

Proceeding as in the proof of Theorem 3.4 we can show the opposite inequality for volumes. Since the body L is infinitely smooth, one can use Parseval's formula in the form of Lemma 3.2:

$$\begin{aligned}
&(2\pi)^n\int_{S^{n-1}} \frac{\|x\|_L^{-1}}{1-\ \|x\|_L^{-2}}\int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}}dr\,dx\\
&= \int_{S^{n-1}}\left(\frac{\|x\|_L^{-1}}{1-\ (\frac{|x|}{\|x\|_L})^2}\right)^\wedge(\theta)\left(|x|^{-n+1}\int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}}dr\right)^\wedge(\theta)\,d\theta\\
&= \int_{S^{n-1}}\left(\frac{\|x\|_L^{-1}}{1-\ (\frac{|x|}{\|x\|_L})^2}\right)^\wedge(\theta)\left(|x|^{-n+1}\int_0^{\frac{|x|}{\|x\|_L}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}}dr\right)^\wedge(\theta)\,d\theta\\
&\quad+\int_{S^{n-1}}\left(\frac{\|x\|_L^{-1}}{1-(\frac{|x|}{\|x\|_L})^2}\right)^\wedge(\theta)\varepsilon v\,(\theta)\ d\theta\\
&> (2\pi)^n\int_{S^{n-1}} \frac{\|x\|_L^{-1}}{1-\ \|x\|_L^{-2}}\int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}}dr\,dx. \qquad \square
\end{aligned}$$

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