Center manifolds under nonuniform hyperbolicity – maximal regularity

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We establish the existence of (invariant) center manifolds with maximal $C^r$ regularity for a nonautonomous dynamics with discrete time. We consider the general case of perturbations of a nonuniform exponential trichotomy. Our proof uses the fiber contraction principle and allows linear perturbations without any further effort.

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1. Introduction

Our main aim is to establish the existence of $C^r$ center manifolds for the nonautonomous difference equation

$$v_{m+1} = A_m v_m + f_m(v_m), \quad m \in \mathbb{N}$$

in $X = \mathbb{R}^p$, assuming that the (nonautonomous) linear dynamics

$$v_{m+1} = A_m v_m, \quad m \in \mathbb{N}$$

defines a nonuniform exponential trichotomy, and that the maps $f_m$ are of class $C^r$ and satisfy some additional assumptions to be made precise later on. It is well known that center manifold theorems are powerful tools in the analysis of the behavior of dynamical systems. In particular, when the linear dynamics defined by Eq. (2) has no unstable directions, all solutions of Eq. (1) converge exponentially to the center manifold. This implies that the stability of a system without unstable directions is completely determined by the behavior on any center manifold. Thus, one often considers a reduction to a center manifold (see [6] for details and references). An exposition of the theory of center manifolds in the case of autonomous equations is given in [14], adapting results from [16]. See [11,15] for the case of equations in infinite-dimensional spaces. We refer the reader to [8–10,14] for more details and further references. See also [3] for a center manifold theorem in the nonuniform setting, although without establishing the maximal regularity that to the best of our knowledge is obtained here for the first time.

The following is our center manifold theorem when $r = 1$. We denote by $d_v f_m$ the (directional) derivative of $f_m$ at the point $v$, and by $\|d_v f_m\|$ its operator norm.

Theorem 1. If Eq. (2) defines a nonuniform exponential trichotomy, and $f_m, m \in \mathbb{Z}$ are $C^1$ maps with $f_m(0) = f_m(u) = 0$ and $\|d_v f_m\| \leq \kappa e^{-m/\kappa}$ for every $m \in \mathbb{N}$ and $u, v \in X$ with $\|u\| \geq c$, for some constant $c > 0$ and some sufficiently small $\kappa > 0$, then the zero solution of Eq. (1) has a $C^1$ center manifold.
2. Lipschitz center manifolds

We consider invertible operators $A_m \in B(X)$ for each $m \in \mathbb{Z}$, where $B(X)$ is the set of bounded linear operators in a Banach space $X$. Each sequence $v_m \in X$ satisfying $v_{m+1} = A_m v_m$ can be written in the form

$$v_m = A(m, n)v_n \quad \text{for every } m, n \in \mathbb{Z},$$

where

$$A(m, n) = \begin{cases} 
A_{m-1} \cdots A_n & \text{if } m > n, \\
Id & \text{if } m = n, \\
A_{n-1} \cdots A_{m-1} & \text{if } m < n.
\end{cases}$$

We say that $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential trichotomy if there exist projections $P_m, Q_{1m}, Q_{2m} \in B(X)$ satisfying

$$P_m + Q_{1m} + Q_{2m} = Id,$$

and

$$P_mA(m, n) = A(m, n)P_n, \quad Q_{1m}A(m, n) = A(m, n)Q_{1m}, \quad i = 1, 2$$

(3)

for every $m, n \in \mathbb{Z}$, and there exist constants

$$\tilde{a} < \zeta \leqslant \alpha < b \quad \text{and} \quad \epsilon, D > 0$$

such that for every $m \geq n$ we have

$$\|A(m, n)P_n\| \leqslant De^{\zeta(m-n+1)+\epsilon|n-1|},$$

$$\|A(m, n)^{-1}Q_{2m}\| \leqslant De^{-\beta(m-n-1)+\epsilon|m-1|},$$

and for every $m \leq n$ we have

$$\|A(m, n)P_n\| \leqslant De^{-\zeta(n-m-1)+\epsilon|m-1|},$$

$$\|A(m, n)^{-1}Q_{1m}\| \leqslant De^{\delta(n-m+1)+\epsilon|m-1|}.$$  

(4)

(5)

In this case we define center, stable and unstable subspaces for each $m \in \mathbb{Z}$ by

$$E_m = P_m X \quad \text{and} \quad F_{1m} = Q_{1m} X, \quad i = 1, 2.$$

It follows readily from (3) that for every $m, n \in \mathbb{Z}$ we have

$$A(m, n)E_n = E_m \quad \text{and} \quad A(m, n)F_{1n} = F_{1m}, \quad i = 1, 2.$$

We also consider continuous functions $f_m : X \to X$ with $f_m(0) = 0$ for every $m \in \mathbb{Z}$, and we assume that there exists a constant $\delta > 0$ such that

$$\|f_m(u) - f_m(v)\| \leqslant \delta e^{-\zeta|m|\|u - v\|}$$

(6)

for every $m \in \mathbb{Z}$ and $u, v \in X$. Given $n \in \mathbb{Z}$ and an initial condition $v_n = (\xi, \eta_1, \eta_2) \in E_n \times F_{1n} \times F_{2n}$, we denote by

$$(x_n, y_{1n}, y_{2n}) = (x_n(n, v_n), y_{1n}(n, v_n), y_{2n}(n, v_n)) \in E_m \times F_{1m} \times F_{2m}$$

the sequence satisfying
\( v_{m+1} = A_m v_m + f_m(v_m), \quad m \in \mathbb{Z} \)

(provided that it is well defined), or equivalently satisfying

\[
\begin{align*}
x_m &= A(m, n)\xi + \sum_{l=n}^{m-1} P_m A(m, l + 1) f_l(x_l, y_{1l}, y_{2l}), \\
y_{im} &= A(m, n)\eta_i + \sum_{l=n}^{m-1} Q_{im} A(m, l + 1) f_l(x_l, y_{1l}, y_{2l}), \quad i = 1, 2
\end{align*}
\]

for \( m \geq n \), and

\[
\begin{align*}
x_m &= A(m, n)\xi - \sum_{l=m}^{n-1} P_m A(m, l + 1) f_l(x_l, y_{1l}, y_{2l}), \\
y_{im} &= A(m, n)\eta_i - \sum_{l=m}^{n-1} Q_{im} A(m, l + 1) f_l(x_l, y_{1l}, y_{2l}), \quad i = 1, 2
\end{align*}
\]

for \( m \leq n \). For each \( l \in \mathbb{Z} \) we set

\[
\Psi_l(n, v_n) = (n + l, x_{n+l}(n, v_n), y_{1,n+l}(n, v_n), y_{2,n+l}(n, v_n)).
\]

Let also \( \mathcal{X} \) be the space of sequences \( \phi = (\phi_m)_{m \in \mathbb{Z}} \) of continuous functions

\[
\phi_m = (\phi_1, \phi_2): E_m \to F_1 \times F_2
\]

such that \( \phi_1(0) = 0 \), and

\[
\|\phi_l(\xi) - \phi_l(\xi')\| \leq \|\xi - \xi'\|
\]

for every \( n \in \mathbb{Z} \) and \( \xi, \xi' \in E_n \). Given \( \phi \in \mathcal{X} \), for each \( m, n \in \mathbb{Z} \) and \( \xi \in E_n \) we write \( u_{mn}(\xi) = \mathcal{F}(m, n)(\xi, \phi_1(\xi)) \), where

\[
\mathcal{F}(m, n) = \begin{cases} F_{m-1} \circ \cdots \circ F_n & \text{if } m > n, \\
Id & \text{if } m = n, \\
F_{m-1} \circ \cdots \circ F_{n-1} & \text{if } m < n,
\end{cases}
\]

and where \( F_n = A_n + f_n \). We also write

\[
\mathcal{V}_{m, \phi} = \{(\xi, \phi_l(\xi)) : \xi \in E_\xi\}, \quad n \in \mathbb{Z}.
\]

The following Lipschitz center manifold theorem was established in [2].

**Theorem 2.** If the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential trichotomy with

\[
\bar{c} - \bar{b} + \varepsilon < 0 \quad \text{and} \quad \bar{a} - \bar{c} + \varepsilon < 0,
\]

\( f_m(0) = 0 \) for every \( m \in \mathbb{Z} \), and (6) holds with \( \delta \) sufficiently small, then there exists a unique \( \phi \in \mathcal{X} \) such that:

1. \( \mathcal{F}(m, n)\mathcal{V}_{m, \phi} = \mathcal{V}_{m, \phi} \) for every \( m, n \in \mathbb{Z} \);
2. there exists \( K > 0 \) such that for every \( m, n \in \mathbb{Z} \) and \( \xi, \xi' \in E_n \),

\[
\| \mathcal{V}_{mn}(\xi) - \mathcal{V}_{mn}(\xi') \| \leq \begin{cases} K e^{c(4A_D)(1+\varepsilon)(m-n)+\varepsilon|m|} \|\xi - \xi'\|, & \text{if } m \geq n, \\
K e^{(-c-4A_D)(1-\varepsilon)(n-m)+\varepsilon|m|} \|\xi - \xi'\|, & \text{if } m \leq n.
\end{cases}
\]

It should be noted that in [2] we obtain \( C^{k+1,\text{lip}} \) center manifolds for each \( k \in \mathbb{N} \), although the same argument (with some simplifications) can also be used when \( k = 0 \). For the proof of the regularity of the sets \( \mathcal{V}_m = \mathcal{V}_{m, \phi} \) we need to recall several elements from the proof of Theorem 2. So that \( \mathcal{F}(m, n)\mathcal{V}_m = \mathcal{V}_n \) for every \( m, n \in \mathbb{Z} \), we must have

\[
\begin{align*}
x_m &= A(m, n)\xi + \sum_{l=n}^{m-1} A(m, l + 1) f_l(x_l, \phi_l(x_l)), \\
\phi_{im}(x_m) &= A(m, n)\phi_l(\xi) + \sum_{l=n}^{m-1} A(m, l + 1) f_l(x_l, \phi_l(x_l)), \quad i = 1, 2.
\end{align*}
\]
for each $m \geq n$, and

$$x_m = A(m, n)\xi - \sum_{l=m}^{n-1} A(m, l + 1) f_l(x_l, \phi_l(x_l)).$$

$$\phi_{im}(x_m) = A(m, n)\phi_{in}(\xi) - \sum_{l=m}^{n-1} A(m, l + 1) f_l(x_l, \phi_l(x_l)), \quad i = 1, 2 \quad (10)$$

for each $m \leq n$. The following statement can be obtained repeating the proof of Lemma 4 in [2] in the particular case when $k = 0$.

**Lemma 1.** Given a sufficiently small $\delta > 0$, given $\phi \in \mathcal{X}$ the following properties are equivalent:

1. for every $n \in \mathbb{Z}$, $\xi \in E_n$, and $i = 1, 2$, for $m \geq n$ we have

$$\phi_{im}(x_m(\xi)) = Q_{im}A(m, n)\phi_{in}(\xi) + \sum_{l=n}^{m-1} Q_{im}A(m, l + 1) f_l(x_l(\xi), \phi_l(x_l(\xi))).$$

and for $m \leq n$ we have

$$\phi_{im}(x_m(\xi)) = Q_{im}A(m, n)\phi_{in}(\xi) - \sum_{l=m}^{n-1} Q_{im}A(m, l + 1) f_l(x_l(\xi), \phi_l(x_l(\xi))).$$

2. for every $n \in \mathbb{Z}$ and $\xi \in E_n$ we have

$$\phi_{1n}(\xi) = \sum_{l=-\infty}^{n-1} Q_{1n}(l + 1, n)^{-1} f_l(x_l(\xi), \phi_l(x_l(\xi))).$$

$$\phi_{2n}(\xi) = \sum_{l=n}^{+\infty} Q_{2n}(l + 1, n)^{-1} f_l(x_l(\xi), \phi_l(x_l(\xi))). \quad (11)$$

Finally, provided that $\delta$ is sufficiently small, it is shown in [2] that there exists a unique $\phi \in \mathcal{X}$ such that (11) holds for every $n \in \mathbb{Z}$ and $\xi \in E_n$. It is found as a fixed point of a contraction operator $T$ in the space $\mathcal{X}$, which is a complete metric space with the norm

$$\|\phi\| = \sup \left\{ \frac{\|\phi_m(\xi)\|}{\|\xi\|} : m \in \mathbb{Z} \text{ and } \xi \in E_m \setminus \{0\} \right\}.$$ 

The operator $T$ is defined for each $\phi \in \mathcal{X}$ by

$$(T\phi)_n(\xi) = \left( \sum_{l=-\infty}^{n-1} Q_{1n}(l + 1, n)^{-1} f_l(x_l(\xi), \phi_l(x_l(\xi))) - \sum_{l=n}^{+\infty} Q_{2n}(l + 1, n)^{-1} f_l(x_l(\xi), \phi_l(x_l(\xi))) \right) \quad (12)$$

for $(n, \xi) \in \mathbb{Z} \times E_n$.

3. **Smoothness of the center manifolds**

For $\mathcal{X} = \mathbb{R}^p$, we show in this section that the Lipschitz manifolds $\mathcal{W}_{m, \phi}$ in Theorem 2 are of class $C^1$.

**Theorem 3.** Let $f_m$ be $C^1$ functions for each $m \in \mathbb{Z}$. If $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential trichotomy satisfying (8),

$$f_m(0) = f_m(u) = 0 \quad (13)$$

for every $m \in \mathbb{Z}$ and $u \in X$ with $\|u\| \geq c$, for some constant $c > 0$, and condition (6) holds with $\delta$ sufficiently small, then for the unique $\phi$ in Theorem 2 the functions $\phi_n$ are of class $C^1$. If in addition $f_m(0) = 0$ for every $m \in \mathbb{Z}$, then $\phi_n(0) = 0$ for every $n \in \mathbb{Z}$.
Proof. We first recall the fiber contraction principle. Let \( X \) and \( Y \) be metric spaces. We say that a transformation \( S : X \times Y \to X \times Y \) defined by

\[
S(x, y) = (T(x), A(x, y)),
\]

for some functions \( T : X \to X \) and \( A : X \times Y \to Y \), is a fiber contraction if there exists \( \lambda \in (0, 1) \) such that

\[
d_Y(A(x, y), A(x, \tilde{y})) \leq \lambda d_Y(y, \tilde{y})
\]

for every \( x \in X \) and \( y, \tilde{y} \in Y \), where \( d_Y \) is the distance in \( Y \). We also say that a fixed point \( x_0 \in X \) of \( T \) is attracting if \( T^n(x) \to x_0 \) when \( n \to \infty \), for every \( x \in X \).

Lemma 3 (Fiber contraction principle). If \( S \) is a continuous fiber contraction, \( x_0 \in X \) is an attracting fixed point of \( T \), and \( y_0 \in Y \) is a fixed point of \( A(x_0, \cdot) \), then \( (x_0, y_0) \) is an attracting fixed point of \( S \).

We consider the space \( \mathcal{F} \) of sequences of continuous functions \( \Phi = (\Phi_n)_{n \in \mathbb{Z}} \) such that each \( \Phi_n \) is a linear transformation from \( E_n \) to \( F_{1n} \times F_{2n} \), with

\[
\| \Phi \| := \sup \{ \| \Phi_n(\xi) \| : (n, \xi) \in \mathbb{Z} \times E_n \} \leq 1.
\]  

We also consider the subset \( \mathcal{F}_0 \subset \mathcal{F} \) composed of the sequences \( \Phi \in \mathcal{F} \) such that \( \Phi_n(0) = 0 \) for every \( n \in \mathbb{Z} \). We can easily verify that \( \mathcal{F} \) and \( \mathcal{F}_0 \) are complete metric spaces with the distance induced by the norm in (14).

We define a linear operator \( A(\phi, \Phi) = (A_1(\phi, \Phi), A_2(\phi, \Phi)) \) for each pair \( (\phi, \Phi) \in X \times \mathcal{F} \) by

\[
A_1(\phi, \Phi)(\xi) = \sum_{l=-\infty}^{n-1} Q_{1n} A(l + 1, n)^{-1} \left( \frac{\partial f_l}{\partial x}(y_{1\phi}) W_l + \frac{\partial f_l}{\partial y}(y_{1\phi}) \Phi_l(z_{1\phi}) W_l \right),
\]  

and

\[
A_2(\phi, \Phi)(\xi) = - \sum_{l=n}^{\infty} Q_{2n} A(l + 1, n)^{-1} \left( \frac{\partial f_l}{\partial x}(y_{1\phi}) W_l + \frac{\partial f_l}{\partial y}(y_{1\phi}) \Phi_l(z_{1\phi}) W_l \right),
\]

where \( (x, y) \in E_l \times (F_{1l} \oplus F_{2l}) \), with the notation

\[
y_{m\phi} = (x_{m\phi}(\xi), \phi_m(x_{m\phi}(\xi))) \quad \text{and} \quad z_{m\phi} = x_{m\phi}(\xi),
\]

and where the linear operators \( W_l = W_{l, \phi, \Phi, \xi} : E_n \to E_m \) are uniquely determined by the identities:

\[
W_m = P_{m} A(m, n) + \sum_{l=n}^{m-1} P_{m} A(m, l + 1) \left( \frac{\partial f_l}{\partial x}(y_{1\phi}) W_l + \frac{\partial f_l}{\partial y}(y_{1\phi}) \Phi_l(z_{1\phi}) W_l \right)
\]  

for \( m \geq n \), and

\[
W_m = P_m A(m, n) - \sum_{l=m}^{n-1} P_m A(m, l + 1) \left( \frac{\partial f_l}{\partial x}(y_{1\phi}) W_l + \frac{\partial f_l}{\partial y}(y_{1\phi}) \Phi_l(z_{1\phi}) W_l \right)
\]

for \( m \leq n \).

Lemma 4. The operator \( A \) is well-defined, and \( A(X \times \mathcal{F}) \subset \mathcal{F} \).

Proof. Set

\[
B_1 = \sum_{l=-\infty}^{n-1} \left\| Q_{1n} A(l + 1, n)^{-1} \left( \frac{\partial f_l}{\partial x}(y_{1\phi}) W_l + \frac{\partial f_l}{\partial y}(y_{1\phi}) \Phi_l(z_{1\phi}) W_l \right) \right\|. 
\]

\[
B_2 = \sum_{l=n}^{\infty} \left\| Q_{2n} A(l + 1, n)^{-1} \left( \frac{\partial f_l}{\partial x}(y_{1\phi}) W_l + \frac{\partial f_l}{\partial y}(y_{1\phi}) \Phi_l(z_{1\phi}) W_l \right) \right\|. 
\]

By (6) we have

\[
\| f_m(u) \| \leq \delta e^{-\varepsilon|m|}
\]  

(19)
for every \( m \in \mathbb{Z} \) and \( u \in X \). It follows from (5) and (19) that
\[
B_1 \leq 2\delta D \sum_{l=-\infty}^{n-1} e^{\delta(n-l)+\varepsilon|l|} W_l \|W_l\| = 2\delta D \sum_{l=-\infty}^{n-1} e^{\delta(n-l)-2\varepsilon|l|} W_l \|W_l\|. \tag{20}
\]
Similarly, it follows from (4) and (19) that
\[
B_2 \leq 2\delta D \sum_{l=n}^{\infty} e^{-b(l-n)+\varepsilon|l|} W_l \|W_l\| = 2\delta D \sum_{l=n}^{\infty} e^{-b(l-n)-2\varepsilon|l|} W_l \|W_l\|. \tag{21}
\]
By (17) and (19), for \( m \geq n \) we have
\[
\|W_m\| \leq De^{\varepsilon(m-n+1)+\varepsilon|n-1|} + 2\delta D \sum_{l=n}^{m-1} e^{\varepsilon(m-l)+\varepsilon|l-1|} W_l \|W_l\|. \tag{22}
\]
Setting \( \Gamma = \sup_{m \geq n} (e^{-\varepsilon(m-n+1)} \|W_m\|) \) we obtain
\[
\Gamma' \leq De^{\varepsilon|n-1|} + 2\delta D \sum_{l=n}^{m-1} e^{-2\varepsilon|l|} \leq De^{\varepsilon n} + \frac{4\delta D}{1-e^{-2\varepsilon}} \Gamma'.
\]
Taking \( \delta \) sufficiently small (independently of \( n \)) we obtain \( \Gamma' \leq 2De^{\varepsilon|n-1|} \), and hence,
\[
\|W_m\| \leq 2De^{\varepsilon(m-n+1)+\varepsilon|n-1|}. \tag{23}
\]
Proceeding in a similar manner we find that for \( m \leq n \),
\[
\|W_m\| \leq 2De^{-\varepsilon(n-m+1)+\varepsilon|n-1|}. \tag{24}
\]
It follows from (24) and (20) that
\[
B_1 \leq C' \delta \sum_{l=-\infty}^{n-1} e^{(\delta-\varepsilon)(n-l)} = C' \delta e^{\delta-\varepsilon} < 1
\]
for some \( C' > 0 \), provided that \( \delta \) is sufficiently small. Similarly, it follows from (23) and (21) that
\[
B_2 \leq C'' \delta \sum_{l=n}^{\infty} e^{(-b+\varepsilon)(l-n)} = C'' \delta e^{-b+\varepsilon} < 1
\]
for some constant \( C'' > 0 \), provided that \( \delta \) is sufficiently small. This shows that \( A(\phi, \Phi) \| \) is well-defined, and that \( \|A(\phi, \Phi)\| \leq 1 \), that is, \( A(\mathcal{X} \times \mathcal{T}) \subset \mathcal{X} \times \mathcal{T} \). \( \square \)

Now we define a transformation \( S : \mathcal{X} \times \mathcal{T} \to \mathcal{X} \times \mathcal{T} \) by
\[
S(\phi, \Phi)_n = (\langle T \phi \rangle_n, A(\phi, \Phi)_n)
\]
with the operator \( T \) as in (12).

**Lemma 5.** For every \( \delta > 0 \) sufficiently small, the operator \( S \) is a fiber contraction.

**Proof.** Given \( \phi \in \mathcal{X} \) and \( \Phi, \Psi \in \mathcal{T} \), let \( W_{1\phi} = W_{1, \phi, x, \varepsilon} \) and \( W_{1\psi} = W_{1, \psi, \varepsilon, \varepsilon} \). Setting \( \alpha = \|\Phi - \Psi\| \), we have
\[
\|A_1(\phi, \Phi)_n(\xi) - A_1(\phi, \Psi)_n(\xi)\| \leq D \sum_{l=-\infty}^{n-1} e^{\alpha(n-l)+\varepsilon|l|} \left| \frac{\partial f_1}{\partial x} W_{1\phi} + \frac{\partial f_1}{\partial y} \Phi_1 W_{1\phi} - \frac{\partial f_1}{\partial x} W_{1\psi} - \frac{\partial f_1}{\partial y} \Psi_1 W_{1\psi} \right|
\]
\[
\leq \delta D \sum_{l=-\infty}^{n-1} e^{\alpha(n-l)-2\varepsilon|l|} (\|W_{1\phi} - W_{1\psi}\| + \|W_{1\phi} - \Phi_1 W_{1\phi} - W_{1\psi}\|)
\]
\[
\leq \delta D \sum_{l=-\infty}^{n-1} e^{\alpha(n-l)-2\varepsilon|l|} (\|W_{1\phi} - W_{1\psi}\| + \|W_{1\phi} - W_{1\psi}\| + \alpha\|W_{1\psi}\|) d\tau
\]
\[
\leq \delta D \sum_{l=-\infty}^{n-1} e^{\alpha(n-l)-2\varepsilon|l|} (2\|W_{1\phi} - W_{1\psi}\| + \alpha\|W_{1\psi}\|), \tag{25}
\]

Lemma 6. For every $\delta > 0$ sufficiently small, the operator $S$ is a fiber contraction. □
Proof. Setting

\[ W_{i\phi} = W_{i,\phi, \xi} \quad \text{and} \quad W_{l\phi} = W_{l,\phi, \xi}, \]

we obtain

\[ \left\| A_1(\phi, \Phi)_{n}(\xi) - A_1(\psi, \Phi)_{n}(\xi) \right\| \leq D \sum_{l=-\infty}^{n-1} e^{\delta(n-l)+\epsilon||l||} \left\| \frac{\partial f_l}{\partial x}(y_{l\phi}) W_{l\phi} + \frac{\partial f_l}{\partial y}(y_{l\phi}) \Phi_l(z_{l\phi}) W_{l\phi} - \frac{\partial f_l}{\partial x}(y_{l\phi}) W_{l\phi} - \frac{\partial f_l}{\partial y}(y_{l\phi}) \Phi_l(z_{l\phi}) W_{l\phi} \right\|, \]

and

\[ \left\| A_2(\phi, \Phi)_{n}(\xi) - A_2(\psi, \Phi)_{n}(\xi) \right\| \leq D \sum_{l=0}^{\infty} e^{2\delta(l-n)+\epsilon||l||} \left\| \frac{\partial f_l}{\partial x}(y_{l\phi}) W_{l\phi} + \frac{\partial f_l}{\partial y}(y_{l\phi}) \Phi_l(z_{l\phi}) W_{l\phi} - \frac{\partial f_l}{\partial x}(y_{l\phi}) W_{l\phi} - \frac{\partial f_l}{\partial y}(y_{l\phi}) \Phi_l(z_{l\phi}) W_{l\phi} \right\|. \]

It follows from (19) that

\[ \left\| A_1(\phi, \Phi)_{n}(\xi) - A_1(\psi, \Phi)_{n}(\xi) \right\| \leq D \sum_{l=-\infty}^{n-1} e^{\delta(n-l)+\epsilon||l||} \left\| \frac{\partial f_l}{\partial x}(y_{l\phi}) - \frac{\partial f_l}{\partial x}(y_{l\phi}) \right\| \cdot \left\| W_{l\phi} \right\| + D \sum_{l=-\infty}^{n-1} e^{\delta(n-l)+\epsilon||l||} \left\| \frac{\partial f_l}{\partial y}(y_{l\phi}) - \frac{\partial f_l}{\partial y}(y_{l\phi}) \right\| \cdot \left\| W_{l\phi} \right\|

[\text{leading terms}]

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for some constant \(D' > 0\). Moreover, given \(\gamma > 0\) there exists \(\sigma \in \mathbb{N}\) (independent of \(n\) and \(\xi\)) such that

\[ 2D' e^{2\epsilon||n||} \sum_{l=-\infty}^{n-1} e^{(-\zeta+\epsilon+\delta)(n-l)} \left\| f'_l(y_{l\phi}) - f'_l(y_{l\phi}) \right\| + 2\delta D' \sum_{l=-\infty}^{n-1} e^{\delta(n-l)-2\epsilon||l||} \left\| W_{l\phi} - W_{l\phi} \right\| \leq 4\delta D' e^{(-\zeta+\epsilon+\delta)\sigma} \frac{1}{1-e^{-\zeta+\epsilon+\delta}} < \gamma, \]

\[ 2\delta D \sum_{l=-\infty}^{n-1} e^{\delta(n-l)-2\epsilon||l||} \left\| W_{l\phi} - W_{l\phi} \right\| \leq \delta D'' \sum_{l=-\infty}^{n-1} e^{(-\zeta+\epsilon+\delta)(n-l)} < \gamma, \]
for some constant $D'' > 0$, and
\[
\delta D' \sum_{l=-\infty}^{n-\sigma} e^{-(\xi+e+\delta)(n-l)-\varepsilon l} \| \Phi_l(z_{\Phi\varepsilon}) - \Phi_l(z_{\Phi}) \| \leq 2 \delta D' \sum_{l=-\infty}^{n-\sigma} e^{-(\xi+e+\delta)(n-l)} < \gamma. \tag{32}
\]

Now we consider the sums from $n - \sigma + 1$ to $n$. For this we consider the sequences
\[
B(p, \phi, n)(\xi) = 2D' e^{2\varepsilon|p|e^{-(\xi+e+\delta)}} f_{\xi}(y_{n-p, \phi}),
\]
\[
C(p, \phi, n)(\xi) = 2\delta D e^{\delta p - 2\varepsilon|p-n|} \| W_{n-p, \phi} \|,
\]
\[
D(p, \phi, n)(\xi) = \delta D' e^{-(\xi+e+\delta)p - \varepsilon|p-n|} \| \Phi_{n-p}(z_{n-p, \phi}) \|
\]
indexed by $p \in \{1, \ldots, \sigma - 1\}$ and $\phi \in X$. Since the functions $\Phi_m, X_m\Phi, \text{ and } (\phi, \xi) \mapsto W_{m, \phi, \xi}$ are continuous, the functions
\[
(\phi, \xi) \mapsto B(p, \phi, n)(\xi), \ C(p, \phi, n)(\xi), \ D(p, \phi, n)(\xi)
\]
are also continuous. Furthermore, by (19), for each $p \in \{1, \ldots, \sigma - 1\}$ and $\phi \in X$ we have
\[
sup_{\xi \in E_n} \| B(p, \phi, n)(\xi) \| \leq 2 \delta D' e^{(\xi+e+\delta)p - \varepsilon|p-n|} \leq 2 \delta D' e^{-\varepsilon|p-n|},
\]
\[
sup_{\xi \in E_n} \| C(p, \phi, n)(\xi) \| \leq 2 \delta D^2 e^{(\xi+e+\delta)p - \varepsilon|p-n|} \leq 4 \delta D' e^{-\varepsilon|p-n|},
\]
\[
sup_{\xi \in E_n} \| D(p, \phi, n)(\xi) \| \leq \delta D' e^{-(\xi+e+\delta)p - \varepsilon|p-n|} \leq \delta D' e^{-\varepsilon|p-n|},
\]
with the norm in (14). In particular, $B(p, \phi), C(p, \phi), \text{ and } D(p, \phi)$ are in $\mathcal{F}$ provided that $\delta$ is sufficiently small. We note that there exists $N \in \mathbb{N}$ such that
\[
sup_{|n| > N} \sup_{\xi \in E_n} \| B(p, \phi, n)(\xi) - B(p, \psi, n)(\xi) \| \leq 8 \delta D' e^{-\varepsilon|p-n|} < \gamma,
\]
\[
sup_{|n| > N} \sup_{\xi \in E_n} \| C(p, \phi, n)(\xi) - C(p, \psi, n)(\xi) \| \leq 8 \delta D' e^{-\varepsilon|p-n|} < \gamma,
\]
\[
sup_{|n| > N} \sup_{\xi \in E_n} \| D(p, \phi, n)(\xi) - D(p, \psi, n)(\xi) \| \leq 8 \delta D' e^{-\varepsilon|p-n|} < \gamma.
\]

Now we consider the case when $|n| \leq N$. Given $n \in \mathbb{Z}$ and $(\phi, \xi) \in X \times E_n$, due to the continuity in (33) there exists $\delta > 0$ such that
\[
\| B(p, \phi, n)(\xi) - B(p, \psi, n)(\xi) \| < \gamma \tag{34}
\]
whenever $d(\phi, \psi) < \delta$ and $\| \xi - \xi \| < \delta$. Since $u \mapsto f_{\xi}(u)$ vanishes for $\|u\| \geq c$, it is sufficient to establish the continuity inside a certain ball in $X$ (possibly depending on $p$ and $n$). This shows that it is sufficient to consider $\xi$ in some compact set $K$ (since now $p$ and $n$ run over a finite set). We can cover $K$ with a finite number of balls $B_i, i = 1, \ldots, r$ centered at points in this set, such that (34) holds whenever $d(\phi, \psi) < \delta_i$ and $\xi, \xi \in B_i$, for $i = 1, \ldots, r$ and some numbers $\delta_i > 0$. Therefore,
\[
\| B(p, \phi, n)(\xi) - B(p, \psi, n)(\xi) \| < \gamma \text{ whenever } d(\phi, \psi) < \min\{\delta_1, \ldots, \delta_r\} \text{ and } (\xi, \lambda) \in K. \text{ This shows that}
\]
\[
\sup_{|n| \leq N} \sup_{\xi \in E_n} \| B(p, \phi, n)(\xi) - B(p, \psi, n)(\xi) \| \leq \gamma
\]
whenever $d(\phi, \psi) < \delta$. Together with (30), (31), and (32) this implies that $\phi \mapsto A_1(\phi, \Phi)$ is continuous. Similarly, since
\[
\| A_2(\phi, \Phi, n)(\xi) - A_2(\psi, \Phi, n)(\xi) \| \leq D \sum_{l=-n}^{\infty} e^{-b(l-n)\varepsilon l} \left\| \frac{\partial f_{\xi}}{\partial x}(y_{l\Phi}) - \frac{\partial f_{\xi}}{\partial x}(y_{l\Phi}) \right\| \cdot \| W_{l\Phi} \|
\]
\[
\quad + D \sum_{l=-n}^{\infty} e^{-b(l-n)\varepsilon l} \left\| \frac{\partial f_{\xi}}{\partial y}(y_{l\Phi}) \right\| \cdot \| W_{l\Phi} - W_{l\Phi} \|
\]
\[
\quad + D \sum_{l=-n}^{\infty} e^{-b(l-n)\varepsilon l} \left\| \frac{\partial f_{\xi}}{\partial y}(y_{l\Phi}) - \frac{\partial f_{\xi}}{\partial y}(y_{l\Phi}) \right\| \cdot \| \Phi_l(z_{\Phi\varepsilon}) W_{l\Phi} \|
\]
we can show that \( \phi \mapsto A_{2}(\phi, \Phi) \) is also continuous. Since the operator \( T \) is a contraction, we conclude that \( S \) is continuous. \( \square \)

Now we observe that if each \( \phi_n \) is of class \( C^{1} \) and \( \Phi_n = \phi_n' \), then \( W_m = x_{m, \Phi}^{\phi} \) in (17) for each \( m \in \mathbb{Z} \). Therefore,

\[
A(\phi, \phi')_{n} = \left( \sum_{l=-\infty}^{\infty} \frac{d}{d\xi} \left[ Q_{n} A(l + 1, n)^{-1} f_{l}(y_{l, \Phi}) \right] \right)_{n} = (T \phi_{n})'. \tag{35}
\]

To complete the proof, we consider the pair \((\phi^1, \Phi^1) = (0, 0) \in \mathcal{X} \times \mathcal{F}\). Clearly, \( \phi_{n}^1 = (\phi_n^1)' \) for each \( n \in \mathbb{Z} \). We define recursively a sequence \((\phi_n^m, \Phi_n^m) \in \mathcal{X} \times \mathcal{F}\) by

\[
(\phi_n^{m+1}, \Phi_n^{m+1}) = S(\phi_n^{m}, \Phi_n^{m}) = (T \phi_n^{m}, A(\phi_n^{m}, \Phi_n^{m})). \tag{36}
\]

Assuming that \( \phi_n^m \) is of class \( C^{1} \) with \( \Phi_n^m = (\phi_n^m)' \) for each \( n \in \mathbb{Z} \), it follows from (35) that

\[
(\phi_n^{m+1})' = (T \phi_n^{m})'_{n} = A(\phi_n^{m}, \Phi_n^{m})_{n} = \phi_n^{m+1}. \tag{37}
\]

Furthermore, if \( \phi_0^0 \) is the fixed point of \( T \), and \( \phi_0^0 \) is the fixed point of \( \phi \mapsto A(\phi^0, \Phi) \), then by Lemma 3 the sequences \( \phi_n^m \) and \( \Phi_n^m \) converge uniformly respectively to \( \phi_0^0 \) and \( \Phi_0^0 \) on bounded subsets, for each \( n \in \mathbb{Z} \). It follows from (37) that each function \( \phi_0^0 \) is of class \( C^{1} \), and that

\[
(\phi_n^{0})' = \phi_n^{0}, \quad n \in \mathbb{Z} \tag{38}
\]

(we recall that if a sequence \( f_m \) of \( C^{1} \) functions converges uniformly, and the sequence \( f'_m \) also converges uniformly, then the limit of \( f_m \) is of class \( C^{1} \), and its derivative is the limit of \( f'_m \)).

Now we assume that \( f_m(0) = 0 \) for every \( m \in \mathbb{Z} \). It follows from (15) and (16) that \( A(\phi, \Phi)_{n}(0) = 0 \) for every \( \phi \in \mathcal{X} \times \mathcal{F}_0 \) and \( n \in \mathbb{Z} \). Therefore, \( A(\mathcal{X} \times \mathcal{F}_0) \subset \mathcal{X} \times \mathcal{F}_0 \). Since \( (\phi^1, \Phi^1) = (0, 0) \in \mathcal{X} \times \mathcal{F}_0 \), and \( S(\mathcal{X} \times \mathcal{F}_0) \subset \mathcal{X} \times \mathcal{F}_0 \), the sequence \( (\phi_n^{m}, \Phi_n^{m}) \) in (36) is also in \( \mathcal{X} \times \mathcal{F}_0 \). Therefore, \( \phi_0^0(0) = 0 \) for every \( n \in \mathbb{Z} \), and it follows from (38) that \( (\phi_n^{0})'(0) = 0 \) for every \( n \in \mathbb{Z} \). \( \square \)

4. Higher regularity of the manifolds

Again for \( X = \mathbb{R}^{p} \), we establish the \( C^{r} \) regularity of the stable manifolds \( \mathcal{W}_{m, \Phi} \) when the maps \( f_m \) are of class \( C^{r} \). We assume in this section that the space \( X \) admits smooth cutoff functions.

**Theorem 4.** Let \( f_{m} \) be \( C^{r} \) functions for some \( r \geq 2 \). If \( (A_{n})_{n \in \mathbb{Z}} \) admits a nonuniform exponential dichotomy satisfying (8), condition (13) holds, and

\[
\|d_{u} f_{m}\| \leq \delta e^{-3|x| |m|} \quad \text{and} \quad \|d_{u}^{2} f_{m}\| \leq \delta e^{-3|x| |m|} \tag{39}
\]

for every \( m \in \mathbb{Z} \) and \( u \in X \), and some sufficiently small \( \delta \) (depending on \( r \)), then for the unique \( \phi \) in Theorem 2 the maps \( \phi_n \) are of class \( C^{r} \).

**Proof.** Let \( \alpha : X \to [0, 1] \) be a \( C^{r} \) function with compact support, such that \( \alpha(z) = 1 \) when \( \|z\| \leq 1 \), and satisfying

\[
\|\alpha(z)z\| \leq C \quad \text{and} \quad \left\| \frac{d}{dz}[\alpha(z)z] \right\| \leq C \tag{40}
\]

for every \( z \in X \) and some constant \( C > 0 \). We consider the maps \( G_{n} : X \times X \to X \times X \) for \( n \in \mathbb{Z} \) given by

\[
G_{n}(u, z) = (F_{n}(u), A_n z + \alpha(z) d_{u} f_{n} z),
\]

and we define \( \mathcal{W}(m, n) \) in a similar manner to that in (7) with each \( F_{j} \) replaced by \( G_{j} \). We also consider the maps \( \tilde{G}_{n} : X \times X \to X \times X \) given by

\[
\tilde{G}_{n}(u, z) = (F_{n}(u), d_{u} F_{n} z).
\]
Clearly, the maps \( g_m(u, z) = (f_m(u), \alpha(z)d_u f_m z) \) are of class \( C^{-1} \), and \( g_m(0, 0) = g_m(u, z) = 0 \) for every \( m \in \mathbb{Z} \) and \( (u, z) \in X \times X \) with \( \| (u, z) \| > c' \), for some constant \( c' > 0 \). Moreover, it follows from (39) and (40) that
\[
\| g_m(u,1, z_1) - g_m(u,2, z_2) \| \leq e^{-\delta e^{|m|}} \| (u,1, z_1) - (u,2, z_2) \|
\]
for every \( m \in \mathbb{Z} \) and \( u_1, u_2, z_1, z_2 \in X \), for some constant \( c > 0 \).

Now let \( \mathcal{Y} \) be the space of sequences \( \psi = (\psi_n)_{n \in \mathbb{Z}} \) of continuous functions \( \psi_n : E_n \times E_n \to F_{1n} \times F_{2n} \) such that for each \( n \in \mathbb{Z} \):

1. the function \( v \mapsto \psi_n(\xi, v) \) is linear for each \( \xi \in E_n \);
2. for each \( \xi, v \in E_n \) we have
\[
\| \psi_n(\xi, v) \| \leq \| v \|. \tag{41}
\]

Lemma 7. There exists a unique \( \psi \in \mathcal{Y} \) such that the sets
\[
T_n \psi = \{ (\xi, v, \phi_n(\xi), \psi_n(\xi, v)) : \xi, v \in E_n \}, \quad n \in \mathbb{Z},
\]
satisfy
\[
\mathcal{S}(m, n)(T_n \psi) = T_m \psi \quad \text{for every } m \geq n. \tag{42}
\]

Proof. By Theorem 2, for each \( n \in \mathbb{Z} \) and \( \xi \in E_n \) there exists a unique \( \phi_n \in X \) such that the sets
\[
V_{\phi_n} = \{ (v, \phi_n(v)) : v \in E_n \}
\]
are invariant under the maps
\[
H_m(z) = A_m z + \alpha(z)d_{u_m} f_m z,
\]
where \( u_m = \mathcal{S}(m, n)u_n \) with \( u_n = (\xi, \phi_n(\xi)) \), that is,
\[
H_n(V_{\phi_n}) = V_{\phi_{n+1}}, \quad n \in \mathbb{Z}.
\]

Since the maps \( H_n \) are linear in a neighborhood of zero, for each \( n \in \mathbb{Z} \) the function \( v \mapsto \phi_n(v) \) is linear in a neighborhood of zero (possibly depending on \( n \)). We set \( \psi_n(\xi, v) = \phi_n(v) \) for any sufficiently small \( v \), and we extend \( v \mapsto \psi_n(\xi, v) \) linearly. It follows that \( \psi = (\psi_n)_{n \in \mathbb{Z}} \in \mathcal{Y} \), and that (42) holds. The uniqueness of \( \psi \) follows from the uniqueness of \( \phi \) for each \( (n, \xi) \in \mathbb{Z} \times E_n \). \( \square \)

Now we proceed by induction on \( r \). Let us assume that the statement in Theorem 4 holds for \( r = l \) and that the maps \( f_m \) are of class \( C^{l+1} \). Then for the unique sequence in Lemma 7 the maps \( \psi_n \) are of class \( C^l \) in \( \xi \). Indeed, since \( \psi_n(\xi, v) = \phi_n(v) \) for any sufficiently small \( v \), and \( (\xi, v) \mapsto \phi_n(v) \) is of class \( C^l \) for each \( n \) (by the induction hypothesis), the functions \( \psi_n \) are also of class \( C^l \).

Let \( z_m = (v_m, w_m) \in E_m \times F_m \) be the sequence obtained from (43) with \( u_n = (\xi, \phi_n(\xi)) \). We write \( w_m = \psi_m(u_m, v_m) = \psi_m(u_m) \psi_m \), and we note that \( \psi \in \mathcal{F} \). Indeed, it follows from (41) that \( \| \psi_n(\xi) \| \leq 1 \) for each \( (n, \xi) \in \mathbb{Z} \times E_n \). Moreover, for \( m \geq n \) we have
\[
v_m = A(m, n) v_n + \sum_{l=n}^{m-1} A(m, l+1) P_{l+1} \left( \frac{\partial f_l}{\partial x} (y_{l}) v_l + \frac{\partial f_l}{\partial y} (y_{l}) \psi_l(u_l) v_l \right), \tag{44}
\]
and for \( m \leq n \),
\[
v_m = A(m, n) v_n - \sum_{l=m}^{n-1} A(m, l+1) P_{l+1} \left( \frac{\partial f_l}{\partial x} (y_{l}) v_l + \frac{\partial f_l}{\partial y} (y_{l}) \psi_l(u_l) v_l \right). \tag{45}
\]
Furthermore, for \( m \geq n \) we have
\[
\psi_n(u_m) v_m = A(m, n) \psi_n(\xi) v_n + \sum_{l=n}^{m-1} A(m, l+1) P_{l+1} \left( \frac{\partial f_l}{\partial x} (y_{l}) v_l + \frac{\partial f_l}{\partial y} (y_{l}) \psi_l(u_l) v_l \right) \tag{46}
\]
and for \( m \leq n \),
\[\psi(u_m)v_m = A(m,n)\psi_0(\xi)v_n - \sum_{l=m}^{n-1} A(m, l + 1) P_{l+1} \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right). \]  

(47)

Comparing (44) and (45) with (17) and (18) we find that \(v_m = W_m v_n\).

On the other hand, (46) is equivalent to

\[\psi_1(n) v_n = \sum_{l=-\infty}^{n-1} Q_{1n} A(n, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right). \]  

(48)

and

\[\psi_2(n) v_n = -\sum_{l=n}^{\infty} Q_{2n} A(n, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right). \]  

(49)

We first show that the series converge. By (23), for each \(l \geq n\) we have

\[\left\| \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right\| \leq 2e^{-3\varepsilon l} \|v_l\| \leq D' e^{c(l-n)-2\varepsilon l} \|v_n\|,\]

and for \(l < n\),

\[\left\| \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right\| \leq 2e^{-3\varepsilon l} \|v_l\| \leq D' e^{-c(l-n)-2\varepsilon l} \|v_n\|,\]

for some constant \(D' > 0\). It follows from the second inequality in (5) that

\[\sum_{l=-\infty}^{n-1} \left\| Q_{1n} A(n, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right) \right\| \leq D' \sum_{l=-\infty}^{n-1} e^{c(l-n)+2\varepsilon(n-l)} < \infty,\]

and it follows from the second inequality in (4) that

\[\sum_{l=n}^{\infty} \left\| Q_{2n} A(n, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right) \right\| \leq D' \sum_{l=n}^{\infty} e^{c(l-n)+2\varepsilon(n-l)} < \infty.\]

Now we assume that identities (46) and (47) hold, and we write them in the equivalent form

\[\psi_1(n) v_n = A(m,n)^{-1} \psi_1(u_m) v_m - \sum_{l=m}^{n-1} Q_{1n} A(n, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right). \]  

(50)

and

\[\psi_2(n) v_n = A(m,n)^{-1} \psi_2(u_m) v_m - \sum_{l=m}^{n-1} Q_{2n} A(n, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right). \]  

(51)

For \(m \leq n\) we have

\[\|A(m,n)^{-1} \psi_m(u_m) v_m\| = \|A(m,n) Q_{1m} \psi_m(u_m) v_m\| \leq C e^{c(n-m)+2\varepsilon m} \|v_m\| \leq C' e^{c(n-m)+2\varepsilon m} \|v_n\|,\]

and for \(m \geq n\) we have

\[\|A(m,n)^{-1} \psi_m(u_m) v_m\| = \|A(m,n) Q_{2m} \psi_m(u_m) v_m\| \leq C e^{c(n-m)+2\varepsilon m} \|v_m\| \leq C'' e^{c(n-m)+2\varepsilon m} \|v_n\|,\]

for some constants \(C', C'' > 0\). By (8), letting \(m \to -\infty\) in (50) and \(m \to \infty\) in (51) we obtain (48) and (49).

Now we assume that (48) and (49) hold. We have

\[A(m,n) \psi_1(n) v_n + \sum_{l=m}^{m-1} Q_{1m} A(m, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right)\]

\[= \sum_{l=-\infty}^{m-1} Q_{1m} A(m, l + 1) \left( \frac{\partial f_1}{\partial x}(y_{\phi(l)}) v_l + \frac{\partial f_1}{\partial y}(y_{\phi(l)}) \psi_l(u_l) v_l \right). \]

(52)

for \(m \geq n\), and
\[ A(m, n)\psi_{2n}(\xi) v_n - \sum_{l=m}^{n-1} Q_{2m} A(m, l + 1) \left( \frac{\partial f_i}{\partial x}(y_{l\phi}) v_l + \frac{\partial f_i}{\partial y}(y_{l\phi}) \psi_l(z_{l\phi}) v_l \right) \]
\[ = - \sum_{l=m}^{\infty} Q_{2m} A(m, l + 1) \left( \frac{\partial f_i}{\partial x}(y_{l\phi}) v_l + \frac{\partial f_i}{\partial y}(y_{l\phi}) \psi_l(u_l) v_l \right) \] \hspace{1cm} (53)

for \( m \leq n \). Again by (48) and (49), with \( (n, \xi) \) replaced by \( (m, u_m) \), we obtain
\[ \psi_m(u_m) v_m = \sum_{l=-\infty}^{m-1} Q_{1m} A(m, l + 1) \left( \frac{\partial f_i}{\partial x}(y_{l\phi}) v_l + \frac{\partial f_i}{\partial y}(y_{l\phi}) \psi_l(u_l) v_l \right), \]

and
\[ \psi_m(u_m) v_m = - \sum_{l=m}^{\infty} Q_{2m} A(m, l + 1) \left( \frac{\partial f_i}{\partial x}(y_{l\phi}) v_l + \frac{\partial f_i}{\partial y}(y_{l\phi}) \psi_l(u_l) v_l \right), \]

which together with (52) and (53) yield identities (46) and (47).

Comparing (48) and (49) with (15) and (16) we find that
\[ \psi_n(\xi) = A(\phi, \phi) \psi_n(\xi) \]

for every \( (n, \xi) \in \mathbb{Z} \times E_n \). Since \( \psi \in S \), and \( \Phi \mapsto A(\phi, \Phi) \) has \( (\phi_n)_{n \in \mathbb{Z}} \) as its unique fixed point, we have \( \psi_n = \phi'_n \) for each \( n \in \mathbb{Z} \). Since each function \( \psi_n \) is of class \( C^1 \), we conclude that \( \phi_n \) is of class \( C^{l+1} \) for each \( n \in \mathbb{Z} \). This completes the proof of the theorem. \( \Box \)

References