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J. Math. Anal. Appl. 338 (2008) 1364–1377

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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Analytical solution for the time-fractional telegraph equation by the method of separating variables

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Received 15 September 2006

Available online 22 June 2007

Submitted by I. Podlubny

Abstract

In this paper, a method of separating variables is effectively implemented for solving a time-fractional telegraph equation (TFTE). We discuss and derive the analytical solution of the TFTE with three kinds of nonhomogeneous boundary conditions, namely, Dirichlet, Neumann and Robin boundary conditions.

Keywords: Fractional telegraph equation; Multivariate Mittag-Leffler function; Method of separating variables

1. Introduction

In recent years, there has been a great deal of interest in fractional differential equations. Historical summaries of the developments of fractional calculus can be found in Oldham and Spanier [1], Miller and Ross [2], Samko et al. [3] and Podlubny [4]. A number of numerical methods have been proposed for fractional differential equations [9–15].

Suspension flows are traditionally modelled by parabolic partial differential equations. Sometimes they can be better modelled by hyperbolic equations such as the telegraph equation, which have parabolic asymptotics. In particular the experimental data described in [16,17] seem to be better modelled by the telegraph equation than by the heat equation. Some of the related mathematics was discussed in [17]. The time-fractional telegraph equations have recently been considered by many authors. Orsingher and Beghin [18] studied the fundamental solutions to time-fractional telegraph equations of order 2α. They obtained the Fourier transforms of the solutions for any α and gave a representation of their inverses in terms of stable densities. For the special case $\alpha = 1/2$, they also showed that the fundamental solution is the probability density of a telegraph process with Brownian time. Beghin and Orsingher [19] considered the fractional telegraph equation with partial fractional derivatives of rational order $\alpha = m/n$ with $m < n$. They proved that the fundamental solution to the Cauchy problem for the time-fractional telegraph equation can be

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.023

expressed as the density of the composition of two processes, one depending on m and the other depending on n. Recently Momani [20] derived the analytic and approximate solutions of the space- and time-fractional telegraph equation with some special initial and boundary conditions using Adomian decomposition.

In this paper, we derive the analytical solution of the nonhomogeneous time-fractional telegraph equation under three types of nonhomogeneous boundary conditions using the method of separation of variables.

In Section 2, we give some relevant definitions and a related theorem. In Section 3 we derive the analytical solution of the nonhomogeneous time-fractional telegraph equation with Dirichlet boundary condition. In Sections 4 and 5, we discuss the analytical solution of the nonhomogeneous time-fractional telegraph equation with Neumann and Robin boundary conditions, respectively. Some conclusions are drawn in Section 6.

2. Basic concepts and theorem

We consider the following nonhomogeneous time-fractional telegraph equation:

$$D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t)=k\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\quad 0<x<L,\ t>0,\ \frac{1}{2}<\alpha\leqslant 1, \tag{1}$$

where $D_t^{2\alpha}$ and D_t^{α} are Caputo fractional derivatives with respect to t, the rate a is an arbitrary nonnegative constant and k is an arbitrary positive constant, x and t are the space and time variables, $f(x,t)$ is a sufficiently smooth function. The Caputo fractional derivative of order α is defined as [4]

$$D_t^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^m(\tau)}{(t-\tau)^{1+\alpha-m}}\,d\tau, & m-1<\alpha<m,\\ \frac{d^m}{dt^m}f(t), & \alpha=m,\end{cases} \tag{2}$$

for $m\in\mathbb{N}$.

When $a=0$, Eq. (1) is the fractional counterpart of the nonhomogeneous wave equation. In fact, without the forcing term $f(x,t)$, and with $a=0$, Eq. (1) is known as the fractional diffusion–wave equation studied in Schneider and Wyss [5], Mainardi [6], and [24]. Daftardar-Gejji and Jafari [23] investigated boundary value problems for the equation. Anh and Leonenko [7,8] considered the fractional diffusion–wave equation under some random initial conditions and obtained their renormalized solutions. Gaussian and non-Gaussian scaling laws are established for these solutions in terms of multiple stochastic integrals. The present paper considers the nonstochastic situation, and the solutions obtained here, in terms of the multivariate Mittag-Leffler function, contain corresponding results for boundary and initial value problems for the wave equation, namely, when $\alpha=1$.

For convenience, we introduce the following definitions and theorem, which are used further in this paper.

Definition 2.1. (See [21].) A real or complex-valued function $f(x)$, $x>0$, is said to be in the space $C_\alpha,\alpha\in\mathcal{R}$, if there exists a real number $p>\alpha$ such that

$$f(x)=x^p f_1(x) \tag{3}$$

for a function $f_1(x)$ in $C[0,\infty]$.

Definition 2.2. (See [22].) A function $f(x)$, $x>0$, is said to be in the space C_α^m, $m\in N_0=N\cup\{0\}$, if and only if $f^m\in C_\alpha$.

Definition 2.3. (See [22].) A multivariate Mittag-Leffler function is defined as

$$E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n):=\sum_{k=0}^{\infty}\sum_{\substack{l_1+\dots+l_n=k\\ l_1\geqslant 0,\dots,l_n\geqslant 0}}\frac{k!}{l_1!\times\dots\times l_n!}\frac{\prod_{i=1}^{n}z_i^{l_i}}{\Gamma(b+\sum_{i=1}^{n}a_il_i)} \tag{4}$$

in which $b>0$, $a_i>0$, $|z_i|<\infty$, $i=1,\dots,n$.

In particular, if $n=1$, the multivariate Mittag-Leffler function is reduced to the Mittag-Leffler function

$$E_{a_1,b}(z_1)=\sum_{k=0}^{\infty}\frac{z_1^k}{\Gamma(b+ka_1)},\quad a_1,b>0,\ |z_1|<\infty. \tag{5}$$

Theorem 2.4. *Let $\mu>\mu_1>\cdots>\mu_n\geqslant 0$, $m_i-1<\mu_i\leqslant m_i$, $m_i\in N_0=N\cup\{0\}$, $\lambda_i\in\mathcal{R}$, $i=1,\dots,n$. The initial value problem*

$$\begin{cases}\big(D^{\mu}y\big)(x)-\displaystyle\sum_{i=1}^{n}\lambda_i\big(D^{\mu_i}y\big)(x)=g(x),\\ y^k(0)=c_k\in\mathcal{R},\quad k=0,\dots,m-1,\ m-1<\mu\leqslant m,\end{cases} \tag{6}$$

where the function $g(x)$ is assumed to lie in C_{-1} if $\mu\in\mathcal{N}$, in C_{-1}^1 if $\mu\notin\mathcal{N}$, and the unknown function $y(x)$ is to be determined in the space C_{-1}^m, has the solution

$$y(x)=y_g(x)+\sum_{k=0}^{m-1}c_k u_k(x),\quad x\geqslant 0, \tag{7}$$

where

$$y_g(x)=\int_0^x t^{\mu-1}E_{(.),\mu}(t)g(x-t)\,dt, \tag{8}$$

and

$$u_k(x)=\frac{x^k}{k!}+\sum_{i=l_k+1}^{n}\lambda_i x^{k+\mu-\mu_i}E_{(.),k+1+\mu-\mu_i}(x),\quad k=0,\dots,m-1, \tag{9}$$

fulfills the initial conditions $u_k^{(l)}(0)=\delta_{kl}$, $k,l=0,\dots,m-1$. The function

$$E_{(.),\beta}(x)=E_{\mu-\mu_1,\dots,\mu-\mu_n,\beta}\big(\lambda_1x^{\mu-\mu_1},\dots,\lambda_nx^{\mu-\mu_n}\big). \tag{10}$$

The natural numbers l_k, $k=0,\dots,m-1$, are determined from the condition

$$\begin{cases}m_{l_k}\geqslant k+1,\\ m_{l_k+1}\leqslant k.\end{cases} \tag{11}$$

In the case $m_i\leqslant k$, $i=1,\dots,m-1$, we set $l_k:=0$, and if $m_i\geqslant k+1$, $i=1,\dots,m-1$, then $l_k:=n$.

Proof. See [22]. □

3. Nonhomogeneous time-fractional telegraph equation with Dirichlet boundary condition

In this section, we determine the solution of the following time-fractional telegraph equation:

$$D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t)=k\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\quad 0<x<L,\ t>0,\ \frac{1}{2}<\alpha\leqslant 1, \tag{12}$$

with the initial conditions

$$u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x),\quad 0\leqslant x\leqslant L, \tag{13}$$

and the nonhomogeneous boundary conditions

$$u(0,t)=\mu_1(t),\quad u(L,t)=\mu_2(t),\quad t>0, \tag{14}$$

using the method of separating variables, where $\phi(x)$, $\psi(x)$ are continuous functions satisfying $\phi(0)=\mu_1(0)$, $\phi(L)=\mu_2(0)$, $\mu_1(t)$ and $\mu_2(t)$ are nonzero smooth functions with order-one continuous derivative.

In order to solve the problem with nonhomogeneous boundary, we firstly transform the nonhomogeneous boundary into a homogeneous boundary condition. Let

$$u(x,t)=W_1(x,t)+V_1(x,t)$$

where $W_1(x,t)$ is a new unknown function and

$$V_1(x,t)=\mu_1(t)+\frac{(\mu_2(t)-\mu_1(t))x}{L} \tag{15}$$

satisfies the boundary conditions

$$V_1(0,t)=\mu_1(t), \quad V_1(L,t)=\mu_2(t). \tag{16}$$

The function $W_1(x,t)$ then satisfies the problem with homogeneous boundary conditions:

$$\begin{cases} D_t^{2\alpha}W_1(x,t)+aD_t^{\alpha}W_1(x,t)=k\frac{\partial^2 W_1(x,t)}{\partial x^2}+\tilde{f}(x,t), & 0<x<L,\ t>0,\\ W_1(x,0)=\phi_1(x), \quad \frac{\partial W_1(x,0)}{\partial t}=\psi_1(x), & 0\leqslant x\leqslant L,\\ W_1(0,t)=W_1(L,t)=0, & t\geqslant 0, \end{cases} \tag{17}$$

in which

$$\begin{aligned} \tilde{f}(x,t)&=-D_t^{2\alpha}V_1(x,t)-aD_t^{\alpha}V_1(x,t)+f(x,t),\\ \phi_1(x)&=\phi(x)-\mu_1(0)-\frac{1}{L}\big[\mu_2(0)-\mu_1(0)\big]x,\\ \psi_1(x)&=\psi(x)-\mu_1'(0)-\frac{1}{L}\big[\mu_2'(0)-\mu_1'(0)\big]x. \end{aligned} \tag{18}$$

We solve the corresponding homogeneous equation in (17) ($\tilde{f}(x,t)$ being replaced by 0) with the boundary conditions by the method of separation of variables.

If we let $W_1(x,t)=X(x)T(t)$ and substitute for $W_1(x,t)$ in (17), we obtain an ordinary linear differential equation for $X(x)$:

$$X''(x)+\frac{\lambda}{k}X(x)=0, \quad X(0)=X(L)=0, \tag{19}$$

and a fractional ordinary linear differential equation with the Caputo derivative for $T(t)$,

$$D_t^{2\alpha}T(t)+aD_t^{\alpha}T(t)+\lambda T(t)=0, \tag{20}$$

where the parameter λ is a positive constant.

The Sturm–Liouville problem given by (19) has eigenvalues

$$\lambda_n=\frac{n^2\pi^2k}{L^2}, \quad n=1,2,\ldots,$$

and corresponding eigenfunctions

$$X_n(x)=\sin\frac{n\pi x}{L}, \quad n=1,2,\ldots.$$

Now we seek a solution of the nonhomogeneous problem in (17) of the form

$$W_1(x,t)=\sum_{n=1}^{\infty}B_n(t)\sin\frac{n\pi x}{L}. \tag{21}$$

We assume that the series can be differentiated term by term. In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $\{\sin\frac{n\pi x}{L}\}$:

$$\tilde{f}(x,t)=\sum_{n=1}^{\infty}\tilde{f}_n(t)\sin\frac{n\pi x}{L}, \tag{22}$$

where

$$\tilde{f}_n(t)=\frac{2}{L}\int_0^L \tilde{f}(x,t)\sin\frac{n\pi x}{L}\,dx. \tag{23}$$

Substituting (21), (22) into (17) yields

$$\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}D_t^{2\alpha}B_n(t)+a\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}D_t^{\alpha}B_n(t)=\frac{-n^2\pi^2k}{L^2}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}B_n(t)+\sum_{n=1}^{\infty}\tilde{f}_n(t)\sin\frac{n\pi x}{L}. \tag{24}$$

By equating the coefficients of both members we get

$$D_t^{2\alpha}B_n(t)+aD_t^{\alpha}B_n(t)+\frac{n^2\pi^2k}{L^2}B_n(t)=\tilde{f}_n(t). \tag{25}$$

Since $W_1(x,t)$ satisfies the initial conditions in (17), we must have

$$\begin{cases}\sum_{n=0}^{\infty}B_n(0)\sin\frac{n\pi x}{L}=\phi_1(x), & 0<x<L,\\ \sum_{n=0}^{\infty}B_n'(0)\sin\frac{n\pi x}{L}=\psi_1(x), & 0<x<L,\end{cases} \tag{26}$$

which yields

$$\begin{cases}B_n(0)=\frac{2}{L}\int_0^L\phi_1(x)\sin\frac{n\pi x}{L}\,dx, & n=1,2,\ldots,\\ B_n'(0)=\frac{2}{L}\int_0^L\psi_1(x)\sin\frac{n\pi x}{L}\,dx, & n=1,2,\ldots.\end{cases} \tag{27}$$

For each value of n, (25) and (27) make up a fractional initial value problem.

According to Theorem 2.4, the fractional initial value problem has the solution

$$B_n(t)=\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\left(-a\tau^{\alpha},\frac{-k\pi^2n^2}{L^2}\tau^{2\alpha}\right)\tilde{f}_n(t-\tau)\,d\tau+B_n(0)u_0(t)+B_n'(0)u_1(t), \tag{28}$$

in which

$$u_0(t)=1-\frac{k\pi^2n^2}{L^2}t^{2\alpha}E_{(\alpha,2\alpha),1+2\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right), \tag{29}$$

$$u_1(t)=t-at^{1+\alpha}E_{(\alpha,2\alpha),2+\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)-\frac{k\pi^2n^2}{L^2}t^{1+2\alpha}E_{(\alpha,2\alpha),2+2\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right), \tag{30}$$

where the multivariate Mittag-Leffler function is given in Definition 2.3. Hence we get the solution of the initial-boundary value problem (17) in the form

$$\begin{aligned}W_1(x,t)&=\sum_{n=1}^{\infty}B_n(t)\sin\frac{n\pi x}{L}\\&=\sum_{n=1}^{\infty}\left[\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\left(-a\tau^{\alpha},-\frac{k\pi^2n^2}{L^2}\tau^{2\alpha}\right)\tilde{f}_n(t-\tau)\,d\tau+B_n(0)u_0(t)+B_n'(0)u_1(t)\right]\sin\frac{n\pi x}{L},\end{aligned} \tag{31}$$

where functions $u_0(t)$ and $u_1(t)$ are given in (29) and (30), respectively. Therefore, we obtain the solution of problem (12)–(14) as

$$u(x,t)=\sum_{n=1}^{\infty}\Bigg[\int_0^t \tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\left(-a\tau^{\alpha},-\frac{k\pi^2n^2}{L^2}\tau^{2\alpha}\right)\tilde{f}_n(t-\tau)\,d\tau$$
$$+B_n(0)u_0(t)+B_n'(0)u_1(t)\Bigg]\sin\frac{n\pi x}{L}+\mu_1(t)+\frac{(\mu_2(t)-\mu_1(t))x}{L}. \tag{32}$$

3.1. Special cases

In this subsection, we consider a number of special cases with results already available in the literature. We aim to show that the solution obtained above agree with those established in these special cases. This indicates that the concept of fractional derivatives extends the concept of derivatives of integer order.

Let us first consider $a=0$ in (32). Then

$$\begin{aligned}u_0(t)&=1-\frac{k\pi^2n^2}{L^2}t^{2\alpha}E_{(\alpha,2\alpha),1+2\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\\&=1-\frac{k\pi^2n^2}{L^2}t^{2\alpha}E_{2\alpha,1+2\alpha}\left(-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\\&=1-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\sum_{m=0}^{\infty}\frac{(-\frac{k\pi^2n^2}{L^2}t^{2\alpha})^m}{\Gamma(2\alpha m+1+2\alpha)}\\&=\sum_{m=0}^{\infty}\frac{(-\frac{k\pi^2n^2}{L^2}t^{2\alpha})^m}{\Gamma(2\alpha m+1)}\\&=E_{2\alpha,1}\left(-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\end{aligned}\tag{33}$$

and

$$\begin{aligned}u_1(t)&=t-at^{1+\alpha}E_{(\alpha,2\alpha),2+\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)-\frac{k\pi^2n^2}{L^2}t^{1+2\alpha}E_{(\alpha,2\alpha),2+2\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\\&=t-\frac{k\pi^2n^2}{L^2}t^{1+2\alpha}E_{2\alpha,2+2\alpha}\left(-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\\&=t-\frac{k\pi^2n^2}{L^2}t^{1+2\alpha}\sum_{m=0}^{\infty}\frac{(-\frac{k\pi^2n^2}{L^2}t^{2\alpha})^m}{\Gamma(2\alpha(m+1)+2)}\\&=t\sum_{m=0}^{\infty}\frac{(-\frac{k\pi^2n^2}{L^2}t^{2\alpha})^m}{\Gamma(2\alpha m+2)}\\&=tE_{2\alpha,2}\left(-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right).\end{aligned}\tag{34}$$

Substituting (33) and (34) into (32), we obtain

$$u(x,t)=\sum_{n=1}^{\infty}\Bigg[\int_0^t \tau^{2\alpha-1}E_{2\alpha,2\alpha}\left(-\frac{k\pi^2n^2}{L^2}\tau^{2\alpha}\right)\tilde{f}_n(t-\tau)\,d\tau$$
$$+B_n(0)E_{2\alpha,1}\left(-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)+tB_n'(0)E_{2\alpha,2}\left(-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\Bigg]\sin\frac{n\pi x}{L}$$
$$+\mu_1(t)+\frac{(\mu_2(t)-\mu_1(t)x)}{L}, \tag{35}$$

where $B_n(0)$, $B_n'(0)$ are given by (27), $\tilde{f}_n(t)$ is given by (23).

If we let $\psi(x)=0$, $\mu_1(t)=\mu_2(t)=0$, $L=\pi$, $f(x,t)=q(t)$ in (35) (to compare with the results in [23]), and note $1=\sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n\pi}\sin nx$, $B_n'(0)=0$, then we have

$$u(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}E_{2\alpha,1}\left(-kn^2t^{2\alpha}\right)\sin nx\int_0^{\pi}\phi(x)\sin nx\,dx$$
$$+\sum_{n=1}^{\infty}\sin nx\cdot\frac{2[1-(-1)^n]}{n\pi}\int_0^{t}\tau^{2\alpha-1}E_{2\alpha,2\alpha}\left(-kn^2t^{2\alpha}\right)q(t-\tau)\,d\tau. \tag{36}$$

This result is in accord with the result obtained in [23].

We next look at the case of zero forcing, i.e. $f(x,t)=0$ in (36). The corresponding solution is

$$u(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}E_{2\alpha,1}\left(-kn^2t^{2\alpha}\right)\sin nx\int_0^{\pi}\phi(x)\sin nx\,dx. \tag{37}$$

The result agrees with that discussed by Agrawal [24].

In particular, let $a=0$, $\alpha=1$, $\mu_1(t)=\mu_2(t)=0$ in (32), then

$$\begin{aligned}
u_0(t)&=1-\frac{k\pi^2n^2}{L^2}t^{2\alpha}E_{(\alpha,2\alpha),1+2\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\\
&=1-\frac{k\pi^2n^2}{L^2}t^2E_{2,3}\left(-\frac{k\pi^2n^2}{L^2}t^2\right)\\
&=1-\frac{k\pi^2n^2}{L^2}t^2\sum_{m=0}^{\infty}\frac{(-\frac{k\pi^2n^2}{L^2}t^2)^m}{\Gamma(2m+3)}\\
&=1-\frac{k\pi^2n^2}{L^2}t^2\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{\sqrt{k}\pi nt}{L})^{2m}}{(2m+2)!}\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{\sqrt{k}\pi nt}{L})^{2m}}{(2m)!}\\
&=\cos\frac{\sqrt{k}\pi nt}{L}
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
u_1(t)&=t-at^{1+\alpha}E_{(\alpha,2\alpha),2+\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)-\frac{k\pi^2n^2}{L^2}t^{1+2\alpha}E_{(\alpha,2\alpha),2+2\alpha}\left(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\right)\\
&=t-\frac{k\pi^2n^2}{L^2}t^3E_{2,4}\left(-\frac{k\pi^2n^2}{L^2}t^2\right)\\
&=t+t\cdot\frac{-k\pi^2n^2t^2}{L^2}\sum_{m=0}^{\infty}\frac{(-\frac{k\pi^2n^2t^2}{L^2})^m}{(2m+3)!}\\
&=\frac{L}{\sqrt{k}n\pi}\sin\frac{\sqrt{k}n\pi t}{L}.
\end{aligned} \tag{39}$$

Substituting (38) and (39) into (32), we obtain

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^{L}\phi(x)\sin\frac{n\pi x}{L}\,dx\cdot\cos\frac{n\pi}{L}\sqrt{k}t\right.$$

$$+\frac{L}{\sqrt{k}\pi n}\cdot\frac{2}{L}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx\cdot\sin\frac{n\pi}{L}\sqrt{k}t\Bigg)\cdot\sin\frac{n\pi}{L}x$$

$$+\sum_{n=1}^{\infty}\frac{L}{\sqrt{k}\pi n}\cdot\sin\frac{n\pi}{L}x\int_0^t\left(\sin\frac{\sqrt{k}\pi n}{L}(t-\tau)\cdot\frac{2}{L}\int_0^L f(\xi,\tau)\sin\frac{n\pi\xi}{L}\,d\xi\right)d\tau. \tag{40}$$

This result is indeed the solution of the integer order wave equation [25].

4. Nonhomogeneous time-fractional telegraph equation with Neumann boundary condition

In this section, we determine solution of the following time-fractional telegraph equation with Neumann boundary condition using the method of separating variables

$$\begin{cases} D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t)=k\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), & 0<x<L,\ t>0,\\ u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x), & 0\leqslant x\leqslant L,\\ u_x(0,t)=\mu_1(t), & t\geqslant 0,\\ u_x(L,t)=\mu_2(t), & t\geqslant 0, \end{cases} \tag{41}$$

where $f(x,t),\phi(x),\psi(x),\mu_1(t),\mu_2(t)$ are as defined in Section 3.

In a similar manner, we transform the nonhomogeneous boundary condition into a homogeneous boundary condition. Let

$$u(x,t)=W_2(x,t)+V_2(x,t),$$

where $V_2(x,t)=\frac{[\mu_2(t)-\mu_1(t)]x^2}{2L}+\mu_1(t)x$ fulfills the boundary condition

$$\begin{aligned} &V_x(0,t)=\mu_1(t),\\ &V_x(L,t)=\mu_2(t), \end{aligned} \tag{42}$$

and the function $W_2(x,t)$ satisfies the homogeneous boundary-value problem

$$\begin{cases} D_t^{2\alpha}W_2(x,t)+aD_t^{\alpha}W_2(x,t)=k\frac{\partial^2 W_2(x,t)}{\partial x^2}+\tilde f(x,t), & 0\leqslant x\leqslant L,\ t\geqslant 0,\\ W_2(x,0)=\phi_2(x), & 0\leqslant x\leqslant L,\\ \frac{\partial W_2(x,0)}{\partial t}=\psi_2(x), & 0\leqslant x\leqslant L,\\ \frac{\partial W_2(0,t)}{\partial x}=\frac{\partial W_2(L,t)}{\partial x}=0, & t\geqslant 0, \end{cases} \tag{43}$$

where

$$\begin{aligned} \tilde f(x,t)&=-D_t^{2\alpha}V_2(x,t)-aD_t^{\alpha}V_2(x,t)+\frac{k[\mu_2(t)-\mu_1(t)]}{L}+f(x,t),\\ \phi_2(x)&=\phi(x)-\frac{[\mu_2(0)-\mu_1(0)]x^2}{2L}-\mu_1(0)x,\\ \psi_2(x)&=\psi(x)-\frac{[\mu_2'(0)-\mu_1'(0)]x^2}{2L}-\mu_1'(0)x. \end{aligned} \tag{44}$$

We now solve the corresponding homogeneous equation (41) (supposing that $\tilde f(x,t)=0$). We assume that the solution of the homogeneous equation takes the form

$$W_2(x,t)=X(x)T(t).$$

Substituting this representation into (41) (letting $\tilde f(x,t)=0$), we get the following ODE with boundary values:

$$\begin{cases} X''(x)+\frac{\lambda}{k}X(x)=0,\\ X'(0)=X'(L)=0. \end{cases} \tag{45}$$

A straightforward calculation shows that the eigenvalues of the Sturm–Liouville problem (45) are

$$\lambda_n = \frac{k\pi^2 n^2}{L^2}, \quad n = 1, 2, \ldots,$$

and corresponding eigenfunctions

$$X_n(x) = \cos\frac{n\pi}{L}x, \quad n = 1, 2, \ldots.$$

Hence, the solution of the nonhomogeneous equation (43) has the form

$$W_2(x,t) = \sum_{n=1}^{\infty} B_n(t)\cos\frac{n\pi}{L}x. \tag{46}$$

We assume that the series is convergent. In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ in a Fourier series by the eigenfunction $\{\cos\frac{n\pi}{L}x\}$,

$$\tilde{f}(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n(t)\cos\frac{n\pi}{L}x, \tag{47}$$

where the Fourier coefficients are

$$\tilde{f}_n(t) = \frac{2}{L}\int_0^L \tilde{f}(x,t)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots. \tag{48}$$

Inserting (46) and (47) into (43), we obtain the following fractional differential equation:

$$D_t^{2\alpha} B_n(t) + aD_t^{\alpha} B_n(t) + \frac{k\pi^2 n^2}{L^2} B_n(t) = \tilde{f}_n(t). \tag{49}$$

Since $W_2(x,t)$ satisfies the initial conditions in (43)

$$\begin{cases} \sum_{n=1}^{\infty} B_n(0)\cos\frac{n\pi x}{L} = \phi_2(x), & 0 < x < L, \\ \sum_{n=1}^{\infty} B_n'(0)\cos\frac{n\pi x}{L} = \psi_2(x), & 0 < x < L, \end{cases} \tag{50}$$

we obtain

$$\begin{cases} B_n(0) = \frac{2}{L}\int_0^L \phi_2(x)\cos\frac{n\pi x}{L}, & n = 1, 2, \ldots, \\ B_n'(0) = \frac{2}{L}\int_0^L \psi_2(x)\cos\frac{n\pi x}{L}, & n = 1, 2, \ldots. \end{cases} \tag{51}$$

According to Theorem 2.4, the solution of the fractional differential equation (49) with initial value (51) is

$$B_n(t) = \int_0^t \tau^{2\alpha-1} E_{(\alpha,2\alpha),2\alpha}\left(-a\tau^{\alpha}, -\frac{k\pi^2 n^2}{L^2}\tau^{2\alpha}\right)\tilde{f}_n(t-\tau)\,d\tau + B_n(0)u_0(t) + B_n'(0)u_1(t), \tag{52}$$

where the functions $u_0(t)$, $u_1(t)$ are given in (29) and (30). Thus, we obtain the solution to the problem (43) as

$$\begin{aligned} W_2(x,t) &= \sum_{n=1}^{\infty} B_n(t)\cos\frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty}\left[\int_0^t \tau^{2\alpha-1} E_{(\alpha,2\alpha),2\alpha}\left(-a\tau^{\alpha}, -\frac{k\pi^2 n^2}{L^2}\tau^{2\alpha}\right)\tilde{f}_n(t-\tau)\,d\tau \right. \\ &\quad \left. + B_n(0)u_0(t) + B_n'(0)u_1(t)\right]\cos\frac{n\pi x}{L}, \end{aligned} \tag{53}$$

where $B_n(0)$, $B_n'(0)$ are given in (51), $\tilde{f}_n(t)$ is given in (48), $u_0(t)$, $u_1(t)$ are given in (29), (30), respectively. Then the solution of (41) is obtained in terms of the multivariate Mittag-Leffler type function $E_{(.),\beta}(x)$ as

$$u(x,t)=\sum_{n=1}^{\infty}\Bigg[\int_0^t \tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\bigg(-a\tau^{\alpha},-\frac{k\pi^2n^2}{L^2}\tau^{2\alpha}\bigg)\tilde{f}_n(t-\tau)\,d\tau \\ +B_n(0)u_0(t)+B_n'(0)u_1(t)\Bigg]\cos\frac{n\pi x}{L}+\frac{[\mu_2(t)-\mu_1(t)]x^2}{2L}+\mu_1(t)x. \tag{54}$$

In particular, let $a=0$, $k=1$, $\alpha=1$ in (41); then the fractional order equation is reduced to an integer order wave equation with Neumann boundary.

Now, applying (54), we get

$$\begin{aligned}\int_0^t x^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\bigg(-ax^{\alpha},-\frac{kn^2\pi^2}{L^2}x^{2\alpha}\bigg)\tilde{f}_n(t-x)\,dx&=\int_0^t xE_{2,2}\bigg(-\frac{n^2\pi^2}{L^2}x^2\bigg)\tilde{f}_n(t-x)\,dx\\&=\int_0^t x\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{n\pi}{L}x)^{2m}}{(2m+1)!}\tilde{f}_n(t-x)\,dx\\&=\frac{L}{n\pi}\int_0^t\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{n\pi}{L}x)^{2m+1}}{(2m+1)!}\tilde{f}_n(t-x)\,dx\\&=\frac{L}{n\pi}\int_0^t\sin\frac{n\pi(t-\tau)}{L}\tilde{f}_n(t-\tau)\,d\tau.\end{aligned}\tag{55}$$

Furthermore

$$\begin{aligned}u_0(t)&=1-\frac{kn^2\pi^2}{L^2}t^{2\alpha}E_{(\alpha,2\alpha),1+2\alpha}\bigg(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\bigg)\\&=1-\frac{n^2\pi^2}{L^2}t^2E_{2,3}\bigg(-\frac{n^2\pi^2}{L^2}t^2\bigg)\\&=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{n\pi t}{L})^{2m}}{(2m)!}\\&=\cos\frac{n\pi t}{L}\end{aligned}\tag{56}$$

and

$$\begin{aligned}u_1(t)&=t-at^{1+\alpha}E_{(\alpha,2\alpha),2+\alpha}\bigg(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\bigg)\\&\quad-\frac{kn^2\pi^2}{L^2}t^{1+2\alpha}E_{(\alpha,2\alpha),2+2\alpha}\bigg(-at^{\alpha},-\frac{k\pi^2n^2}{L^2}t^{2\alpha}\bigg)\\&=t-\frac{n^2\pi^2}{L^2}t^3E_{2,4}\bigg(-\frac{n^2\pi^2}{L^2}t^2\bigg)\\&=t+t\cdot\frac{-n^2\pi^2t^2}{L^2}\sum_{m=0}^{\infty}\frac{(-\frac{n^2\pi^2t^2}{L^2})^m}{(2m+3)!}\\&=\frac{L}{n\pi}\sin\frac{n\pi t}{L}.\end{aligned}\tag{57}$$

Substituting (55)–(57) into (54), we obtain

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{L}{n\pi}\int_0^t \sin\frac{n\pi(t-\tau)}{L}\tilde{f}_n(\tau)\,d\tau+B_n(0)\cos\frac{n\pi t}{L}\right.$$
$$\left.+\frac{L}{n\pi}B_n'(0)\sin\frac{n\pi(t)}{L}\right]\cos\frac{n\pi x}{L}+\frac{[\mu_2(t)-\mu_1(t)]x^2}{2L}+\mu_1(t)x, \tag{58}$$

where $\tilde{f}_n(t)$ is given by (48), and $B_n(0)$, $B_n'(0)$ are given as (51).

5. Nonhomogeneous time-fractional telegraph equation with Robin boundary condition

Let us now consider the time-fractional telegraph equation with Robin boundary condition

$$\begin{cases} D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t)=k\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), & 0<x<L,\ t>0,\\ u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x), & 0\leqslant x\leqslant L,\\ u(0,t)+\alpha_1 u_x(0,t)=\mu_1(t), & t\geqslant 0,\\ u(L,t)+\beta_1 u_x(L,t)=\mu_2(t), & t\geqslant 0, \end{cases} \tag{59}$$

where α_1, β_1 are nonzero constants. We assume

$$u(x,t)=W_3(x,t)+V_3(x,t),$$

where

$$V_3(x,t)=\frac{\mu_1(t)-\mu_2(t)}{\alpha_1-\beta_1-L}x-\frac{(L+\beta_1)\mu_1(t)-\alpha_1\mu_2(t)}{\alpha_1-\beta_1-L}, \tag{60}$$

fulfills the boundary condition

$$\begin{cases} V_3(0,t)+\alpha_1\dfrac{\partial V_3(0,t)}{\partial x}=\mu_1(t),\\ V_3(L,t)+\beta_1\dfrac{\partial V_3(L,t)}{\partial x}=\mu_2(t), \end{cases} \tag{61}$$

and the function $W_3(x,t)$ is the solution of the following problem:

$$\begin{cases} D_t^{\alpha}W_3(x,t)+aD_t^{\alpha}W_3(x,t)=k\frac{\partial^2 W_3(x,t)}{\partial x^2}+\tilde{f}(x,t), & 0<x<L,\ t>0,\\ W_3(x,0)=\phi_3(x),\quad \frac{\partial W_3(x,0)}{\partial t}=\psi_3(x), & 0\leqslant x\leqslant L,\\ W_3(0,t)+\alpha_1\frac{\partial W_3(0,t)}{\partial x}=0, & t\geqslant 0,\\ W_3(L,t)+\beta_1\frac{\partial W_3(L,t)}{\partial x}=0, & t\geqslant 0, \end{cases} \tag{62}$$

with

$$\tilde{f}(x,t)=-D_t^{\alpha}V_3(x,t)-aD_t^{2\alpha}V_3(x,t)+f(x,t),$$
$$\phi_3(x)=\phi(x)-\frac{\mu_1(0)-\mu_2(0)}{\alpha_1-\beta_1-L}x+\frac{(L+\beta_1)\mu_1(0)-\alpha_1\mu_2(0)}{\alpha_1-\beta_1-L},$$
$$\psi_3(x)=\psi(x)-\frac{\mu_1'(0)-\mu_2'(0)}{\alpha_1-\beta_1-L}x+\frac{(L+\beta_1)\mu_1'(0)-\alpha_1\mu_2'(0)}{\alpha_1-\beta_1-L}. \tag{63}$$

Similarly to the analysis in Section 4, we firstly assume that the solution of the homogeneous equation in (62) (putting $\tilde{f}(x,t)=0$) has the form

$$W_3(x,t)=X(x)T(t).$$

Inserting this expression into (62) (putting $\tilde{f}(x,t)=0$), we get the Sturm–Liouville problem

$$\begin{cases} X''(x) + \dfrac{\lambda}{k} X(x) = 0, \\ X(0) + \alpha_1 X'(0) = 0, \quad X(L) + \beta_1 X'(L) = 0. \end{cases} \tag{64}$$

Through a simple calculus, we know the eigenvalues satisfy the equation

$$\tan \mu L = \frac{(\alpha_1 - \beta_1)\mu}{1 + \alpha_1 \beta_1 \mu^2}, \quad \mu = \sqrt{\frac{\lambda}{k}} > 0,\ \lambda = k\mu^2. \tag{65}$$

We can obtain the solution of this equation by numerical method. For convenience we denote the solution in (65) as μ_n, $n = 1, 2, \ldots$. Then the eigenvalues are $\lambda_n = k\mu_n^2$, $n = 1, 2, \ldots$. Note that the eigenvalues are countable and can be listed in a sequence

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

with

$$\lim_{n \to \infty} \lambda_n = \infty.$$

So the corresponding eigenfunctions are

$$X_n(x) = -\alpha_1 \mu_n \cos \mu_n x + \sin \mu_n x, \quad n = 1, 2, \ldots.$$

A straightforward calculation shows that

$$\begin{aligned} \int_0^L X_n^2(x)\,dx &= \int_0^L (-\alpha_1 \mu_n \cos \mu_n x + \sin \mu_n x)^2\,dx \\ &= \frac{\alpha_1^2 \mu_n - 1}{2} \cdot \frac{\tan \mu_n L}{1 + \tan^2 \mu_n L} - \frac{\alpha_1 \tan^2 \mu_n L}{1 + \tan^2 \mu_n L} + \frac{(\alpha_1^2 \mu_n^2 + 1)L}{2} \\ &=: b_n, \end{aligned} \tag{66}$$

where $\tan \mu_n L = \frac{(\alpha_1 - \beta_1)\mu_n}{1 + \alpha_1 \beta_1 \mu_n^2}$. Then the formal solution of the boundary value problem (62) is

$$W_3(x,t) = \sum_{n=1}^{\infty} B_n(t) X_n(x). \tag{67}$$

We can determine $B_n(0),\ B_n'(0)$ by the following expressions:

$$\begin{cases} W_3(x,0) = \displaystyle\sum_{n=1}^{\infty} B_n(0)(-\alpha_1 \mu_n \cos \mu_n x + \sin \mu_n x) = \phi_3(x), \\ \dfrac{\partial W_3(x,0)}{\partial t} = \displaystyle\sum_{n=1}^{\infty} B_n'(0)(-\alpha_1 \mu_n \cos \mu_n x + \sin \mu_n x) = \psi_3(x). \end{cases} \tag{68}$$

We assume that $\phi_3(x), \psi_3(x)$ are continuous functions with order-one derivative. Multiplying (68) by $(-\alpha_1 \mu_n \cos \mu_n x + \sin \mu_n L)$ and integrating from 0 to L with respect to x, we get

$$\begin{cases} B_n(0) = b_n^{-1} \displaystyle\int_0^L \phi_3(\xi)(-\alpha_1 \mu_n \cos \mu_n \xi + \sin \mu_n \xi)\,d\xi, \\ B_n'(0) = b_n^{-1} \displaystyle\int_0^L \psi_3(\xi)(-\alpha_1 \mu_n \cos \mu_n \xi + \sin \mu_n \xi)\,d\xi. \end{cases} \tag{69}$$

Here b_n is given in (66).

As in Section 4 we also expand $\tilde{f}(x,t)$ in a Fourier series in the interval $[0, L]$ by the eigenfunctions $\{-\alpha_1 \mu_n \cos \mu_n x + \sin \mu_n x\}$,

$$\tilde{f}(x,t)=\sum_{n=1}^{\infty}\tilde{f}_n(t)(-\alpha_1\mu_n\cos\mu_n x+\sin\mu_n x), \tag{70}$$

where

$$\tilde{f}_n(t)=b_n^{-1}\int_0^L \tilde{f}(\xi,t)(-\alpha_1\mu_n\cos\mu_n\xi+\sin\mu_n\xi)\,d\xi. \tag{71}$$

Substituting (67), (70) and (71) into Eq. (62) we obtain the fractional linear differential equation fulfilling conditions (69)

$$D_t^{2\alpha}B_n(t)+aD_t^{2\alpha}B_n(t)+k\mu_n^2B_n(t)=\tilde{f}_n(t),\quad n=1,2,\ldots. \tag{72}$$

According to Theorem 2.4,

$$B_n(t)=\int_0^t x^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\big(-ax^{\alpha},-k\mu_n^2x^{2\alpha}\big)\tilde{f}_n(t-x)\,dx+B_n(0)u_0(t)+B_n'(0)u_1(t) \tag{73}$$

in which

$$\begin{aligned}
&u_0(t)=1-k\mu_n^2t^{2\alpha}E_{(\alpha,2\alpha),1+2\alpha}\big(-at^{\alpha},-k\mu_n^2t^{2\alpha}\big),\\
&u_1(t)=t-at^{1+\alpha}E_{(\alpha,2\alpha),2+\alpha}\big(-at^{\alpha},-k\mu_n^2t^{2\alpha}\big)-k\mu_n^2t^{1+2\alpha}E_{(\alpha,2\alpha),2+2\alpha}\big(-at^{\alpha},-k\mu_n^2t^{2\alpha}\big).
\end{aligned} \tag{74}$$

Consequently the solution of the nonhomogeneous time-fractional telegraph equation with Robin boundary condition is

$$\begin{aligned}
u(x,t)&=W_3(x,t)+V_3(x,t)\\
&=\sum_{n=1}^{\infty}B_n(t)(-\alpha_1\mu_n\cos\mu_n x+\sin\mu_n x)+\frac{\mu_1(t)-\mu_2(t)}{\alpha_1-\beta_1-L}x-\frac{(L+\beta_1)\mu_1(t)-\alpha_1\mu_2(t)}{\alpha_1-\beta_1-L}.
\end{aligned} \tag{75}$$

In particular, letting $a=0$, $k=1$, $\alpha=1$, $\alpha_1=\beta_1=1$ in (59), the fractional order equation is reduced to an integer order wave equation with Robin boundary.

Now, noting that $\tan\mu L=\frac{(\alpha_1-\beta_1)\mu}{1+\alpha_1\beta_1\mu^2}=0$, $\mu=\sqrt{\lambda}$, then $\mu_n=\frac{n\pi}{L}$, $n=1,2,\ldots$. From (56) and (57), we then get

$$\begin{aligned}
u_0(t)&=1-k\mu_n^2t^{2\alpha}E_{(\alpha,2\alpha),1+2\alpha}\big(-at^{\alpha},-k\mu_n^2t^{2\alpha}\big)\\
&=1-\frac{n^2\pi^2}{L^2}t^2E_{2,3}\left(-\frac{n^2\pi^2}{L^2}t^2\right)\\
&=\cos\frac{n\pi t}{L},
\end{aligned} \tag{76}$$

and

$$\begin{aligned}
u_1(t)&=t-at^{1+\alpha}E_{(\alpha,2\alpha),2+\alpha}\big(-at^{\alpha},-k\mu_n^2t^{2\alpha}\big)\\
&\qquad-k\mu_n^2t^{1+2\alpha}E_{(\alpha,2\alpha),2+2\alpha}\big(-at^{\alpha},-k\mu_n^2t^{2\alpha}\big)\\
&=\frac{L}{n\pi}\sin\frac{n\pi t}{L}.
\end{aligned} \tag{77}$$

Inserting (76) and (77) into (75) we obtain

$$\begin{aligned}
u(x,t)=&\sum_{n=1}^{\infty}\left[\frac{L}{n\pi}\int_0^t\sin\frac{n\pi(t-\tau)}{L}\tilde{f}_n(\tau)\,d\tau+B_n(0)\cos\frac{n\pi t}{L}+\frac{L}{n\pi}B_n'(0)\sin\frac{n\pi t}{L}\right]\\
&\times\left(-\frac{n\pi}{L}\cos\frac{n\pi t}{L}+\sin\frac{n\pi}{L}x\right)+\frac{[\mu_2(t)-\mu_1(t)]x}{L}+\frac{(L+1)\mu_1(t)-\mu_2(t)}{L},
\end{aligned} \tag{78}$$

where $\tilde{f}_n(t)$ is given by (71), and $B_n(0)$, $B_n'(0)$ are given by (69).

6. Conclusions

We have derived the analytical solutions of the nonhomogeneous time-fractional telegraph equation under three kinds of boundary conditions using the separation-of-variables method. The time fractional derivative is considered in the Caputo sense. The solutions, which are given in the form of the multivariate Mittag-Leffler function, reduce to those of the integer order telegraph equation and corresponding wave and diffusion equations.

Acknowledgments

The work of this paper was partially funded by the Australian Research Council grant LP0348653 and the National Natural Science Foundation of China grant 10271098. The authors wish to thank the referee for his constructive comments and suggestions.

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