Chebyshev Subspaces in the Space of Bounded Linear Operators from $c_0$ to $c_0$

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1. INTRODUCTION

The space $c_0$ is the linear space of all sequences $x = \{x_j\}$ converging to zero, with the norm of $x$ given by $\|x\| = \sup |x_j|$, the supremum taken over all $j$. The symbol $[c_0, c_0]$ will denote the linear space of all bounded linear operators from $c_0$ to $c_0$. If $T \in [c_0, c_0]$, then the norm of $T$ is the standard operator norm given by

$$\|T\| = \sup \{\|T(x)\|: x \in c_0, \|x\| \leq 1\}.$$ 

If $M$ is a subset of the normed linear space $X$ and $x \in X$, then a point $x_0$ in $M$ is said to be a best approximation to $x$ from $M$ if $\|x - x_0\| = \inf \{\|x - y\|: y \in M\}$. If each $x$ in $X$ has a unique best approximation in $M$, then $M$ is called a Chebychev subset of $X$.

In this paper we are concerned with the characterization of best approximations in a finite dimensional subspace $M$ of $[c_0, c_0]$, and the determination of conditions under which $M$ is Chebychev. An element $x$ in $X$ has $x_0$ as a best approximation in a subspace $M$ if and only if $x - x_0$ has 0 as a best approximation in $M$. Therefore, to characterize best approximations in $M$, it suffices to provide conditions under which an element has 0 as a best approximation in $M$. It is known (see, e.g., [2, p. 20]) that if $M$ is a finite dimensional subspace of $X$, then each $x \in X$ has a best approximation in $M$. Thus, if $M$ is non-Chebychev, there exists some element $x \in X$ with two best approximations in $M$.

By a result in [5], each bounded linear operator in $[c_0, c_0]$ can be represented by an infinite matrix of scalars. We use this fact in Section 2 to characterize best approximations in a finite dimensional subspace of $[c_0, c_0]$. In

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Section 3 we will characterize one-dimensional Chebychev subspaces in $[c_0, c_0]$, and in Section 4 present a necessary condition and a sufficient condition for a finite dimensional subspace of $[c_0, c_0]$ to be non-Chebychev. Finally, we will show that if a bounded linear operator $T$ in $[c_0, c_0]$ is represented by an infinite matrix, then the second adjoint $T^{**}$ in $[l_\infty, l_\infty]$ may also be represented by that matrix. This permits the reformulation of previous results in terms independent of the operator's matrix representation.

Unless otherwise stated, all notation will correspond to that of [3]. All scalars will be assumed to be real. Let $X$ be a linear space with norm $\| \cdot \|$. The conjugate space $X^*$ will be assumed to have the usual operator norm. For each $x$ in $X$, $\check{x}$ will denote that functional in $X^{**}$ defined by $\check{x}(x^*) = x^*(x)$ for all $x^*$ in $X^*$, and $\hat{X} = \{ \check{x} : x \in X \}$. The norm closed unit sphere of $X$ will be denoted by $S(X)$. For any set $A$, $cl(A)$ will mean the norm closure of $A$.

The annihilator $M^\perp$ of a subspace $M$ of $X$ is defined by

$$M^\perp = \{ x^* \in X^* : x^*(y) = 0 \text{ for all } y \in M \}.$$

If $x_1, \ldots, x_n$ are vectors in the linear space $X$, then $[x_1, \ldots, x_n]$ will denote the linear subspace of $X$ spanned by these vectors. We will assume, unless otherwise stated, that $[x_1, \ldots, x_n]$ has dimension $n$. If $Y$ is a normed linear space, then by $(Y \times \cdots \times Y)_n$ ($n$ summands), we will mean the linear space of all ordered $n$-tuples of the form $y = (y_1, \ldots, y_n)$ for $y_i \in Y$, $i = 1, \ldots, n$ with norm defined by $\| y \| = \max(\| y_i \| : 1 \leq i \leq n)$. The symbol $(Y \times \cdots \times Y)_1$ ($n$ summands) is defined similarly, with the norm in this case defined by $\| y \| = \sum_{i=1}^n \| y_i \|$. The following lemma is then easily seen.

**Lemma 1.1.** Let $Y$ be a normed linear space. If for $f = (f_1, \ldots, f_n) \in (Y^* \times \cdots \times Y^*)_1$ ($n$ summands), we write $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$, for all $(x_1, \ldots, x_n) \in E = (Y \times \cdots \times Y)_n$ ($n$ summands), then $E^*$ can be identified with $(Y^* \times \cdots \times Y^*)_1$ ($n$ summands).

We state here for convenience a known result which may be found in [4].

**Theorem 1.2.** Let $M$ be a subspace of the normed linear space $X$, and let $x \in X - cl(M)$. Then $x$ has 0 as a best approximation in $M$ if and only if there exists $f$ in $M^\perp$, $\| f \| = 1$, such that $f(x) = \| x \|$.

### 2. Characterization of Best Approximations

Each bounded linear operator in $[c_0, c_0]$ may be represented by an infinite matrix of scalars according to the following theorem found in [5, p. 217].
THEOREM 2.1. If $A \in [c_0, c_0]$, there exists a unique infinite matrix of scalars $(a_{ij})$ $(i, j = 1, 2, \ldots)$ such that

(i) $\|A\| = \sup_i \sum_{j=1}^{\infty} |a_{ij}|$,

(ii) $\lim_{i \to \infty} a_{ij} = 0$, $j = 1, 2, \ldots$,

(iii) if $x = \{x_i\}$, $y = \{y_i\} \in c_0$ with $y = Ax$, then $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$, $i = 1, 2, \ldots$.

Conversely, if $(a_{ij})$ $(i, j = 1, 2, \ldots)$ is an infinite matrix of scalars such that $\sup \sum_{j=1}^{\infty} |a_{ij}|$ (supremum taken over $i$) is finite and such that (ii) holds, then the equation in (iii) defines a member $A$ of $[c_0, c_0]$ whose norm is given by (i).

A problem in considering the space $[c_0, c_0]$ is that there is no convenient way to represent the bounded linear functionals on the space. However, if we restrict the matrices in $[c_0, c_0]$ to a fixed finite number of rows, we obtain the restricted space $E = (l_1 \times \cdots \times l_1)$. It is known (see [3]) that $l_1^*$ can be identified with $l_1$. Then since $E^*$ can be identified with $(l_1 \times \cdots \times l_1)_1$ by Lemma 1.1, we know what form the bounded linear functionals on the restricted space take. Hence, to obtain some of the following results, we consider a selected finite number of rows in the matrices.

In order to characterize best approximations in a finite dimensional subspace $M$ of $[c_0, c_0]$, we will need the next lemma. If $K$ is a set of positive integers and $A = (a_{ij})$ is an infinite matrix, then $A | K$ denotes the matrix whose rows are precisely the rows $(a_{ij})$ $(j = 1, 2, \ldots)$ of $A$ for which $i \in K$.

LEMMA 2.2. Let $A_1, \ldots, A_n$ be linearly independent operators in $[c_0, c_0]$ with $M = [A_1, \ldots, A_n]$. Then

(a) there exists a positive integer $p$ such that if $K_p = \{i: 1 \leq i \leq p\}$ and $A_i = A_i | K_p \in (l_1 \times \cdots \times l_1)_\infty$ $(p$ summands), $i = 1, \ldots, n$, then $A_1, \ldots, A_n$ are linearly independent.

(b) given $B \in [c_0, c_0]$, there exists a nonnegative constant $Q$ such that for any positive integer $s \geq p$, if $i_1 < \cdots < i_{s-p}$ denote any fixed positive integers with $i_1 > p$, $K_s = \{i: 1 \leq i \leq p$ or $i = i_1, \ldots, i_{s-p}\}$, $A_i = A_i | K_s$, $B^s = B | K_s$, and $A^s = \sum_{i=1}^{n} \lambda_i A_i^s$ is a best approximation to $B^s$ in $[A_1^s, \ldots, A_n^s]$, then we have $|\lambda_i| \leq Q$, $i = 1, \ldots, n$.

Proof. For each positive integer $n$, let $K_n = \{i: 1 \leq i \leq n\}$ and define the mapping $\varphi_n$ on $M$ by

$\varphi_n(A) = A | K_n \quad \text{for} \quad A \in M$.

We have $\overline{A^n} \in (l_1 \times \cdots \times l_1)_\infty$ $(n$ summands). For any $n$, $\varphi_n$ is a bounded linear transformation and is, thus, continuous. Let $A = (a_{ij}) \in M$ with
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\[ || A || = 1. \]

Then there exists \( i = n_A \) such that \( \sum_{j=1}^{\infty} | a_{n_A j} | > (1/2). \) Therefore, \( \| \varphi_A (A) \| > (1/2). \)

Let \( \mathcal{C} = \{ \varphi_A (C) : C \in M \text{ with } \| \varphi_A (C) \| > (1/2) \}. \) Let \( U_A = \varphi_A ^{-1} (\mathcal{C}) \), so \( U_A \) is open in \( M \). Let \( M' = \{ A \in M : \| A \| = 1 \} \). Then \( M' \) is a closed subset of \( S(M) \), which is compact since \( M \) is finite dimensional. Therefore, \( M' \) is compact. Since \( \{ U_A : A \in M' \} \) is an open covering of \( M' \), there exists a finite subcover \( \{ U_{B_1}, \ldots, U_{B_e} \} \) of \( M' \) for \( B_1, \ldots, B_e \in M' \). Let \( p = \max \{ n_{B_1}, \ldots, n_{B_e} \} \). If \( A \in M' \), then \( A \in U_{B_i} \) for some \( i = 1, \ldots, e \). Therefore, \( \| \varphi_A (A) \| > (1/2). \) Hence, for all \( A \in M' \), we have \( \| \varphi_A (A) \| > (1/2). \)

Now \( \varphi_A (A_i) = \bar{A}_i \), so \( \bar{A}_1, \ldots, \bar{A}_n \) are linearly dependent. Thus there exists \( A \in M \), \( A \neq 0 \), such that \( \varphi_A (A) = 0. \) However, \( A / \| A \| \in M' \), so \( \| \varphi_A (A/ \| A \| ) \| > (1/2) \), a contradiction. Therefore, \( \bar{A}_1, \ldots, \bar{A}_n \) are linearly independent, and (a) is proved.

To prove (b), let \( \varphi_p \) be defined on \( M \) by \( \varphi_p (A) = \bar{A} = A / K_p \) for \( A \in M \). Then \( \varphi_p \) is a continuous linear transformation from \( M \) onto \( [A_1, \ldots, A_n] \).

Also \( \varphi_p \) is one-to-one since \( \bar{A}_1, \ldots, \bar{A}_n \) are linearly independent by (a). Thus, \( \varphi_p \) has an inverse \( \varphi_p ^{-1} \) which is clearly a linear transformation. By the open mapping theorem, \( \varphi_p ^{-1} \) is continuous and, hence, bounded.

Now define a new norm \( \| \cdot \|_1 \) on \( [A_1, \ldots, A_n] \) by

\[
\max_{1 \leq i \leq n} \| \delta_i A_i \|_1 = \max_{1 \leq i \leq n} | \delta_i |,$

where the maximum is taken over \( 1 \leq i \leq n. \) Since in a finite dimensional space all norms are equivalent, there exists a positive constant \( c \) such that \( \| A \|_1 \leq c \| A \| \) for all \( A \in M. \) Let \( B \in [c_0, c_0] \). Then let \( Q = 2c \| \varphi_p ^{-1} \| \| B \|. \)

Let \( s \) be a positive integer such that \( s \geq p, \) and let \( \bar{A}^s, B^s, \) and \( A^s \) be as given in (b). Define \( \varphi_s \) on \( M \) by \( \varphi_s (C) = \bar{C}^s = C / K_s \) for any \( C \in M. \) We can easily see that \( \| \varphi_s ^{-1} \| \) exists and \( \| \varphi_s ^{-1} \| \leq \| \varphi_p ^{-1} \|. \) Since \( \bar{A}^s \) is a best approximation to \( B^s \) in \( [A_1^s, \ldots, A_n^s] \), we have \( \| \bar{A}^s \| \leq \| \bar{B}^s - \bar{A}^s \| + \| \bar{B}^s \| \leq 2 \| B \|. \) Thus,

\[
\max | \lambda_i | = \max_{1 \leq i \leq n} \| \sum_{i=1}^{n} \lambda_i A_i \|_1 = \| A \|_1 \leq c \| A \|
\]

\[
\leq c \| \varphi_p ^{-1} \| \| \bar{A}^s \| \leq 2c \| \varphi_p ^{-1} \| \| B \| = Q.
\]

Therefore, \( | \lambda_i | \leq Q, i = 1, \ldots, n, \) and (b) is proved.

Now we are ready to give necessary and sufficient conditions for an element to have 0 as a best approximation in a finite dimensional subspace of \( [c_0, c_0] \).

Without loss of generality, we may assume that each of the operators generating the subspace has norm equal to 1.

**Theorem 2.3.** Let \( A_k = (a_{k j}) \in [c_0, c_0] \) with \( \| A_k \| = 1, k = 1, \ldots, n, \) and
let $B = (b_{ij}) \in [c_0, c_0]$. Then $B$ has 0 as a best approximation in $[A_1, \ldots, A_n]$ if and only if for all $\epsilon > 0$, there exist $m$ positive integers $k_1, \ldots, k_m$, $m$ $l_\infty$ sequences $f_1^i, \ldots, f_m^i$ with $\|f_i\|^1 = 1$, $i = 1, \ldots, m$ and $m$ scalars $r_1, \ldots, r_m$ with $r_i > 0$, $i = 1, \ldots, m$ and $\sum_{i=1}^m r_i = 1$ such that

(i) $\sum_{i=1}^m r_i \sum_{j=1}^\infty f_i^j a_k^j = 0$ \hspace{0.5cm} $k = 1, \ldots, n$, \\
(ii) $\sum_{i=1}^m r_i \sum_{j=1}^\infty f_i^j b_{k,i} - \|B\| < \epsilon$.

Proof. Choose $p$ and $Q$ as in Lemma 2.2. Suppose $B$ has 0 as a best approximation in $[A_1, \ldots, A_n]$. Let $\lambda_1, \ldots, \lambda_n \in [-Q, Q]$ and $\epsilon > 0$. Then there exists $i = k(\lambda_1, \ldots, \lambda_n) = k(\lambda)$ such that

$$\sum_{j=1}^\infty |b_{k(\lambda),j} - (\lambda_1 a_{k(\lambda),j}^1 + \cdots + \lambda_n a_{k(\lambda),j}^n)|$$

$$- \|B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| < \epsilon/6.$$  \hspace{0.5cm} (1)

Let $\mu_1, \ldots, \mu_n \in [-Q, Q]$. It is easily seen that the function $\varphi(\mu_1, \ldots, \mu_n) = \sum_{j=1}^\infty |b_{k(\lambda),j} - (\mu_1 a_{k(\lambda),j}^1 + \cdots + \mu_n a_{k(\lambda),j}^n)|$ is continuous at $(\lambda_1, \ldots, \lambda_n)$. Therefore, for each $i = 1, \ldots, n$ there exists an open interval

$I_{\lambda_i} = \{\mu: |\mu - \lambda_i| < \epsilon/6n\}$

such that for $\mu_1, \ldots, \mu_n \in [-Q, Q]$, if $\mu_i \in I_{\lambda_i}$ for each $i = 1, \ldots, n$, then

$$\sum_{j} |b_{k(\lambda),j} - (\mu_1 a_{k(\lambda),j}^1 + \cdots + \mu_n a_{k(\lambda),j}^n)|$$

$$- \sum_{j} |b_{k(\lambda),j} - (\lambda_1 a_{k(\lambda),j}^1 + \cdots + \lambda_n a_{k(\lambda),j}^n)| < \epsilon/6.$$  \hspace{0.5cm} (2)

Now using (1) and (2) we obtain for $\mu_1, \ldots, \mu_n \in [-Q, Q]$ and $\mu_i \in I_{\lambda_i}$, $i = 1, \ldots, n$

$$\sum_{j} |b_{k(\lambda),j} - (\mu_1 a_{k(\lambda),j}^1 + \cdots + \mu_n a_{k(\lambda),j}^n)|$$

$$- \|B - (\mu_1 A_1 + \cdots + \mu_n A_n)\| < \epsilon/2.$$  \hspace{0.5cm} (3)

For a scalar $\lambda \in [-Q, Q]$, let $I_\lambda = \{\mu: |\mu - \lambda| < (\epsilon/6)n\}$. Then $\{I_\lambda: \lambda \in [-Q, Q]\}$ is an open covering of the compact interval $[-Q, Q]$. Consequently, there exists a finite number of scalars $\lambda_1, \ldots, \lambda_s$ in $[-Q, Q]$ such that $[-Q, Q] \subseteq \bigcup_{\lambda=1}^s I_{\lambda}$. Consider $k(\lambda_{p_1}) - k(\lambda_{p_2}, \ldots, \lambda_{p_n})$, where $\lambda_{p_i}$ may be chosen from $\lambda_1$ to $\lambda_s$ for $i = 1, \ldots, n$. Thus, we have $s^n$ rows. Now add to these the first $p$ rows as obtained by Lemma 2.2 (a). Let $m$ be the number of distinct rows obtained, so $m \leq p + s^n$. Label these rows by $k_1, \ldots, k_m$. 
Let $\lambda_1, \ldots, \lambda_n$ be arbitrary scalars in $[-Q, Q]$. Then for each $i = 1, \ldots, n$, $\lambda_i \in I_{\ell_i}$ for some $\ell_i = 1, \ldots, s$. Now $k(\lambda_1, \ldots, \lambda_n) = k_\ell$ for some $\ell = 1, \ldots, m$. Then we may apply (3) to obtain
\[ \left| \sum_j b_{k_\ell}^j - (\lambda_1 a_{k_\ell}^1 + \cdots + \lambda_n a_{k_\ell}^n) \right| = \left| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n) \right| < (\epsilon/2). \]

It follows that
\[ \max_{1 \leq \ell \leq m} \sum_j b_{k_\ell}^j - (\lambda_1 a_{k_\ell}^1 + \cdots + \lambda_n a_{k_\ell}^n) \leq \left| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n) \right| < (\epsilon/2). \] (4)

Define, respectively, $\bar{A}_1, \ldots, \bar{A}_n$, $\bar{B} = A_1, \ldots, A_n$, $B = \{ k \}_{1 \leq k \leq m} \in E = (I_1 \times \cdots \times I_n)$ (m summands). By Lemma 2.2 $\bar{A}_1, \ldots, \bar{A}_n$ are linearly independent. By (4) we obtain
\[ \left| \bar{B} - (\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n) \right| = \left| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n) \right| < (\epsilon/2). \] (5)

Consider the quotient space $E/\{\bar{A}_1, \ldots, \bar{A}_n\}$. We have $\| \pi \bar{B} \| = \inf \| \bar{B} - (\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n) \|$, where the infimum is taken over $\lambda_i \in [-Q, Q]$, $i = 1, \ldots, n$ by Lemma 2.2 (b). We know $\| \pi \bar{B} \| \leq \| \bar{B} \| \leq \| B \|$, so $\| \pi \bar{B} \| - \| B \| < \epsilon$. Utilizing (5) and the fact that $B$ has 0 as a best approximation in $\{A_1, \ldots, A_n\}$, we see that
\[ \| B \| \leq (\epsilon/2) + \inf \| \bar{B} - (\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n) \|, \]

the infimum taken over $\lambda_i \in [-Q, Q]$, $i = 1, \ldots, n$. Thus, $\| B \| - \| \pi \bar{B} \| < \epsilon$. Hence,
\[ \| \pi \bar{B} \| - \| B \| < \epsilon. \] (6)

Suppose $B \neq 0$. Then by the Hahn–Banach theorem there exists $F$ in $E^*$, $\| F \| = 1$, such that $F(\bar{A}_1, \ldots, \bar{A}_n) = 0$ and $F(B) = \| \pi \bar{B} \|$. By Lemma 1.1, $E^*$ can be identified with $(l_\infty \times \cdots \times l_\infty)$ (m summands). Hence, $F$ can be represented by an m-tuple of $l_\infty$ sequences $(g^1, \ldots, g^m)$ where $1 = \| F \| = \sum_{i=1}^{m} \| g^i \|$. Without loss of generality, assume $\| g^i \| > 0$ for $i = 1, \ldots, m$. Then define
\[ f^i = \| g^i \| g^i, \quad r_i = \| g^i \| \quad i = 1, \ldots, m. \]

Then $\sum_{i=1}^{m} r_i = 1$ and $\| f^i \| = 1$ for $i = 1, \ldots, m$. For $T = (t_{ij}) \in [c_0, c_0]$, $F(T) = \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f^i_j k_{ij}$. Then (i) holds since $F(\bar{A}_1, \ldots, \bar{A}_n) = 0$. Since $F(B) = \| \pi \bar{B} \|$, we have $\| F(B) - \| B \| < \epsilon$ by (6). Hence, (ii) holds.
Now if \( B = 0 \), select \( C \in E, \bar{C} \notin [\tilde{A}_1, ..., \tilde{A}_n] \). Again apply the Hahn–Banach theorem to obtain \( F \) in \( E^* \), \( \| F \| = 1 \) such that \( F([\tilde{A}_1, ..., \tilde{A}_n]) = 0 \) and \( F(\bar{C}) = \| \pi \bar{C} \| \). By the same argument as above, (i) holds, and (ii) holds since \( \bar{B} = 0 = \| B \| \).

To show sufficiency, let \( \varepsilon > 0 \). Then there exist \( k_i, f^i, \) and \( r_i, i = 1, ..., m \) as stated in the theorem such that (i) and (ii) hold. For \( T = (t_{ij}) \in [c_0, c_0] \), define \( F \) on \([c_0, c_0]^*\) by

\[
F(T) = \sum_{i=1}^n r_i \sum_{j=1}^\infty f^i_j t_{kj}.
\]

Now \( |F(T)| \leq \| T \| \), so \( F \in [c_0, c_0]^* \) and \( \| F \| \leq 1 \).

We have \( F(A_k) = 0 \) for \( k = 1, ..., n \) by (i), and \( |F(B) - \| B \| | < \varepsilon \) by (ii).

Let \( \lambda_1, ..., \lambda_n \) be arbitrary scalars. Then \( F[B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)] = F(B) \).

Hence,

\[
\| B - \| \leq \varepsilon \cdot F[B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)] \leq \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\|
\]

But this can be shown for all \( \varepsilon > 0 \). Therefore,

\[
\| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| \geq \| B \|.
\]

Since the scalars were arbitrary, \( B \) has 0 as a best approximation in \([A_1, ..., A_n]\). This completes the proof.

3. Characterization of One-Dimensional Chebyshev Subspaces

**Definition.** If \( P_1 \) and \( P_2 \) are subsets of the set \( I \) of all positive integers such that \( P_1 \cap P_2 = \emptyset \) and \( P_1 \cup P_2 = I \), then we say that \( P_1 \) and \( P_2 \) form a partition of \( I \). Let \( A = (a_{ij}) \in [c_0, c_0] \). Then \( A \) satisfies the partition property if and only if there exists \( \delta > 0 \) such that for all \( i \) and for all partitions \( P_1, P_2 \) of \( I \), we have

\[
\left| \sum_{j \in P_1} |a_{ij}| - \sum_{j \in P_2} |a_{ij}| \right| \geq \delta.
\]

We will show that if \( A \neq 0 \), then \([A]\) is a Chebyshev subspace of \([c_0, c_0]\) if and only if \( A \) satisfies the partition property. Before we can prove this result, however, we will need two lemmas.

**Lemma 3.1.** Let \( A = (a_{ij}) \in [c_0, c_0] \). Then the partition property fails to hold for \( A \) if and only if either there exists \( i \) and a partition \( P_1, P_2 \) of \( I \) such that

\[
\left| \sum_{j \in P_1} |a_{ij}| - \sum_{j \in P_2} |a_{ij}| \right| = 0,
\]
or there exists a sequence \( i_1 < \cdots < i_n < \cdots \) and corresponding partitions \( P_1^n, P_2^n \) such that \( \left| \sum_{j \in P_1^n} a_{i_j} - \sum_{j \in P_2^n} a_{i_j} \right| < (1/n) \).

**Proof.** If either of the conditions hold, \( A \) clearly fails to satisfy the partition property.

Conversely, suppose the partition property fails to hold for \( A \). To simplify notation in this proof, let \( Q(i, P_1, P_2) = \left| \sum_{j \in P_1} a_{i_j} - \sum_{j \in P_2} a_{i_j} \right| \). If the first condition holds, we are finished. Therefore, assume there does not exist \( i \) and a partition \( P_1, P_2 \) such that \( Q(i, P_1, P_2) = 0 \). Since the partition property fails to hold for \( A \), there exists \( i \) and \( P_1, P_2 \) such that \( Q(i, P_1, P_2) < 1 \). Let \( i_1 \) be the smallest such \( i \) where this occurs, i.e., for all \( i < i_1 \) and for all \( P_1, P_2 \) we have \( Q(i, P_1, P_2) \geq 1 \), but there exists a partition \( P_1^1, P_2^1 \) such that \( Q(i_1, P_1^1, P_2^1) < 1 \).

Now there exists \( i \) and \( P_1, P_2 \) such that \( Q(i, P_1, P_2) < (1/2) \). We know \( i < i_1 \) is impossible. Suppose the only possible choice is \( i = i_1 \). Then we must have one row \( i \) and a sequence of partitions \( P_1^n, P_2^n \) such that \( Q(i, P_1^n, P_2^n) < 1 \) but there exists a partition \( P_1^n, P_2^n \) such that \( Q(i, P_1^n, P_2^n) < 1 \).

For each \( n \), define \( g^n = \{ g_j^n \} \) by

\[
\begin{align*}
g_j^n &= 1 \quad &\text{for } j \in P_1^n, &a_{ij} \geq 0, \text{ or } j \in P_2^n, &a_{ij} < 0, \\
&= -1 \quad &\text{for } j \in P_1^n, &a_{ij} < 0, \text{ or } j \in P_2^n, &a_{ij} \geq 0.
\end{align*}
\]

Then \( Q(i, P_1^n, P_2^n) = \left| \sum_{j=1}^n g_j^n a_{ij} \right| < (1/n) \). For the fixed \( i \) above, let \( x = \{a_{ij}\} \) in \( l_1 \). Consider the function \( f: R \to R \) defined by \( f(x) = |x| \).

Now define \( G: l_\infty \to R \) by \( G(g) = \left| \sum_{j=1}^n a_{ij} g_j \right| \). For \( i = 1, 2, \ldots \), let \( e_i \) be that sequence in \( l_1 \) which has \( 1 \) in the \( i \)th place and \( 0 \) elsewhere. Define \( F_i: l_\infty \to R \) by \( F_i = f e_i^* \) for \( i = 1, 2, \ldots \).

Since \( \|g^n\| = 1 \) for each \( n \), we have \( g^n \in S(l_\infty) = S(l_1^*) \), which is compact in the weak* topology by Alaoglu's theorem (see [3, p. 424]). Let

\[
A = \{ g = \{ g_j \} \in S(l_\infty): |g_j| = 1 \text{ for all } j \}.
\]

Then \( g^n \in A \) for all \( n \), so \( \inf G(g) = 0 \), where the infimum is taken over all \( g \in A \). Now \( F_i(g) = |g(e_i)| \). Then \( \{ g \in S(l_\infty): |g(e_i)| = 1 \} \) is weak* closed since \( F_i \) is weak* continuous. Therefore, \( A = \bigcap_i \{ g \in S(l_\infty): |g(e_i)| = 1 \} \) is weak* compact. Since \( G \) is a weak* continuous function on \( A \), there exists \( g \in A \) such that \( G(g) = 0 \), i.e., there exists \( g = \{ g_j \} \) where \( |g_j| = 1 \) for all \( j \), and \( |\sum_j a_{ij} g_j| = 0 \). Let those \( j \) which give rise to terms \( |a_{ij}| \) for the product \( a_{ij} g_j \) be in \( P_1 \), and the remaining \( j \) in \( P_2 \). Then we have exhibited a partition \( P_1, P_2 \) such that \( Q(i, P_1, P_2) = 0 \), a contradiction. Therefore, there does not exist one row \( i \) and a sequence of partitions \( P_1^n, P_2^n \) such that \( Q(i, P_1^n, P_2^n) < (1/n) \).

It is, therefore, possible to select \( i > i_1 \) and \( P_1, P_2 \) such that \( Q(i, P_1, P_2) < \)
Let \( i_2 \) be the smallest \( i > i_1 \) where this is possible. Assume we have in this way selected \( i_1 < \cdots < i_n \) so that for each \( n \), there exists \( P_1^n, P_2^n \) such that \( Q(i_n, P_1^n, P_2^n) < (1/n) \). We know there exists \( i \) and \( P_1, P_2 \) such that \( Q(i, P_1, P_2) < [1/(n+1)] \) since the partition property fails to hold for \( A \). Suppose the only possible choice for this \( i \) is \( i \leq i_n \). Then for all \( k \geq n+1 \), there exists \( i_k < i_n \) and \( P_1^k, P_2^k \) such that \( Q(i_k, P_1^k, P_2^k) < (1/k) \). For each \( s = 1, \ldots, n \), if \( i = i_s \), there exists a positive integer \( k_s \) such that for all \( P_1, P_2 \), \( Q(i_s, P_1, P_2) > (1/k_s) \), since we have previously shown that for any one row \( i \), it is impossible to obtain for all \( n \) a partition \( P_1^n, P_2^n \) such that \( Q(i, P_1^n, P_2^n) < (1/n) \). Let \( k = \max\{k_1, \ldots, k_n, n+1\} \). Then \( k \geq n+1 \), so there exists \( i_k < i_n \) and \( P_1^k, P_2^k \) such that \( Q(i_k, P_1^k, P_2^k) < (1/k) \). However, for all \( i < i_n \) and all \( P_1, P_2 \), in particular for \( i_k \) and \( P_1^k, P_2^k \), we have \( Q(i_k, P_1^k, P_2^k) > (1/k) \), a contradiction. This completes the inductive argument and proves the lemma.

**Lemma 3.2.** If \( \| B - \lambda A \| = \| B - \lambda_1 A \| \) for all scalars \( \lambda \) in \( [\lambda_1, \lambda_2] \) (where \( \lambda_1 < \lambda_2 \)), then \( B \) has \( \lambda A \) as a best approximation in \([A]\) for all \( \lambda \) in \([\lambda_1, \lambda_2]\).

**Proof.** Let \( \| B - \lambda A \| = \| B - \lambda_1 A \| \) for all \( \lambda \) in \([\lambda_1, \lambda_2]\). Suppose \( \lambda_1 A \) is not a best approximation to \( B \) in \([A]\). Then there exists \( \mu \in R \) such that \( \| B - \mu A \| < \| B - \lambda_1 A \| \). Either \( \mu > \lambda_2 \) or \( \mu < \lambda_1 \). Assume \( \mu > \lambda_2 \) since the other case is similar. Define \( \alpha = [(\lambda_2 - \mu)/(\lambda_2 - \lambda_1)] \). Then \( \alpha \in (0, 1) \) and \( \lambda_2 = \alpha \lambda_1 + (1 - \alpha) \mu \). Hence,

\[
\| B - \lambda_2 A \| \leq \| B - \lambda_1 A \| + (1 - \alpha)\| B - \mu A \| < \| B - \lambda_1 A \|,
\]

which is a contradiction. This proves the lemma.

**Theorem 3.3.** Let \( A = (a_{ij}) \in [c_0, c_0] \), \( A \neq 0 \). Then \([A]\) is a Chebychev subspace of \([c_0, c_0]\) if and only if \( A \) satisfies the partition property.

**Proof.** Suppose the partition property fails to hold for \( A \). First, let us consider the case where there exists \( i_0 \) such that \( a_{i_0 j} = 0 \) for all \( j \). Define \( B = (b_{ij}) \), where \( b_{i_0 j} = \| A \| \), and \( b_{ij} = 0 \) for \( i \neq i_0 \), for all \( j \), and for \( i = i_0 \), \( j = 2, 3, \ldots \). Then \( B \in [c_0, c_0] \) and \( \| B \| = \| A \| = \| B - A \| \). For all \( \lambda \) in \( R \), \( \sum_{j=1}^{\infty} |b_{i_0 j} - \lambda a_{i_0 j}| = \| A \| \). Hence, for all \( \lambda \) in \( R \), we have \( \| B - \lambda A \| \geq \| A \| = \| B \| \). Thus, \( B \) has 0 and \( A \neq 0 \) as best approximations in \([A]\), so \([A]\) is not Chebychev.

Now suppose that \( \sum_{i=1}^{\infty} |a_{i j}| > 0 \) for all \( i \). Since \( A \) fails to satisfy the partition property, there are two possible cases by Lemma 3.1.

Case 1. There exists \( i_1 < \cdots < i_n < \cdots \) and corresponding partitions \( P_1^n, P_2^n \) such that \( \| \sum_{i \in R_1^n} |a_{i, n}| - \sum_{j \in R_2^n} |a_{n, j}| \| < (1/n) \). Therefore, for
each $n$ there exists $\epsilon^n = \{\epsilon^n_j\}$ in $l_\infty$ with $|\epsilon^n_j| = 1$ for all $j$ and all $n$ such that $0 < \sum_{j=1}^{\infty} \epsilon^n_j a_{i,j} < (1/n)$. Then $\lim_{n \to \infty} \sum_{j=1}^{\infty} \epsilon^n_j a_{i,j} = 0$.

Let $s = \sup \sum_{j=1}^{\infty} |a_{i,j}|$, the supremum taken over all $n$, so $0 < s \leq \|A\| < \infty$. Now there exists $\lambda_0$, $0 < \lambda_0 < 1$ such that $\lambda_0 \|A\| < s$. Define $\sigma_n = s - \sum_{j=1}^{\infty} |a_{i,j}| \geq 0$. Now let $B = (b_{i,j})$, where $b_{i,j}$ is defined as follows:

$$b_{i,j} = \begin{cases} 
0 & \text{for } i \neq i_n \text{ for any } n = 1, 2, \ldots, \\
\epsilon^n_j |a_{i,j}| + \epsilon^n_j (\sigma_n/i_n) & \text{for } j = 1, \ldots, i_n \text{ and } n = 1, 2, \ldots, \\
\epsilon^n_j |a_{i,j}| & \text{for } j > i_n \text{ and } n = 1, 2, \ldots.
\end{cases}$$

It is easy to see that $\lim_{n \to \infty} b_{i,j} = 0$ for all $j$. The next part of the proof will show that $\|B\| = s < \infty$, which will show by Theorem 2.1 that $B \in [c_0, c_0]$.

Let $\lambda$ be given, $0 < \lambda \leq \lambda_0$. If $i \neq i_n$ for any $n$, then

$$\sum_{j=1}^{\infty} |b_{i,j} - \lambda a_{i,j}| \leq \lambda_0 \|A\| < s.$$

If $i = i_n$, we have

$$\sum_{j=1}^{\infty} |b_{i,j} - \lambda a_{i,j}| = \sum_{j=1}^{i_n} |a_{i,j}| + \sum_{j=i_n+1}^{\infty} |(\sigma_n/i_n) + \lambda \epsilon^n_j a_{i,j}|$$

$$+ \sum_{j=i_n+1}^{\infty} |a_{i,j}| - \lambda \epsilon^n_j a_{i,j}|$$

$$= \sum_{j=1}^{i_n} |a_{i,j}| + \sum_{j=i_n+1}^{\infty} (\sigma_n/i_n) - \lambda \sum_{j=1}^{i_n} \epsilon^n_j a_{i,j}$$

$$+ \sum_{j=i_n+1}^{\infty} |a_{i,j}| - \lambda \sum_{j=i_n+1}^{\infty} \epsilon^n_j a_{i,j}$$

$$= s - \lambda \sum_{j=1}^{\infty} \epsilon^n_j a_{i,j} \leq s.$$

Then since $\lim_{n \to \infty} \sum_{j=1}^{\infty} \epsilon^n_j a_{i,j} = 0$, we must have $\|B - \lambda A\| = s$ for all $\lambda$ in $[0, \lambda_0]$. Then by Lemma 3.2, $B$ has $\lambda A$ as a best approximation in $[A]$ for all $\lambda$ in $[0, \lambda_0]$. Therefore, $[A]$ is not Chebychev.

**Case 2.** There exists $i_0$ and partition $P_1, P_2$ such that

$$\left| \sum_{j \in P_1} a_{i_0,j} \right| - \left| \sum_{j \in P_2} a_{i_0,j} \right| = 0.$$

Therefore, there exists $\{\epsilon_j\}$ in $l_\infty$ with $|\epsilon_j| = 1$ for all $j$ such that $\sum_{j=1}^{\infty} \epsilon_j a_{i,j} = 0$. 

Let \( s = \sum_{j=1}^{\infty} |a_{i_0}^j| \), so \( 0 < s \leq \|A\| \). Then there exists \( \lambda_0 \), \( 0 < \lambda_0 < 1 \) such that \( \lambda_0 \|A\| < s \). Let \( B = (b_{ij}) \) be defined by

\[
\begin{align*}
  b_{ij} &= \begin{cases} 
    0 & \text{if } i \neq i_0, \\
    e_j & \text{if } i = i_0.
  \end{cases}
\end{align*}
\]

Then \( B \in [c_0, c_0] \). Let \( \lambda \in [0, \lambda_0] \). For \( i \neq i_0 \), \( \sum_{j=1}^{\infty} |b_{ij} - \lambda a_{ij}| < s \). If \( i = i_0 \),

\[
\sum_{j=1}^{\infty} |b_{i_0j} - \lambda a_{i_0j}| = \sum_{j=1}^{\infty} |a_{i_0j}| = s.
\]

Therefore, \( \|B - \lambda A\| = s \) for all \( \lambda \in [0, \lambda_0] \). By Lemma 3.2, \([A]\) is not Chebychev.

We must now show sufficiency. Without loss of generality, assume \( \|A\| = 1 \). Suppose \([A]\) is not Chebychev. It is easy to see that there exist \( B \in [c_0, c_0] \), \( \|B\| = 1 \), and \( \lambda > 0 \) such that \( B \) has \( 0 \) and \( \pm \lambda A \neq 0 \) as best approximations in \([A]\). Let \( \epsilon > 0 \) and \( \epsilon' = \epsilon \lambda \). Then by Theorem 2.3 there exist \( m \) positive integers \( k_1, \ldots, k_m \), \( m \) \( l_\infty \) sequences \( f_1, \ldots, f_m \) with \( \|f_1\| = 1 \), \( i = 1, \ldots, m \), and \( m \) scalars \( r_1, \ldots, r_m \) with \( r_i > 0 \), \( i = 1, \ldots, m \) and \( \sum_{i=1}^{m} r_i = 1 \) such that

\[
\begin{align*}
  (i) & \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_i^j a_{k_i,j} = 0, \\
  (ii) & |\sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_i^j b_{k_i,j} - 1| < \epsilon'.
\end{align*}
\]

Then there exists \( i \) such that

\[
\left| \sum_{j=1}^{\infty} f_i^j b_{k_i,j} - 1 \right| < \epsilon'.
\]

Define \( \{g_j\} \) in \( l_\infty \) by

\[
\begin{align*}
  g_j &= \begin{cases} 
    |b_{k_i,j}| & \text{if } b_{k_i,j} \neq 0, \\
    1 & \text{if } b_{k_i,j} = 0.
  \end{cases}
\end{align*}
\]

Then \( g_j b_{k_i,j} = |b_{k_i,j}| \) and \( |g_j| = 1 \) for all \( j \) since the scalars are real. Since \( \|f_1\| = 1 \), it follows that

\[
\sum_{j=1}^{\infty} g_j b_{k_i,j} \geq \sum_{j=1}^{\infty} f_i^j b_{k_i,j}.
\]

Next we will show that \( |\sum_{j=1}^{\infty} g_j a_{k_i,j}| < (\epsilon'/\lambda) \) for the selected \( i \). If \( \sum_{j=1}^{\infty} g_j a_{k_i,j} = 0 \), there is nothing to prove. Suppose \( \lambda \sum_{j=1}^{\infty} g_j a_{k_i,j} > 0 \). Now \( |\sum_{j=1}^{\infty} g_j (b_{k_i,j} \pm \lambda a_{k_i,j})| \leq \|B \pm \lambda A\| = \|B\| = 1 \) since \( 0 \) and \( \pm \lambda A \) are best approximations to \( B \) in \([4]\). Then using (1) and (2), we have
If $-\lambda \sum_{j=1}^{\infty} g_j a_{k,j} > 0$, the result follows in a similar fashion. Let $P_1$ be the set of those $j$ where $g_j a_{k,j} = |a_{k,j}|$, and let $P_2$ consist of the remaining $j$. Then we have exhibited $k_i$ and a partition $P_1, P_2$ such that $\sum_{j \in P_1} |a_{k,j}| - \sum_{j \in P_2} |a_{k,j}| < \varepsilon$. Hence, $A$ fails to satisfy the partition property, and the theorem is proved.

4. RESULTS FOR FINITE DIMENSIONAL SUBSPACES

We return in this section to an arbitrary finite dimensional subspace $M$ of $[c_0, c_0]$, and now present a necessary condition for $M$ to be non-Chebychev.

**Theorem 4.1.** Let $M = [A_1, \ldots, A_n]$ be a non-Chebychev subspace of $[c_0, c_0]$, where $A_k = (a_{ij}^k) \in [c_0, c_0]$ with $\| A_k \| = 1$, $k = 1, \ldots, n$. Then there exists $A = (a_{ij}) \in M$, $\| A \| = 1$, such that given $\epsilon > 0$, there exist $m$ positive integers $k_1, \ldots, k_m$ and $\alpha = (a_{ij})$ in $M^\perp$ with $\alpha_{ij} = 0$ for $i \neq k_1, \ldots, k_m$, $\sup_j |\alpha_{ij}| < \infty$ for all $i$ and $\sum_{i=1}^{m} \sup_j |\alpha_{k,i}| = 1$ such that

(i) if $\beta \in [c_0, c_0]^{*}$ and $\| \alpha \pm \beta \| \leq 1$, then $|\beta(A)| < \epsilon$,

(ii) $\sum_{i=1}^{m} \sup_j \alpha_{k,i} a_{ij} < \epsilon$.

**Proof.** Since $M$ is non-Chebychev, it follows that there exists $B$ in $[c_0, c_0]$ such that $B$ has 0 and $\pm A \neq 0$ as best approximations in $M$, with $\| A \| = 1$. Thus, $\| B \| = \| B - A \| = \| B + A \|$. This is the required $A$. Let $\epsilon > 0$. Then by Theorem 2.3 there exist $m$ positive integers $k_1, \ldots, k_m$, $m l_\infty$ sequences $f_1, \ldots, f_m$ with $\| f_i \| = 1$, $i = 1, \ldots, m$ and $m$ scalars $r_1, \ldots, r_m$ with $r_i > 0$, $i = 1, \ldots, m$ and $\sum_{i=1}^{m} r_i = 1$ such that

(i') $\sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j a_{k,j}^i = 0$ $k = 1, \ldots, n$,

(ii') $|\sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j b_{k,j}^i - \| B \| | < (\epsilon/2)$.

Define $\alpha = (a_{ij})$ by $\alpha_{k,j} = r_i f_j^i$ for $i = 1, \ldots, m$ and $\alpha_{ij} = 0$ for $i \neq k_1, \ldots, k_m$. Then $\sup_j |\alpha_{ij}| < \infty$ for all $i$ and $\sum_{i=1}^{m} \sup_j |\alpha_{k,i}| = 1$. For $T = (t_{ij}) \in [c_0, c_0]$, let $\alpha(T) = \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j t_{k,j}^i$. Then it is easy to see that $\alpha \in [c_0, c_0]^{*}$. By (i'), $\alpha \in M^\perp$. By (ii') we have

$$|\alpha(B) - \| B \| | < (\epsilon/2).$$

(1)
To prove (i), let $\beta \in [c_0, c_0]^*$ with $\|\alpha \pm \beta\| \leq 1$. Then

$$\|\alpha \pm \beta\| \leq \|B\| \leq \|\alpha \pm \beta\| \leq \|B\|,$$

so by (1), $|\beta(B)| < (\epsilon/2)$. Similarly, since $\alpha \in M^\perp$, we obtain

$$\alpha(B) \pm \beta(B - A) \leq \|B - A\| = \|B\|.$$

Thus, $|\beta(B - A)| < (\epsilon/2)$. It follows that $|\beta(A)| < \epsilon$.

We must now show (ii). Let

$$P = \left\{ i: \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} \geq 0 \right\}, \quad P' = \left\{ i: \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} > 0 \right\}, \quad \text{and} \quad N = \left\{ i: \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} < 0 \right\}.$$

Since $\alpha \in M^\perp$, $\alpha(A) = \sum_{i=1}^{\infty} \alpha_{ij}a_{ij} = 0$. Thus, if $N = \emptyset$, then $P' = \emptyset$, and conversely. In this case (ii) holds trivially. Therefore, assume $P' \neq \emptyset$ and $N \neq \emptyset$. Now

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} + \sum_{i \in N} \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} = 0.$$

Therefore,

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} = \sum_{i \in N} \sum_{j=1}^{\infty} \alpha_{ij}a_{ij}.$$

Now suppose (ii) is false. Then

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} \geq (\epsilon/2). \quad (2)$$

Also, $\sum_{i \in N} \sum_{j=1}^{\infty} \alpha_{ij}a_{ij} \geq (\epsilon/2)$. For each $i = 1, 2, \ldots$, let $\lambda_i = \sup_{j} |\alpha_{ij}| \geq 0$, the supremum taken over all $j$. Let $\lambda_P = \sum_{i \in P} \lambda_i > 0$ and $\lambda_N = \sum_{i \in N} \lambda_i > 0$.

Then

$$\lambda_P + \lambda_N = 1. \quad \text{Let} \quad E_1 = \sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}b_{ij} \quad \text{and} \quad E_2 = \sum_{i \in N} \sum_{j=1}^{\infty} \alpha_{ij}b_{ij}.$$

Then

$$E_1 + E_2 = \alpha(B) \geq \|B\| - (\epsilon/2) \quad \text{by (1)}. \quad \text{This implies that either} \quad (a) \quad E_1 > \lambda_P[\|B\| - (\epsilon/2)] \quad \text{or} \quad (b) \quad E_2 > \lambda_N[\|B\| - (\epsilon/2)] \quad \text{must hold. Suppose (a) is true.} \quad \text{Then using (2), we obtain}$$

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}(b_{ij} + a_{ij}) > \lambda_P \|B\| - (\epsilon/2) \lambda_P + (\epsilon/2) > \lambda_P \|B\| = \lambda_P \|B + A\|.$$

But for each $i$ in $P$, $\sum_{j=1}^{\infty} \alpha_{ij}(b_{ij} + a_{ij}) \leq \lambda_i \|B + A\|$. Thus,

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}(b_{ij} + a_{ij}) \leq \lambda_P \|B + A\|.$$

Thus, we have been led to a contradiction. If (b) holds, we obtain a contradiction in a similar manner. Therefore, (ii) is proved.

**Theorem 4.2.** Let $M = [A_1, \ldots, A_n]$ be an $n$-dimensional Chebychev subspace of $X = [c_0, c_0]$. Let $i_1 < \cdots < i_s$ denote a fixed but arbitrary finite
The adjoint $T^*$ of a bounded linear operator $T$ from $c_0$ to $c_0$ is the mapping from $c_0$ to $c_0^*$ defined by $T^*y^* = y^*T$. By [5, p. 201], $c_0^*$ can be identified with $c_0$ sphere, and $c_0^*$ is the dual space of $c_0$. The adjoint $T^*$ of a bounded linear operator $T$ from $c_0$ to $c_0$ is the mapping from $c_0$ to $c_0^*$ defined by $T^*y^* = y^*T$. By [5, p. 201], $c_0^*$ can be identified with $c_0$ sphere, and $c_0^*$ is the dual space of $c_0$.
with $l_1$. By [3, p. 478] we can see that $T^* \in [l_1, l_1]$ and $\|T^*\| = \|T\|$. Recall that $l_1^*$ can be identified with $l_\infty$. Hence, the second adjoint $T^{**} \in [l_\infty, l_\infty]$.

Now $c_0 = c_0 \subseteq l_\infty$. It is known by [3, p. 479] that $T^{**}: l_\infty \to l_\infty$ is an extension of $T$, i.e., for $x \in c_0$, $T^{**}(x) = T(x)$.

Let $T$ be represented by the infinite matrix of scalars $(\alpha_{ij})$, so that by Theorem 2.1, if $x = \{x_i\} \in c_0$, $y = \{y_i\} \in c_0$, then $Tx = y$ can be expressed by the equations

$$y_i = \sum_{j=1}^{\infty} \alpha_{ij}x_j \quad i = 1, 2, \ldots$$

The norm of $T$ is given by $\|T\| = \sup_i \sum_{j=1}^{\infty} |\alpha_{ij}|$. Then by [5, p. 220], the matrix $(\alpha_{ij})$ also defines a bounded linear operator $T'$ on $l_\infty$ into $l_\infty$ with the same defining equations and same norm. Thus, $T'$ is also an extension of $T$, i.e., for $x \in c_0$, $T'(x) = T(x)$.

**Theorem 4.5.** Let the bounded linear operator $T$ on $c_0$ into $c_0$ be represented by the infinite matrix of scalars $(\alpha_{ij})$, and let $T'$ represent $(\alpha_{ij})$ considered as a bounded linear operator from $l_\infty$ to $l_\infty$. Then $T' = T^{**}$.

**Proof.** Let $b = \{b_j\} \in l_\infty$ and let $T'(b) = z = \{z_i\} \in l_\infty$, so $z_i = \sum_{j=1}^{\infty} \alpha_{ij}b_j$ for $i = 1, 2, \ldots$. Let $y^* = \{y_i^*\} \in l_1$. For each $j = 1, 2, \ldots$, define $e_j$ to be that sequence in $c_0$ which has 1 in the $j$th place and 0 elsewhere. Then $T(e_j) = (\alpha_{1j}, \alpha_{2j}, \ldots)$. Therefore, $T^*y^*(e_j) = \sum_{i=1}^{\infty} y_i^*\alpha_{ij}$. Let $x = \{x_i\} \in c_0$. Then $x = \sum_{j=1}^{\infty} x_j e_j$. Hence,

$$T^*y^*(x) = \sum_{j=1}^{\infty} x_j (T^*y^*)(e_j) = \sum_{j=1}^{\infty} x_j \sum_{i=1}^{\infty} y_i^*\alpha_{ij}. \quad (1)$$

Now consider $f = \{f_j\}$ where $f_j = \sum_{i=1}^{\infty} y_i^*\alpha_{ij}$ for $j = 1, 2, \ldots$. Then we have $\sum_{j=1}^{\infty} |f_j| < \infty$, where the interchange of limits is justified by a standard theorem (see, e.g., [1, p. 398]). Hence, $f \in l_1$. We also know $T^*y^* \in l_1$. Moreover, for any $x$ in $c_0$, $f(x) = (T^*y^*)(x)$ by (1). Thus, $T^*y^* = f$. Now, justifying the interchange of limits in the same manner as above, we obtain

$$b(T^*y^*) = \sum_{j=1}^{\infty} b_j \sum_{i=1}^{\infty} y_i^*\alpha_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_j y_i^*\alpha_{ij} = \sum_{i=1}^{\infty} y_i^*z_i.$$
As an example of this, we give an equivalent formulation of Theorem 3.3 in the following corollary.

**Corollary 4.6.** Let $A \in [c_0, c_0], A \neq 0$. Then $[A]$ is a Chebyshev subspace of $[c_0, c_0]$ if and only if there exists $\delta > 0$ such that if $x = \{x_j\} \in l_\infty$ with $|x_j| = 1$ for all $j$, and $A**(x) = \{y_i\}$, then $|y_i| \geq \delta$ for all $i$.

**Proof.** By Theorem 4.5, if $A = (a_{ij})$, then $A**(a_{ij})$, so $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$. Then the given condition holds if and only if $A$ satisfies the partition property. The result then follows by Theorem 3.3.

**References**