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Approximate Solution of Singular Integral Equations

A. CHAKRABARTI*
Department of Mathematics, Indian Institute of Science
Bangalore 560012, India

G. VANDEN BERGHE
Vakgroep Toegepaste Wiskunde en Informatica, Universiteit Ghent
Krijgslaan 281 - S9, B-9000 Gent, Belgium

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Abstract—An approximate method is developed for solving singular integral equations of the first kind, over a finite interval. The singularity is assumed to be of the Cauchy type, and the four basically different cases of singular integral equations of practical occurrence are dealt with simultaneously. The presently obtained results are found to be in complete agreement with the known analytical solutions of simple equations. The methodology of the present work is expected to be useful for solving singular integral equations of the first kind, involving partly singular and partly regular kernels, as well as equations of the second kind involving similar kernels, with appropriate adjustments regarding the endpoint behaviours of the unknown function. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Singular integral equations of the first kind, with a Cauchy type singular kernel, over a finite interval can be represented by the general equation

$$\int_{-1}^{1} f(t)[k_0(t,x) + k(t,x)]\,dt = g(x), \qquad -1 < x < 1, \tag{1.1}$$

where

$$k_0(t,x) = \frac{\hat{k}(t,x)}{t-x}, \qquad \left(\hat{k}(t,t) \neq 0\right), \tag{1.2}$$

$\hat{k}$ and k are regular square-integrable functions of the two variables t and x, and the kernel k_0 clearly involves the singularity of the Cauchy type. Integral equations of form (1.1) and other different forms occur in varieties of mixed boundary value problems of mathematical physics

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*Presently on leave, at the Department of Mathematical Sciences, NJIT, University Heights, Newark, NJ 07102, U.S.A.

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which include problems of two-dimensional deformations of isotropic elastic bodies involving cracks (see [1–3]) and scattering of two-dimensional surface water waves by vertical barriers (see [4–8]) and other related problems.

The simplest integral equation of the form (1.1) is

$$\int_{-1}^{1} \frac{f(t)}{t-x} dt = g(x), \tag{1.3}$$

for which $\hat{k}(t,x) = 1$ and $k(t,x) = 0$ (see [1,3]), and there are four basically important and interesting cases of equation (1.1), even under such simplifying assumptions on the nature of the kernel (i.e., when $\hat{k} = 1$ and $k = 0$), as given by the following.

CASE (I). $f(x)$ is unbounded at both the endpoints $x = \pm 1$.

CASE (II). $f(x)$ is unbounded at the end $x = -1$, but bounded at the end $x = +1$.

CASE (III). $f(x)$ is bounded at the end $x = -1$, but unbounded at the end $x = +1$.

CASE (IV). $f(x)$ is bounded at both the endpoints $x = \pm 1$.

It is well known (see [1,3]) that the complete analytical solutions of the singular integral equation (1.3) in the above four cases can be determined by using the following formulae:

$$\text{Case (I):} \quad f(x) = \frac{A_0}{(1-x^2)^{1/2}} - \frac{1}{\pi^2 (1-x^2)^{1/2}} \int_{-1}^{1} \frac{(1-t^2)^{1/2} g(t)}{(t-x)} dt, \tag{1.4}$$

where A_0 is an arbitrary constant,

$$\text{Case (II):} \quad f(x) = -\frac{1}{\pi^2} \left(\frac{1-x}{1+x}\right)^{1/2} \int_{-1}^{1} \left(\frac{1+t}{1-t}\right)^{1/2} \frac{g(t)}{t-x} dt, \tag{1.5}$$

$$\text{Case (III):} \quad f(x) = -\frac{1}{\pi^2} \left(\frac{1+x}{1-x}\right)^{1/2} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{1/2} \frac{g(t)}{t-x} dt, \tag{1.6}$$

$$\text{Case (IV):} \quad f(x) = -\frac{(1-x^2)^{1/2}}{\pi^2} \int_{-1}^{1} \frac{g(t)}{(1-t^2)^{1/2}(t-x)} dt, \tag{1.7}$$

the solution existing in Case (IV), if and only if

$$\int_{-1}^{1} \frac{g(t)}{(1-t^2)^{1/2}} dt = 0. \tag{1.8}$$

Guided by the analytical results available, as given by expressions (1.4)–(1.7), for the solution of the simple singular equation (1.3), as well as by utilizing the idea (see [9]) of replacing the integrand by an appropriate approximate function, we explain, in the next section, a numerical scheme that can be developed and implemented, for obtaining the approximate solutions of the general singular integral equation (1.1). The particular case of equation (1.3) follows quite easily and the known analytical solutions are recovered in the cases of simple forms of the forcing function $g(x)$, being polynomials of low degree.

2. THE APPROXIMATE SCHEME

We shall represent the unknown function $f(x)$ in the form

$$f(x) = \frac{h_r(x)\lambda_r(x)}{(1-x^2)^{1/2}}, \qquad (r = 1,2,3,4), \tag{2.1}$$

where $h_r(x)$ is a well-behaved function of x in the interval $-1 \le x \le 1$, and $\lambda_1(x) = 1$, in Case (I), $\lambda_2(x) = 1 - x$, in Case (II), $\lambda_3(x) = 1 + x$, in Case (III), and $\lambda_4(x) = 1 - x^2$, in Case (IV).

Then we approximate the unknown function $h_r(x)$ by means of a polynomial of degree n, given by

$$h_r(x) \approx \left[\sum_{j=0}^{n} c_j^{(r)} x^j\right], \qquad (r = 1, 2, 3, 4), \tag{2.2}$$

in the four cases as mentioned above, by using a "Chebyshev approximation", using the zeros x_j $(j = 0, 1, 2, \ldots, n)$ of the Chebyshev polynomial $T_{n+1}(x) = \cos[(n+1)\arccos(x)]$ (see [10]) in $[-1, 1]$.

Using the approximate form (2.2) of the function $h_r(x)$ along with the representation (2.1), in the original integral equation (1.1), we obtain

$$\sum_{j=0}^{n} c_j^{(r)} \left[\int_{-1}^{1} \frac{\lambda_r(t)\hat{k}(t,x)t^j}{(1-t^2)^{1/2}(t-x)}\, dt + \int_{-1}^{1} \frac{\lambda_r(t)k(t,x)t^j}{(1-t^2)^{1/2}}\, dt\right] = g(x), \tag{2.3}$$
$$(r = 1, 2, 3, 4), \quad (-1 < x < 1).$$

In the above equation (2.3), we next use the following "Chebyshev approximations" to the kernels $\hat{k}(t,x)$ and $k(t,x)$, given by (for fixed x)

$$\begin{aligned} \hat{k}(t,x) &\approx \sum_{p=0}^{m} \hat{k}_p(x)t^p, \\ k(t,x) &\approx \sum_{q=0}^{s} k_q(x)t^q, \end{aligned} \tag{2.4}$$

with known expressions for $\hat{k}_p(x)$ and $k_q(x)$, obtainable in terms of the points t_p, t_q, where $-1 < t_0 < t_1 < \cdots < t_m < 1$ and $-1 < t_0 < t_1 < \cdots < t_s < 1$, $t_0, t_1, \ldots, t_m$ being the zeros of $T_{m+1}(t)$ in $[-1, 1]$. We thus obtain the following functional relation to be solved for the unknown constants c_j $(j = 0, 1, \ldots, n)$:

$$\sum_{j=0}^{n} c_j^{(r)} \left[\sum_{p=0}^{m} \hat{k}_p(x) \int_{-1}^{1} \frac{t^{p+j}\lambda_r(t)}{(t-x)(1-t^2)^{1/2}}\, dt + \sum_{q=0}^{s} k_q(x) \int_{-1}^{1} \frac{t^{q+j}\lambda_r(t)}{(1-t^2)^{1/2}}\, dt\right] = g(x), \tag{2.5}$$
$$(-1 < x < 1).$$

Now, using the notations

$$\int_{-1}^{1} \frac{t^{p+j}\lambda_r(t)}{(t-x)(1-t^2)^{1/2}}\, dt = u_{p+j}^{(r)}(x) \tag{2.6}$$

and

$$\int_{-1}^{1} \frac{t^{q+j}\lambda_r(t)}{(1-t^2)^{1/2}}\, dt = \gamma_{q+j}^{(r)}, \tag{2.7}$$

where the $u_{p+j}^{(r)}(x)$ can be determined to be certain polynomials by using standard contour integration and where $\gamma_{q+j}^{(r)}$ is a constant, obtainable in terms of the β-functions, we obtain, from equation (2.5),

$$\sum_{j=0}^{n} c_j^{(r)} \left[\sum_{p=0}^{m} \hat{k}_p(x) u_{p+j}^{(r)}(x) + \sum_{q=0}^{s} k_q(x) \gamma_{q+j}^{(r)}\right] = g(x), \qquad (r = 1, 2, 3, 4), \quad -1 < x < 1. \tag{2.8}$$

Setting $x = x_l$, $l = 0, 1, 2, \ldots, n$ in relation (2.8), we obtain the following system of $(n+1) \times (n+1)$ linear equations for the determination of the unknown constants $c_j^{(r)}$, $(j = 0, 1, 2, \ldots, n)$:

$$\sum_{j=0}^{n} c_j^{(r)} \alpha_{j,l}^{(r)} = g_l, \qquad (l = 0, 1, 2 \ldots, n), \quad (r = 1, 2, 3, 4), \tag{2.9}$$

where

$$g_l = g(x_l) \tag{2.10}$$

and

$$\alpha_{j,l}^{(r)} = \sum_{p=0}^{m} \hat{k}_p(x_l) u_{p+j}^{(r)}(x_l) + \sum_{q=0}^{s} k_q(x_l) \gamma_{q+j}^{(r)}. \tag{2.11}$$

Solving the system of equations (2.9) and utilizing relations (2.1) and (2.2), we determine the approximate solution of the singular integral equation (1.1) in the form

$$f(x) \approx \lambda_r(x) \sum_{j=0}^{n} \frac{c_j^{(r)} x^j}{(1-x^2)^{1/2}}, \qquad (r = 1, 2, 3, 4). \tag{2.12}$$

We observe that if the functions $\hat{k}_p(x)$, $k_q(x)$, and $g(x)$ in relation (2.8) are replaced by their Chebyshev approximants, then the unknown constants $c_j^{(r)}$ can be determined, any desired accuracy, by comparing coefficients of like power of x from both sides. This is illustrated in the next section through simple problems.

3. PARTICULAR CASES AND EXAMPLES

For the simple equation (1.3), we select in the first instance the forcing term $g(x)$ to be a polynomial of degree one, i.e.,

$$g(x) = b_0 + b_1 x, \qquad (-1 < x < 1), \tag{3.1}$$

with b_0 and b_1 known constants. We first observe that due to (1.8), we must have for Case (IV) that

$$b_0 = 0, \tag{3.2}$$

whereas b_0 can be a nonzero constant in the other three Cases (I)–(III), for the existence of solutions. Then, using the facts that, for equation (1.3),

$$\hat{k}(t, x) = 1 \qquad \text{and} \qquad k(t, x) = 0, \tag{3.3}$$

along with relations (2.4)–(2.7) and (2.11), we obtain (by using a standard contour integral procedure, as explained in Gakhov's book [1, Problem 18, p. 81]) that

$$\alpha_{j,l}^{(r)} = u_j^{(r)}(x_l) = \pi \,\mathrm{PP}\left[x^{j-1} \lambda_r(x) \left(1 - \frac{1}{x^2}\right)^{-1/2} \right]_{x=x_l}, \tag{3.4}$$

($\mathrm{PP}[v(x)]$ representing the principal part of the expansion of $v(x)$ for large x) and equation (2.8) reduces to the simple polynomial relation

$$\sum_{j=0}^{n} c_j^{(r)} u_j^{(r)}(x) = b_0 + b_1 x, \qquad (r = 1, 2, 3, 4), \tag{3.5}$$

so that we can determine the unknown constants $c_j^{(r)}$ directly, by just comparing the coefficients of various powers of x from both sides of equation (3.5). There is thus no need to solve the system of equations (2.9) in this simple situation. The following expressions are easily found:

	$j=0$	$j=1$	$j=2$	$j=3$	$\cdots$
$u_j^{(1)}$	0	π	πx	$\pi\left(\frac{1}{2} + x^2\right)$	$\cdots$
$u_j^{(2)}$	$-\pi$	$\pi(1-x)$	$\pi\left(-\frac{1}{2} + x - x^2\right)$	$\pi\left(\frac{1}{2} - \frac{x}{2} + x^2 - x^3\right)$	$\cdots$
$u_j^{(3)}$	π	$\pi(1+x)$	$\pi\left(\frac{1}{2} + x + x^2\right)$	$\pi\left(\frac{1}{2} + \frac{x}{2} + x^2 + x^3\right)$	$\cdots$
$u_j^{(4)}$	$-\pi x$	$\pi\left(\frac{1}{2} - x^2\right)$	$\pi\left(\frac{x}{2} - x^3\right)$	$\pi\left(\frac{1}{8} + \frac{x^2}{2} - x^4\right)$	$\cdots$

(3.6)

The constants $c_j^{(r)}$ can then be determined easily and the final forms of the unknown function $f(x)$ agree with the known results obtainable from relations (1.4)–(1.7).

Let us illustrate this for the case $r = 4$. We get from equation (3.5)

$$c_0^{(4)} u_0^{(4)}(x) + c_1^{(4)} u_1^{(4)}(x) + c_2^{(4)} u_2^{(4)}(x) + \cdots = b_0 + b_1 x$$

or

$$c_0^{(4)}(-\pi x) + c_1^{(4)} \left[-\pi \left(x^2 - \frac{1}{2}\right)\right] + c_2^{(4)} \left[-\pi \left(x^3 - \frac{x}{2}\right)\right] + \cdots = b_0 + b_1 x.$$

By equating similar powers of x from both sides of the equation, and taking into account (1.8), one obtains

$$\begin{aligned} c_1^{(4)} \frac{\pi}{2} &= b_0 = 0, \\ c_0^{(4)}[-\pi] + c_2^{(4)} \left[\frac{\pi}{2}\right] &= b_1, \\ c_j^{(4)} &= 0, \qquad j = 2, 3 \ldots, \end{aligned}$$

giving as a result $c_0^{(4)} = -b_1/\pi$, $c_j^{(4)} = 0$, $j = 1, 2, \ldots$. The result for $f(x)$ follows from (2.1),(2.2) and $\lambda_4(x) = 1 - x^2$, i.e.,

$$f(x) = -b_1 \frac{\left(1 - x^2\right)^{1/2}}{\pi},$$

which is also the exact value obtained by (1.7) for $g(x)$ given by (3.1).

As a second simple example, we consider the equation

$$\int_{-1}^{1} \left[\frac{1}{t - x} + (t + x)\right] f(t)\, dt = g(t), \qquad -1 < x < 1, \tag{3.7}$$

which corresponds with $\hat{k}(t, x) = 1$ and $k(t, x) = t + x$. So, one gets

$$\begin{aligned} \hat{k}_0(x) &= 1, \qquad \hat{k}_1(x) = \hat{k}_2(x) = \cdots = 0, \\ k_0(x) &= x, \qquad k_1(x) = 1, \qquad k_2(x) = k_3(x) = \cdots = 0, \\ u_{0+j}^{(r)}(x) \equiv u_j^{(r)} &= \int_{-1}^{1} \frac{\lambda_r(t) t^j}{\left(1 - t^2\right)^{1/2} (t - x)}\, dt, \\ \gamma_j^{(r)} &= \int_{-1}^{1} \frac{t^j \lambda_r(t)}{\left(1 - t^2\right)^{1/2}}\, dt, \\ \gamma_{1+j}^{(r)} \equiv \gamma_{j+1}^{(r)} &= \int_{-1}^{1} \frac{t^{j+1} \lambda_r(t)}{\left(1 - t^2\right)^{1/2}}\, dt, \end{aligned}$$

where $j = 0, 1, 2, \ldots$. Let us consider in detail the case $r = 1$. This results in

$$\gamma_j^{(1)} = \int_{-1}^{1} \frac{t^j}{\left(1 - t^2\right)^{1/2}}\, dt.$$

In particular, we find

$$\begin{aligned} \gamma_0^{(1)} &= \pi, \qquad \gamma_1^{(1)} = 0, \\ \gamma_2^{(1)} &= \frac{\pi}{2}, \qquad \gamma_3^{(1)} = 0, \\ \gamma_4^{(1)} &= \frac{3\pi}{8}, \end{aligned} \tag{3.8}$$

and

$$u_0^{(1)} = 0, \qquad u_1^{(1)} = \pi,$$
$$u_2^{(1)} = \pi x, \qquad u_3^{(1)} = \pi\left(x^2 + \frac{1}{2}\right), \qquad \dots.$$

Thus, using (2.11), one gets for $r = 1$,

$$\alpha_{j,l}^{(1)} = u_j^{(1)}(x_l) + x_l \gamma_j^{(1)} + \gamma_{j+1}^{(1)}, \qquad (j, l = 0, 1, 2, 3, \dots),$$

which for $j = 0, 1, 2, 3$, reads as

$$\begin{aligned} \alpha_{0,l}^{(1)} &= x_l \gamma_0^{(1)} + \gamma_1^{(1)}, \\ \alpha_{1,l}^{(1)} &= \pi + x_l \gamma_1^{(1)} + \gamma_2^{(1)}, \\ \alpha_{2,l}^{(1)} &= \pi x_l + x_l \gamma_2^{(1)} + \gamma_3^{(1)}, \\ \alpha_{3,l}^{(1)} &= \pi\left(x_l^2 + \frac{1}{2}\right) + x_l \gamma_3^{(1)} + \gamma_4^{(1)}, \\ &\vdots \end{aligned} \tag{3.9}$$

or by introducing (3.8),

$$\begin{aligned} \alpha_{0,1}^{(1)} &= \pi\, x_l, \\ \alpha_{1,l}^{(1)} &= \frac{3\pi}{2}, \\ \alpha_{2,l}^{(1)} &= \frac{3\pi}{2\, x_l}, \\ \alpha_{3,l}^{(1)} &= \pi\left(x_l^2 + \frac{7}{8}\right). \end{aligned} \tag{3.10}$$

Finally, by choosing $n = 3$, we have to solve system (2.9) for Case (I), $r = 1$, i.e.,

$$\sum_{j=0}^{3} c_j^{(1)} \alpha_{j,l}^{(1)} = g_l, \qquad (l = 0, 1, 2, 3). \tag{3.11}$$

Now in the special situation when $g(x) = 1$, equation (3.7) can be expressed as

$$\int_{-1}^{1} \frac{f(t)\, dt}{t - x} = (1 + \mu_0) + \mu_1\, x, \tag{3.12}$$

with

$$\mu_0 = -\int_{-1}^{1} t f(t)\, dt, \qquad \mu_1 = -\int_{-1}^{1} f(t)\, dt. \tag{3.13}$$

We can easily solve equation (3.12), along with special conditions (3.13), by utilizing relation (1.4), in Case (I), and we find that $f(x)$ is given by

$$f(x) = \frac{B_0}{\left(1 - x^2\right)^{1/2}} + \frac{2}{3\pi}\left[x - B_0 \pi x^2\right], \tag{3.14}$$

where B_0 is an arbitrary constant.

Also, by utilizing the system of equations (3.11), with $g_l = 1$, for $l = 0, 1, 2, 3$, we obtain that, for any four chosen values of x_l $(-1 \le x_l \le 1)$,

$$\begin{aligned} c_0^{(1)} &= B_0, \qquad & c_1^{(1)} &= \frac{2}{3\pi}, \\ c_2^{(1)} &= -\frac{2}{3} B_0 \quad \text{and} & c_3^{(1)} &= 0, \end{aligned} \tag{3.15}$$

where B_0 is an arbitrary constant.

Finally, by using constants (3.15) in relation (1.2) in Case (I), i.e., for $r = 1$, we obtain the same expression (3.14), which is the exact solution for equation (3.7), in the special situation when $g = 1$.

The reason behind the matching of our approximate solutions with the exact ones of the special problems considered here as examples is that the forcing function is a polynomial and that we have taken a sufficiently large enough value for n, in our method.

Numerical solution of system (2.9) has not been pursued in the present paper.

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