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Some Remarks Concerning Exact Solution Numbers for a Class of Nonlinear Boundary Value Problems

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Results concern the exact number of solutions for a certain class of nonlinear boundary value problems together with estimates on their sizes. The methods apply to problems with fairly general boundary conditions. Neumann conditions are given special atttention.

1. INTRODUCTION

In this paper we discuss a method for determining the number (and various qualitative features) of solutions to the nonlinear differential equation $-u'' = f(x, u)$ together with nonhomogeneous boundary conditions. Throughout we assume that both f and its derivative, f_u, belong to $C([0, \pi] \times \mathbb{R}; \mathbb{R})$ and that

$$f_u(x, u) \to \begin{cases} \alpha & \text{as} \quad u \to -\infty \\ \beta & \text{as} \quad u \to \infty \end{cases}$$

with the above limits assumed to be uniform in x. For the sake of illustration the technique is applied specifically to the Neumann problem

$$\begin{gathered} -u'' = f(x, u) \\ u'(0) = \sigma_1, \qquad u'(\pi) = \sigma_2. \end{gathered} \tag{1}$$

The only restrictions placed on the parameters α and β is that they be taken positive as this ensures that the model problem (4) below is an oscillator. The Neumann problem has been emphasized here for two reasons; first that it exhibits interesting bifurcation behavior, a pitchfork or cusp bifurcation, with respect to the boundary data, and second that the shooting maps, which we subsequently give a brief description of, have been worked out in detail, cf. [3].

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There have been a number of papers written on this problem concerning both the ordinary differential equation and the partial differential equation. The references at the end of this paper list several of them [1–11]. A more comprehensive list can be found in the survey article by Lazer and McKenna [8] who summarize much of what is currently known about these kinds of problems and their connection with modeling severe oscillations in suspension bridges. A common theme in these papers has been to obtain lower estimates on solution numbers as a function of both the parameters α and β and the size of the projection of f into the principal eigenspace of the operator $Lu := -\Delta u$ with homogeneous Dirichlet or Neumann boundary conditions. Our motivation here is, mainly, to obtain as much precision on the number of solutions as we can for a given set of parameters, although additional qualitative information on these solutions will be obtained along the way; e.g., the quantity and approximate locations of their zeros, growth rates, estimates of the norms, etc. Again, we focus on the Neumann problem, although the techniques that we use apply to Dirichlet, periodic, and other boundary cases. It is of interest that the methods needed to obtain the results in this paper are rather straightforward in that they involve only the use of a shooting procedure together with estimates on the size of solutions and the size of derivatives with respect to a shooting parameter θ that was introduced in relation to the problem in [3, 2].

2. Preliminaries

The assumptions on the nonlinear term f, stated in the Introduction, imply that it can be written in the form $f(x, u) = \alpha u^- + \beta u^+ + v(x, u)$ where $v(x, u)$ is differentiable with respct to u in $\mathbb{R}\setminus\{0\}$ and $v_u(x, u) \to 0$ as $|u| \to \infty$ uniformly in x. Fix $\tau > 0$ and consider the two differential equations $-u'' = f(x, u)$ and $-u'' = \alpha u^- + \beta u^+ + (1/\tau)\, v(t, \tau u)$. It is easy to check that u solves the former with boundary values $u'(0) = \tau\sigma_1$ and $u'(\pi) = \tau\sigma_2$ if and only if w, where $u = \tau w$, solves the latter with boundary conditions $w'(0) = \sigma_1$ and $w'(\pi) = \sigma_2$. This describes, for each $\tau > 0$ and pair of boundary values σ_1 and σ_2, a bijective correspondence between the solution sets of the following two boundary value problems:

$$\begin{gathered} -u'' = f(x, u) \\ u'(0) = \tau\sigma_1, \qquad u'(\pi) = \tau\sigma_2 \end{gathered} \tag{2}$$

and

$$\begin{gathered} -u'' = -\alpha u^- + \beta u^+ + \frac{1}{\tau} v(x, \tau u) \\ u'(0) = \sigma_1, \qquad u'(\pi) = \sigma_2. \end{gathered} \tag{3}$$

We formally state this situation in the following lemma.

LEMMA 2.1. *Fix σ_1, σ_2, and $\tau > 0$. Then problem* (3) *has exactly n solutions in the ball $B_r(0) \subset C[0, \pi]$ if and only if problem* (2) *has exactly n solutions inside the ball $B_{\tau r}(0) \subset C[0, \pi]$.*

In light of this lemma and our goal of obtaining an exact number of solutions we compare the boundary value problem (3) with the following *model* problem:

$$\begin{aligned} -u'' &= \alpha u^- + \beta u^+ \\ u'(0) = \sigma_1, &\qquad u'(\pi) = \sigma_2. \end{aligned} \tag{4}$$

Note that u is a solution of (4) with $\sigma_1 > 0$ ($\sigma_1 < 0$) if and only if $w = (1/\sigma_1)u$ ($w = -(1/\sigma_1)u$) is a solution of (4) with $w'(0) = 1$ ($w'(0) = -1$). Also, u is a solution of (4) if and only if $w = -u$ solves the problem

$$\begin{aligned} -w'' &= \tilde{\alpha} w^- + \tilde{\beta} w^+ \\ w'(0) = -\sigma_1, &\qquad w'(\pi) = -\sigma_2, \end{aligned}$$

where $\tilde{\alpha} = \beta$ and $\tilde{\beta} = \alpha$. With these symmetries in mind it is sufficient, in the subsequent analysis of (4), to consider only the cases $\sigma_1 = 0$ and $\sigma_1 = 1$.

The analysis of the oscillator (4) begins with writing it as the system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -\alpha u^- - \beta u^+ \end{aligned} \tag{5}$$

and looking at the trajectories of this system in the phase plane. The analyis of the phase plane structure for this problem can be found in [2] for the Dirichlet problem and in [3] for the Neumann problem. We close this section with a short outline of the shooting procedure used in [3] and then summarize in Theorem 2.2 the essential properties of the shooting maps as described in [3].

2.1. *The Shooting Procedure*

For $\theta \in (-\pi/2, \pi/2)$, we let $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right][t; \theta)$ denote the solution of (5) with starting value $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](0; \theta) = (1/c)\left[\begin{smallmatrix} \tan\theta \\ c \end{smallmatrix}\right]$. We assume that $c := \sqrt{\alpha}$ if $\theta \leqslant 0$ and $c := \sqrt{\beta}$ if $\theta > 0$. These choices facilitate certain computations involved in the shooting procedure. Note that $v_0(0; \theta) \equiv 1$.

Define the map $\sigma_{\alpha\beta}$: $(-\pi/2, \pi/2) \to \mathbb{R}$ by $\sigma_{\alpha\beta}(\theta) := v_0(\pi; \theta)$ if and only if $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](0; \theta) = (1/c)\left[\begin{smallmatrix} \tan\theta \\ c \end{smallmatrix}\right]$. Thus $\sigma_{\alpha\beta}(\theta)$ is the slope of the solution of the differential equation in (4) with initial (shooting) conditions $u(0) = \tan\theta/c$ and $u'(0) = 1$. We could also define a map, say, $\tilde{\sigma}_{\alpha\beta}$: $(-\pi/2, \pi/2) \to \mathbb{R}$ by letting $\tilde{\sigma}_{\alpha\beta}(\theta)$ denote the slope of the solution of the differential equation in (4) at $t=\pi$ with initial conditions $u(0)=-\tan\theta/c$ and $u'(0)=-1$. The symmetry discussion above can be used to show that $\tilde{\sigma}_{\alpha\beta}=-\sigma_{\beta\alpha}$. Evidently then, it is sufficient, when looking for solutions to problem (1) with $\sigma_1 \neq 0$, to study either the family of maps $\{\sigma_{\alpha\beta}\}$ for α and β positive or, assume that $\alpha<\beta$, and then study *all* the maps $\{\sigma_{\alpha\beta}, \tilde{\sigma}_{\alpha\beta}\}$.

Since the right hand side of system (5) fails to be C^1 only when $u=0$ the solutions $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right](t;\theta)$ may not be differentiable with respect to θ when $u_0=0$. In what follows this may seem to be a problem but, in fact, does little damage as the zero set of $u_0(t;\theta)$ does not have a qualitative effect on the shooting map $\theta \to v_0(\pi, \theta)$. The zero set of $u_0(t, \theta)$ is also easy to describe. One just observes first that solutions of the oscillator (5) have period determined from two times. $\pi/\sqrt{\beta}$ is the time the solution spends in the right half plane, $u>0$, and $\pi/\sqrt{\alpha}$ is the time the solution spends in the left half plane, $u<0$; thus, $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]$ has period $\pi/\sqrt{\alpha}+\pi/\sqrt{\beta}$. The zeros of u_0 in $(-\pi/2, \pi/2)\times[0, \pi]$ can be determined then by knowing the first time, t_1, that $u_0(t, \theta)=0$. For if we let t_2 denote the second time that $u_0(t,\theta)=0$ and t_3 the third, etc., then

$$t_n=\begin{cases} t_1+\dfrac{k\pi}{\sqrt{\beta}}+\dfrac{(k-1)\pi}{\sqrt{\alpha}} & \text{if } n=2k \text{ and } \theta\leqslant 0\\[2ex] t_1+\dfrac{k\pi}{\sqrt{\alpha}}+\dfrac{(k-1)\pi}{\sqrt{\beta}} & \text{if } n=2k \text{ and } \theta>0\\[2ex] t_1+\dfrac{k\pi}{\sqrt{\beta}}+\dfrac{k\pi}{\sqrt{\alpha}} & \text{if } n=2k+1. \end{cases}$$

It can be checked that $t_1=-\theta/\sqrt{\alpha}$ if $\theta\leqslant 0$ and $t_1=(\pi-\theta)/\sqrt{\beta}$ for $\theta>0$.

For fixed $\tau>0$ we regard the following system as a perturbation of (5):

$$\dot{u}=v, \qquad \dot{v}=-\alpha u^- -\beta u^+ -\frac{1}{\tau}\,v(t, \tau u). \tag{6}$$

For $\theta\in(-\pi/2, \pi/2)$, denote by $\left[\begin{smallmatrix}u_\tau\\ v_\tau\end{smallmatrix}\right](t;\theta)$ solutions of system (6) having the same starting values as the solutions $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right](t,\theta)$ of system (5); that is, $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right](0;\theta)=\left[\begin{smallmatrix}u_\tau\\ v_\tau\end{smallmatrix}\right](0;\theta)=(1/c)\left[\begin{smallmatrix}\tan\theta\\ c\end{smallmatrix}\right]$, where the number c is defined as before. An integration of the difference $(d/dt)\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right](t;\theta)-(d/dt)\left[\begin{smallmatrix}u_\tau\\ v_\tau\end{smallmatrix}\right](t;\theta)$ yields

$$\begin{aligned}&\begin{bmatrix}u_0\\ v_0\end{bmatrix}(t;\theta)-\begin{bmatrix}u_\tau\\ v_\tau\end{bmatrix}(t;\theta)\\ &\quad=\int_0^t\begin{bmatrix}v_1(s;\theta)-v_2(s;\theta)\\ -\alpha(u_1^-(s;\theta)-u_2^-(s;\theta))-\beta(u_1^+(s;\theta)-u_2^+(s;\theta))\end{bmatrix}ds\\ &\qquad+\frac{1}{\tau}\int_0^t\begin{bmatrix}0\\ v(s, \tau u_2(s;\theta))\end{bmatrix}ds.\end{aligned} \tag{7}$$

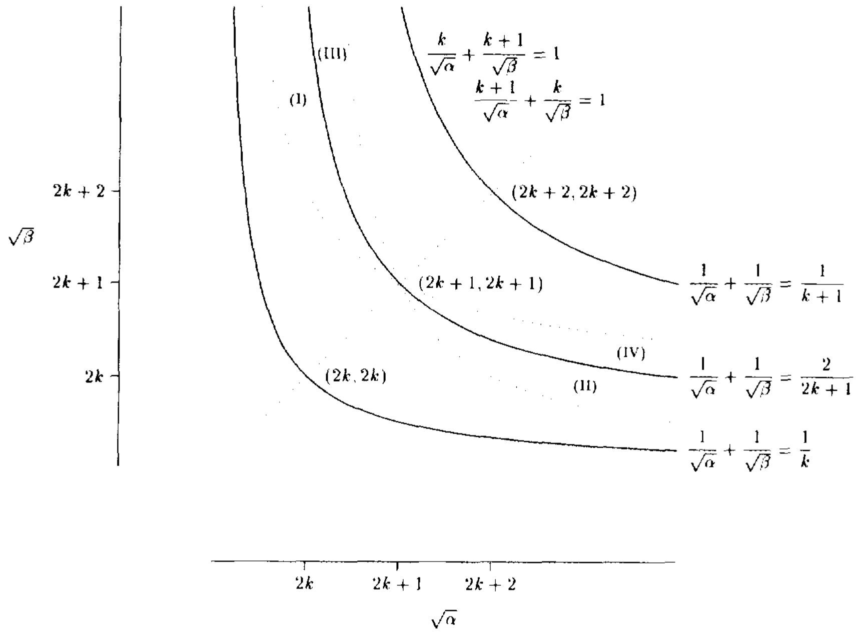

FIG. 1. A piece of the *spectrum* for problem (4).

The goal of this paper is to compare the two maps $\theta \mapsto v_0(\pi, \theta)$, i.e., $\theta \mapsto \sigma_{\alpha\beta}$, and $\theta \mapsto v_\tau(\pi, \theta)$, $\theta \in (-\pi/2, \pi/2)$. The qualitative features of the first map, for positive parameters α and β, is summarized in Theorem 2.2 below. For the second it will be argued that, as $\tau \to \infty$, the map $\theta \mapsto v_\tau(\pi, \theta)$ converges in a C^1 sense to $\theta \mapsto v_0(\pi, \theta)$ on compact subsets of $(-\pi/2, \pi/2)$. The analysis then proceeds in the next section by exploiting Eq. (7) together with a Gronwall inequality used to estimate bounds on the size of the solutions, $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right]$, and their derivatives $(\partial/\partial\theta)\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right]$.

For brevity in what follows we call a piecewise C^1 map h : $(-\pi/2, \pi/2) \to \mathbb{R}$ *cusplike* if it is *onto* $\mathbb{R}$ and has exactly two local extrema, one a strict minimum and the other a strict maximum, with those extrema being the only points x at which either of the one sided derivatives $h'(x+)$ or $h'(x-)$ can vanish.

The qualitative behavior of the map $\theta \to v_0(\pi; \theta)$ depends on the choice of the parameters α and β and is summarized in the theorem below. Figure 1 aided us in visualizing the parameter regions involved and so we include it here. Figure 2 is a sketch of a cusplike shooting map.

THEOREM 2.2. *Assume that $\alpha > 0$ and $\beta > 0$.*

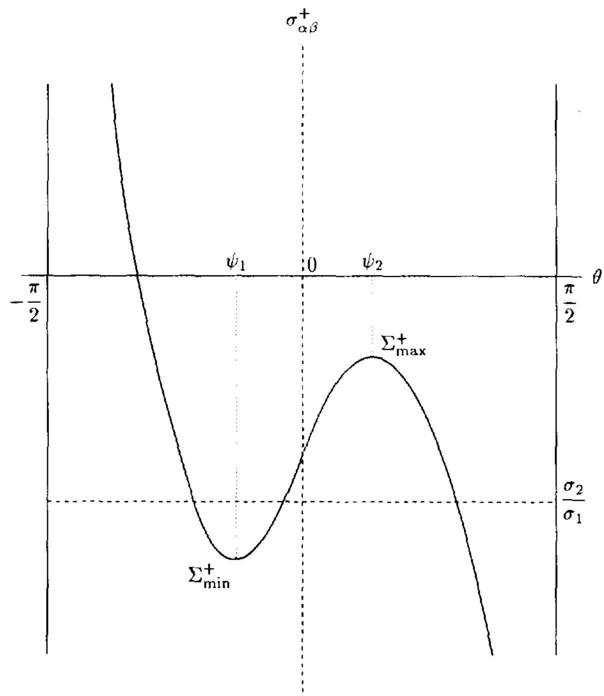

FIG. 2. A cusplike map $\sigma_{\alpha\beta}$ when α and β satisfy inequality (I) of Theorem 2.2.

(A) *Suppose that*

$$\text{(I)}\quad \frac{k}{\sqrt{\alpha}}+\frac{k+1}{\sqrt{\beta}}<1<\frac{k+(1/2)}{\sqrt{\alpha}}+\frac{k+(1/2)}{\sqrt{\beta}}$$

or

$$\text{(IV)}\quad \frac{k+(1/2)}{\sqrt{\alpha}}+\frac{k+(1/2)}{\sqrt{\beta}}<1<\frac{k}{\sqrt{\alpha}}+\frac{k+1}{\sqrt{\beta}};$$

then $\sigma_{\alpha\beta}\colon(-\pi/2, \pi/2)\to\mathbb{R}$ *is cusplike with a negative local minimum* $\sum_{min}$ *at the point* $\theta=\psi_1$ *and a negative local maximum* $\sum_{max}$ *at the point* $\theta=\psi_2$, $\psi_1<\psi_2$.

(B) *If* (α, β) *does not belong to either of the two regions* (I) *or* (II) *in* (A) *above and does not lie on any curve of the form* $1/\sqrt{\alpha}+1/\sqrt{\beta}=2/k$, $k/\sqrt{\alpha}+(k+1)/\sqrt{\beta}=1$ *for* $\alpha<\beta$, $(k+1)/\sqrt{\alpha}+k/\sqrt{\beta}=k$ *for* $\alpha>\beta$, *for some* $k\in\mathbb{N}$ *then* $\sigma_{\alpha\beta}$ *is a homeomorphism onto* $\mathbb{R}$ *with the one sided derivatives* $\sigma'_{\alpha\beta}(\theta^-)$ *and* $\sigma'_{\alpha\beta}(\theta^+)$ *either always positive or always negative.*

(C) *If* $k/\sqrt{\alpha}+(k+1)/\sqrt{\beta}=1$ *with* $\alpha<\beta$ *then* $\sigma_{\alpha\beta}$ *is a* C^1 *homeomorphism onto* $\mathbb{R}$ *with its derivative negative except at the point* $\theta=0$ *where it vanishes.*

(D) *Finally, if* $1/\sqrt{\alpha}+1/\sqrt{\beta}=2/(2k+1)$ *then* $\sigma_{\alpha\beta}$ *maps the interval* $(-\pi/2, \pi/2)$ *homeomorphically onto the interval* $(-\sqrt{\beta/\alpha}, -\sqrt{\alpha/\beta})$ *with* $\sigma'_{\alpha\beta}(\theta^-)$ *and* $\sigma'_{\alpha\beta}(\theta^+)$ *always positive if* $\alpha<\beta$ *and always negative if* $\alpha>\beta$.

Just for completeness, we observe that if $1/\sqrt{\alpha}+1/\sqrt{\beta}=1/k$ *then* $\sigma^+_{\alpha\beta}\equiv 1$.

We remark that the conditions in (C) and, respectively, (D) on α and β are precisely the conditions required on these parameters so that the differential equation $-u''=\alpha u^- + \beta u^+$ together with the homogeneous Dirichlet boundary conditions $u(0)=u(\pi)=0$, and, respectively, the homogeneous Neumann boundary conditions $u'(0)=u'(\pi)=0$ have nontrivial solutions. A proof of this theorem can be found in [3].

3. Estimates

This section contains a sequence of technical lemmas giving the estimates needed to prove Theorem 3.5. In what follows we use the notation $f^\tau(x,u):=\alpha u^- + \beta u^+ + (1/\tau)\, v(x, \tau u)$ and $\|[{u \atop v}]\| := \max\{|u|, |v|\}$.

LEMMA 3.1. *If* $[\theta_1, \theta_2]\subset(-\pi/2, \pi/2)$ *then there is a positive constant* $M=M(\theta_1,\theta_2)$ *such that* $\|[{u_\tau \atop v_\tau}](t;\theta)\|\leqslant M$ *for all* $0\leqslant t\leqslant\pi$, $\theta_1\leqslant\theta\leqslant\theta_2$, *and* $\tau\geqslant 1$.

Proof. Integrate (6) to get

$$\begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix}(t;\theta)=\frac{1}{c}\begin{bmatrix}\tan\theta \\ c\end{bmatrix}+\int_0^t\begin{bmatrix} v_\tau(s;\theta) \\ -f^\tau(s, u_\tau(s;\theta))\end{bmatrix}ds$$

and then estimate

$$\left\|\begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix}(t;\theta)\right\|\leqslant\frac{1}{c}\left\|\begin{bmatrix}\tan\theta \\ c\end{bmatrix}\right\|+\int_0^t\left\|\begin{bmatrix} v_\tau(s;\theta) \\ -f^\tau(s, u_\tau(s;\theta))\end{bmatrix}\right\|ds.$$

If $C:=\max\{\alpha,\beta\}$ then

$$\begin{aligned}|f^\tau(s,u)| &\leqslant C\,|u|+\frac{1}{\tau}\,|v(s,\tau u)| \\ &\leqslant (C+1)\,|u|+\frac{1}{\tau}M_1\end{aligned}$$

because $|v(s,u)| \leqslant |u| + M_1$, for some $M_1 > 0$ and all $(s,u) \in [0,\pi] \times \mathbb{R}$. Consequently,

$$\left\| \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (t;\theta) \right\| \leqslant \frac{1}{c} \left\| \begin{bmatrix} \tan\theta \\ c \end{bmatrix} \right\| + (C+1) \int_0^t \left\| \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (s;\theta) \right\| ds + \frac{1}{\tau} M_1 t.$$

Gronwall's lemma yields the estimate

$$\left\| \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (t;\theta) \right\| \leqslant \left(\frac{1}{\tau} M_1 \pi + \frac{1}{c} \left\| \begin{bmatrix} \tan\theta \\ c \end{bmatrix} \right\| \right) e^{(C+1)t}. \quad \blacksquare$$

LEMMA 3.2. *If* $[\theta_1, \theta_2] \subset (-\pi/2, \pi/2)$ *then* $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right](t;\theta) \to \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](t;\theta)$ *as* $\tau \to \infty$ *uniformly on* $[0,\pi] \times [\theta_1, \theta_2]$.

Proof. From Eq. (7)

$$\left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} (t;\theta) - \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (t;\theta) \right\|$$

$$\leqslant k \int_0^t \left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} (s;\theta) - \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (s;\theta) \right\| ds + \frac{1}{\tau} \int_0^t |v(s, \tau u_\tau(s;\theta))| \, ds$$

for some $k = k(\alpha, \beta) > 0$. It follows that

$$\left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} (t;\theta) - \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (t;\theta) \right\| \leqslant \frac{1}{\tau} e^{kt} \int_0^\pi |v(s, \tau u_\tau(s;\theta))| \, ds.$$

Fix $\varepsilon > 0$ and let M be as in the proof of previous lemma. Let $M_\varepsilon > 0$ be such that $|v(s,u)| \leqslant \varepsilon |u| + M_\varepsilon$ for $(s,u) \in [0,\pi] \times \mathbb{R}$. Then

$$\left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} (t;\theta) - \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} (t;\theta) \right\| \leqslant \frac{1}{\tau} e^{k\pi} \int_0^\pi (\varepsilon\tau \, |u_\tau(s;\theta)| + M_\varepsilon) \, ds$$

$$\leqslant e^{k\pi} \pi \left(M\varepsilon + \frac{M_\varepsilon}{\tau} \right). \quad \blacksquare$$

LEMMA 3.3. *If* $[\theta_1, \theta_2] \in (-\pi/2, \pi/2)$ *then there exists a positive constant* $\tilde{M}$ *such that* $\left\| \left[\begin{smallmatrix} \partial u_\tau/\partial\theta \\ \partial v_\tau/\partial\theta \end{smallmatrix}\right](t;0) \right\| \leqslant \tilde{M}$ *for all* $0 \leqslant t \leqslant \pi$, $\theta_1 \leqslant \theta \leqslant \theta_2$, *and* $\tau \geqslant 1$.

Proof. Integrating (6) again and then differentiating with respect to θ gives

$$\begin{bmatrix} \dfrac{\partial u_\tau}{\partial\theta} \\ \dfrac{\partial v_\tau}{\partial\theta} \end{bmatrix} (t;\theta) = \frac{1}{c} \begin{bmatrix} \sec^2\theta \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} \dfrac{\partial v_\tau}{\partial\theta}(s;\theta) \\ -\dfrac{\partial f^\tau}{\partial u}(s, u_\tau(s;\theta)) \dfrac{\partial u_\tau}{\partial\theta}(s;\theta) \end{bmatrix} ds$$

and hence,

$$\left\|\begin{bmatrix}\dfrac{\partial u_\tau}{\partial\theta}\\[2mm] \dfrac{\partial v_\tau}{\partial\theta}\end{bmatrix}(t;\theta)\right\| \leqslant \frac{1}{c}\sec^2\theta + \int_0^t \left\|\begin{bmatrix}\dfrac{\partial v_\tau}{\partial\theta}(s;\theta)\\[2mm] -\dfrac{\partial f^\tau}{\partial u}(s;u_\tau(s;\theta))\dfrac{\partial u_\tau}{\partial\theta}(s;\theta)\end{bmatrix}\right\| ds.$$

Note that $f^\tau(s,u)=f(s,u)-v(s,u)+(1/\tau)\,v(s,\tau u)$ and so, for $u\neq 0$,

$$\frac{\partial f^\tau}{\partial u}(s,u)=\frac{\partial f}{\partial u}(s,u)-\frac{\partial v}{\partial u}(s,u)+\frac{\partial v}{\partial u}(s,\tau u).$$

Since $(\partial v/\partial u)(s,0^+)=(\partial f/\partial u)(s,0)-\beta$ and $(\partial v/\partial u)(s,0^-)=(\partial f/\partial u)(s,0)-\alpha$, i.e., since right and left hand derivatives of $\partial v/\partial u$ at $u=0$ exist are continuous in the variable s, and since $\partial v/\partial u$ is bounded for $u\neq 0$ then the family $\{\partial f^\tau/\partial u\}_{\tau>0}$ is a uniformly bounded subset of $C([0,\pi]\times\mathbb{R};\mathbb{R})$. Hence, for some positive constant $A>0$,

$$\left\|\begin{bmatrix}\dfrac{\partial u_\tau}{\partial\theta}\\[2mm] \dfrac{\partial v_\tau}{\partial\theta}\end{bmatrix}(t;\theta)\right\|$$

$$\leqslant \frac{1}{c}\sec^2\theta + A\int_0^t \left\|\begin{bmatrix}\dfrac{\partial u_\tau}{\partial\theta}\\[2mm] \dfrac{\partial v_\tau}{\partial\theta}\end{bmatrix}(s;\theta)\right\| ds, \qquad 0\leqslant t\leqslant\pi,\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

It follows that

$$\left\|\begin{bmatrix}\dfrac{\partial u_\tau}{\partial\theta}\\[2mm] \dfrac{\partial v_\tau}{\partial\theta}\end{bmatrix}(t;\theta)\right\| \leqslant \frac{1}{c}\sec^2\theta e^{At}, \qquad 0\leqslant t\leqslant\pi,\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right). \quad \blacksquare$$

LEMMA 3.4. *If* $[\theta_1,\theta_2]\subset(-\pi/2,\pi/2)$ *then for each* $\varepsilon>0$ *there is a* $T>0$ *such that if* $\tau\geqslant T$ *then* $\|\left[\begin{smallmatrix}\partial u_\tau/\partial\theta\\ \partial v_\tau/\partial\theta\end{smallmatrix}\right](t;\theta)-\left[\begin{smallmatrix}\partial u_0/\partial\theta\\ \partial v_0/\partial\theta\end{smallmatrix}\right](t;\theta)\|<\varepsilon$ *for* $(t;\theta)\in[0,\pi]\times[\theta_1,\theta_2]$, *at least wherever the derivative* $\left[\begin{smallmatrix}\partial u_0/\partial\theta\\ \partial v_0/\partial\theta\end{smallmatrix}\right](t;\theta)$ *exists.*

Proof. From Eq. (7) we have

$$\begin{bmatrix} \frac{\partial u_0}{\partial \theta} \\ \frac{\partial v_0}{\partial \theta} \end{bmatrix}(t;\theta) - \begin{bmatrix} \frac{\partial u_\tau}{\partial \theta} \\ \frac{\partial v_\tau}{\partial \theta} \end{bmatrix}(t;\theta)$$

$$= \int_0^t \begin{bmatrix} \frac{\partial v_0}{\partial \theta}(s;\theta) - \frac{\partial v_\tau}{\partial \theta}(s;\theta) \\ -\alpha\left(\frac{\partial u_0^-}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^-}{\partial \theta}(s;\theta)\right) - \beta\left(\frac{\partial u_0^+}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^+}{\partial \theta}(s;\theta)\right) \end{bmatrix} ds$$

$$-\frac{1}{\tau}\int_0^t \begin{bmatrix} 0 \\ v'(s, \tau u_\tau(s;\theta))\frac{\partial u_\tau}{\partial \theta}(s;\theta) \end{bmatrix} ds,$$

at least at points for which $[\begin{smallmatrix} \partial u_0/\partial\theta \\ \partial v_0/\partial\theta \end{smallmatrix}](t;\theta)$ exist. Consequently,

$$\left\| \begin{bmatrix} \frac{\partial u_0}{\partial \theta} \\ \frac{\partial v_0}{\partial \theta} \end{bmatrix}(t;\theta) - \begin{bmatrix} \frac{\partial u_\tau}{\partial \theta} \\ \frac{\partial v_\tau}{\partial \theta} \end{bmatrix}(t;\theta) \right\|$$

$$\leqslant \int_0^t \left\| \begin{bmatrix} \frac{\partial v_0}{\partial \theta}(s;\theta) - \frac{\partial v_\tau}{\partial \theta}(s;\theta) \\ -\alpha\left(\frac{\partial u_0^-}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^-}{\partial \theta}(s;\theta)\right) - \beta\left(\frac{\partial u_0^+}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^+}{\partial \theta}(s;\theta)\right) \end{bmatrix} \right\| ds$$

$$+\int_0^t \left| v'(s, \tau u_\tau(s;\theta))\frac{\partial u_\tau}{\partial \theta}(s;\theta)\right| ds. \tag{8}$$

Now fix $\varepsilon > 0$ and the interval $[\theta_1, \theta_2] \subset (-\pi/2, \pi/2)$. From Lemma 3.2 we can choose $\tau > 0$ large enough to ensure that

$$|u_0(t;\theta) - u_\tau(t;\theta)| < \varepsilon/2, \qquad 0 \leqslant t \leqslant \pi \quad \theta_1 \leqslant \theta \leqslant \theta_2.$$

For such τ, u_0 and u_τ have the same sign on the set $\{(s;\theta) \mid 0 \leqslant s \leqslant \pi, \theta_1 \leqslant \theta \leqslant \theta_2, |u_0(s;\theta)| \geqslant \varepsilon\}$. For each θ and t define $Q_t := \{s \mid 0 \leqslant s \leqslant t, |u_0(s;\theta)| \geqslant \varepsilon\}$ and $\tilde{Q}_t := \{s \mid 0 \leqslant s \leqslant t, |u_0(s;\theta)| < \varepsilon\}$; then

$$\int_0^t \left|\frac{\partial u_0^+}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^+}{\partial \theta}(s;\theta)\right| ds \leqslant \int_{Q_t} \left|\frac{\partial u_0^+}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^+}{\partial \theta}(s;\theta)\right| ds$$

$$+\int_{\tilde{Q}_t} \left|\frac{\partial u_0^+}{\partial \theta}(s;\theta)\right| ds + \int_{\tilde{Q}_t} \left|\frac{\partial u_\tau^+}{\partial \theta}(s;\theta)\right| ds$$

$$\leqslant \int_0^t \left|\frac{\partial u_0}{\partial \theta}(s;\theta) - \frac{\partial u_\tau}{\partial \theta}(s;\theta)\right| ds + M_2\mu(\tilde{Q}_\pi),$$

where M_2 is some positive constant and μ denotes Lebesgue measure on $\mathbb{R}$. The constant M_2 can be chosen so that we also have

$$\int_0^t \left| \frac{\partial u_0^+}{\partial \theta}(s;\theta) - \frac{\partial u_\tau^+}{\partial \theta}(s;\theta) \right| ds \leqslant \int_0^t \left| \frac{\partial u_0}{\partial \theta}(s;\theta) - \frac{\partial u_\tau}{\partial \theta}(s;\theta) \right| ds + M_2 \mu(\tilde{Q}_\pi).$$

Using these estimates in (8) yields, for some constant $k = k(\alpha, \beta)$,

$$\begin{aligned} &\left\| \begin{bmatrix} \dfrac{\partial u_0}{\partial \theta} \\ \dfrac{\partial v_0}{\partial \theta} \end{bmatrix} (t;\theta) - \begin{bmatrix} \dfrac{\partial u_\tau}{\partial \theta} \\ \dfrac{\partial v_\tau}{\partial \theta} \end{bmatrix} (t;\theta) \right\| \\ &\qquad \leqslant k \int_0^t \left\| \begin{bmatrix} \dfrac{\partial u_0}{\partial \theta} \\ \dfrac{\partial v_0}{\partial \theta} \end{bmatrix} (t;\theta) - \begin{bmatrix} \dfrac{\partial u_\tau}{\partial \theta} \\ \dfrac{\partial v_\tau}{\partial \theta} \end{bmatrix} (t;\theta) \right\| ds \\ &\qquad\quad + \tilde{M} \int_0^t |v'(s, \tau u_\tau(s;\theta))| \, ds + k M_2 \mu(\tilde{Q}_\pi), \end{aligned}$$

where $\tilde{M}$ is a bound on the term $\partial u_\tau / \partial \theta$, cf. Lemma 3.3. By Gronwall's inequality

$$\begin{aligned} &\left\| \begin{bmatrix} \dfrac{\partial u_0}{\partial \theta} \\ \dfrac{\partial v_0}{\partial \theta} \end{bmatrix} (t;\theta) - \begin{bmatrix} \dfrac{\partial u_\tau}{\partial \theta} \\ \dfrac{\partial v_\tau}{\partial \theta} \end{bmatrix} (t;\theta) \right\| \\ &\qquad \leqslant k M_2 e^{kt} \mu\{s \in [0, \pi] \mid |u_0(s;\theta)| < \varepsilon\} + \tilde{M} e^{kt} \int_0^\pi |v'(s, \tau u_\tau(s;\theta))| \, ds. \end{aligned}$$

Consider the above integram term $\int_0^\pi |v'(s, \tau u_\tau(s;\theta))| \, ds$. We have

$$\begin{aligned} \int_0^\pi |v'(s, \tau u_\tau(s;\theta))| \, ds &= \int_{\tilde{Q}_\pi} |v'(s, \tau u_\tau(s;\theta))| \, ds + \int_{Q_\pi} |v'(s, \tau u_\tau(s;\theta))| \, ds \\ &\leqslant M_3 \mu(\tilde{Q}_\pi) + \int_{Q_\pi} |v'(s; \tau u_\tau(s;\theta))| \, ds \end{aligned}$$

for some $M_3 > 0$. Using Lemma 3.2 and the fact that $v'(s, u) \to 0$ uniformly in s as $|u| \to \infty$ shows that

$$\int_{Q_\pi} |v'(s, \tau u_\tau(s;\theta))| \, ds \to 0 \qquad \text{as} \quad \tau \to \infty.$$

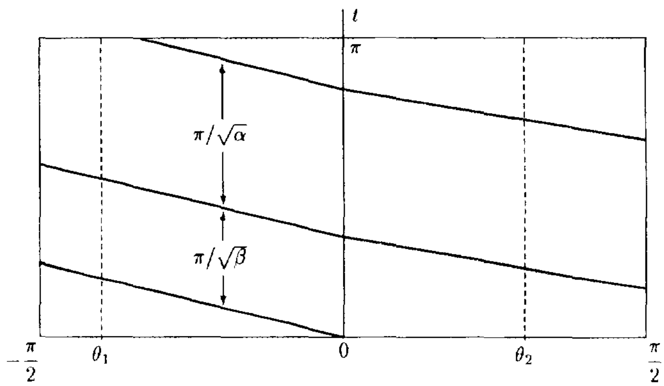

FIG. 3. The zero set of u_0.

Finally, we observe that $\mu\{s \in [0, \pi] \mid |u_0(s; \theta)| < \varepsilon\} \to 0$ as $\varepsilon \to 0$ independently of $\theta \in [\theta_1, \theta_2]$ (see Fig. 3), which finishes the proof. ∎

Lemmas 3.2 and 3.4 together imply the following theorem.

THEOREM 3.5. $v_\tau(\pi; \theta) \to \sigma_{\alpha\beta}(\theta)$ *uniformly on each compact subinterval* $[\theta_1, \theta_2] \subset (-\pi/2, \pi/2)$. *Moreover, if* $[\theta_1, \theta_2]$ *is an interval on which* $\sigma'_{\alpha\beta}(\theta^{\pm})$ *is positive (negative) then, for* τ sufficiently large, $(d/d\theta)\, v_\tau(\pi; \theta)$ *is positive (negative) for all* $\theta \in [\theta_1, \theta_2]$.

4. CONCLUSIONS

The previous results are gathered together in the next theorem about problem (9):

$$\begin{gathered} -u'' = f(x, u) \\ u'(0) = \tau\sigma_1, \qquad u'(\pi) = \tau\sigma_2. \end{gathered} \tag{9}$$

THEOREM 4.1. *Assume that* $\alpha > 0$ *and* $\beta > 0$.

(A) *Suppose that*

$$\text{(I)} \quad \frac{k}{\sqrt{\alpha}} + \frac{k+1}{\sqrt{\beta}} < 1 < \frac{k+(1/2)}{\sqrt{\alpha}} + \frac{k+(1/2)}{\sqrt{\beta}}$$

or

$$\text{(IV)} \quad \frac{k+(1/2)}{\sqrt{\alpha}} + \frac{k+(1/2)}{\sqrt{\beta}} < 1 < \frac{k}{\sqrt{\alpha}} + \frac{k+1}{\sqrt{\beta}};$$

then there exist numbers $\sum_{\min} < \sum_{\max} < 0$, *which define the open region* $\mathscr{R} := \{(x, y) \mid x > 0,\ \sum_{\min} x < y < \sum_{\max} x\}$ *in the right half plane, such that if* $(\sigma_1, \sigma_2) \in \mathscr{R}$ *then problem* (9), *for* τ *sufficiently large, has exactly three solutions,* u_τ^1, u_τ^2, *and* u_τ^3, *such that* $(1/\tau\sigma_1)\, u_\tau^i \to u^i$ *uniformly as* $\tau \to \infty$, *where* u^i *is a solution of problem* (4) *with*

$$(u^i)'(0) = 1 \quad \text{and} \quad (u^i)'(\pi) = \frac{\sigma_1}{\sigma_2}, \qquad i = 1, 2, 3.$$

If $(\sigma_1, \sigma_2) \in \mathbb{R}^2 \setminus \bar{\mathscr{R}}$ *then, for* τ *sufficiently large, problem* (9) *has a unique solution* u_τ^0 *such that* $(1/\tau\, |\sigma_1|)\, u_\tau^0 \to u^0$ *as* $\tau \to \infty$, *where* u^0 *is a solution of problem* (4) *with*

$$(u^0)'(0) = \begin{cases} -1 & \text{if } \sigma_1 < 0 \\ 0 & \text{if } \sigma_1 = 0 \\ 1 & \text{if } \sigma_1 > 0 \end{cases}$$

and

$$(u^0)'(\pi) = \begin{cases} -\dfrac{\sigma_2}{\sigma_1} & \text{if } \sigma_1 < 0 \\ \sigma_2 & \text{if } \sigma_1 = 0 \\ \dfrac{\sigma_2}{\sigma_1} & \text{if } \sigma_1 > 0. \end{cases}$$

(B) *If*

$$\text{(II)} \quad \frac{k+1}{\sqrt{\alpha}} + \frac{k}{\sqrt{\beta}} < 1 < \frac{k+(1/2)}{\sqrt{\alpha}} + \frac{k+(1/2)}{\sqrt{\beta}}$$

or

$$\text{(III)} \quad \frac{k+(1/2)}{\sqrt{\alpha}} + \frac{k+(1/2)}{\sqrt{\beta}} < 1 < \frac{k+1}{\sqrt{\alpha}} + \frac{k}{\sqrt{\beta}}$$

then there are numbers $\sum_{\max} > \sum_{\min} > 0$, *defining the open region* $\tilde{\mathscr{R}} := \{(x, y) \mid x < 0,\ \sum_{\min} x < y < \sum_{\max} x\}$ *in the left half plane, such that if* $(\sigma_1, \sigma_2) \in \tilde{\mathscr{R}}$ *problem* (9), *for* τ *sufficiently large, has exactly three solutions,* u_τ^1, u_τ^2, *and* u_τ^3, such that $(1/\tau)\, u_\tau^i \to -\sigma_1 u^i$ *as* $\tau \to \infty$, *where* u^i *solves problem* (4) *with*

$$(u_i)'(0) = -1 \quad \text{and} \quad (u^i)'(\pi) = -\frac{\sigma_2}{\sigma_1}, \qquad i = 0, 1, 2, 3.$$

If $(\sigma_1, \sigma_2) \in \mathbb{R}^2 \backslash \bar{\mathscr{R}}$ *then, for* τ *sufficiently large, problem* (9) *has a unique solution* u_τ^0 *such that* $(1/\tau\,|\sigma_1|)\, u_\tau^0 \to u^0$ *as* $\tau \to \infty$, *where* u^0 *is a solution of problem* (4) *satisfying the same conditions at the boundary as in part* (A) *above.*

(C) *If* $k/\sqrt{\alpha} + (k+1)/\sqrt{\beta} = 1$ *with* $\alpha < \beta$ *or if* $k/\sqrt{\alpha} + (k+1)/\sqrt{\beta} = 1$ *with* $\alpha > \beta$ *then, if* $\sigma_1 \neq 0$, *problem* (9), *for* τ *sufficiently large, has a unique solution* u_τ.

(D) *if* $1/\sqrt{\alpha} + 1/\sqrt{\beta} = 2/(2k+1)$ *then, provided that the ratio* σ_2/σ_1 *lies between* $-\sqrt{\beta/\alpha}$ *and* $-\sqrt{\alpha/\beta}$, *problem* (9), *for* τ *sufficiently large, has a solution* u_τ. *Moreover, there exist constants* $M_\tau \to \infty$ *as* $\tau \to \infty$, *such that if* w_τ *is another solution then* $(1/\tau)\,\|w_\tau\| \geqslant M_\tau$.

(E) *If pair* (α, β) *does* not *fall into one of the regions or lie on one of the curves covered in parts* (A), (B), (C), *or* (D), *above then, for any choice of* σ_1 *and* σ_2, *problem* (9) *has, for* τ *sufficiently large, and unique solution* u_τ.

(F) *In each of the cases* (C), (D), *and* (E) *above the solution* u_τ *has the property that* $(1/\tau)\, u_\tau \to \sigma_1 u$ *uniformly as* $\tau \to \infty$, *where* u *is a solution of problem* (4).

Proof. In view of Lemma 2.1, in order to argue that the number of solutions stated above is correct, we may assume that the nonlinear term, f, of (9) is of the form $f(x, u) = -\alpha u^- + \beta u^+ + (1/\tau)\, v(x, \tau u)$. For brevity, the arguments below are for the parameter situation described in (A) only, the other cases can be handled in a similar fashion.

So suppose that the pair (α, β) satisfies one of the inequalities in part (A) above and that $(\sigma_1, \sigma_2) \in \mathscr{R}$; i.e., σ_1 and σ_2 are such that $\sigma_1 > 0$ and $\sum_{\text{min}} < \sigma_1/\sigma_2 < \sum_{\text{max}}$. From Theorem 2.2 there are exactly three shooting values, $\theta_1 < \theta_2 < \theta_3$, in the interval $(-\pi/2, \pi/2)$, such that $\sigma_{\alpha\beta}(\theta_i) = \sigma_2/\sigma_1$, $i = 1, 2, 3$. This means that there are exactly three solutions $u_0(\cdot\,; \theta_i)$, $i = 1, 2, 3$, to the problem

$$-u'' = \alpha u^- + \beta u^+$$

$$u'(0) = 1, \qquad u'(\pi) = \frac{\sigma_2}{\sigma_1}.$$

Now, let ψ_1 and ψ_2 be the points at which the two local extrema of $\sigma_{\alpha\beta}$ occur as described in Theorem 2.2, cf. Fig. 2. Choose ϕ_1, ϕ_1', ϕ_2, ϕ_2', ϕ_3, and ϕ_3' such that

$$\frac{\pi}{2} < \phi_1 < \theta_1 < \phi_1' < \psi_1 < \phi_2 < \theta_2 < \phi_2' < \psi_2 < \phi_3 < \theta_3 < \phi_3' < \frac{\pi}{2}.$$

By Theorem 3.5 there is a $T>0$ such that $\tau \geqslant (1/\sigma_1)T$ implies that $\theta \mapsto v_{\sigma_1\tau}(\pi; \theta)$ is strictly decreasing on $[\phi_1, \phi'_1]$, strictly increasing on $[\phi_2, \phi'_2]$, and strictly decreasing again on $[\phi_3, \phi'_3]$. Moreover, Theorem 3.5 also implies that T can be chosen large enough so that $v_{\sigma_1\tau}(\pi; \tilde{\theta}_i) = \sigma_2/\sigma_1$ at *exactly* three points, $\tilde{\theta}_i = \tilde{\theta}_i(\tau)$, $i = 1, 2, 3$, in the interval $[\phi_1, \phi'_3]$ with $\tilde{\theta}_1 \in [\phi_1, \phi'_1]$, $\tilde{\theta}_2 \in [\phi_2, \phi'_2]$, $\tilde{\theta}_3 \in [\phi_3, \phi'_3]$, and $\tilde{\theta}_i \to \theta_i$ as $\tau \to \infty$. In other words, the problem

$$
\begin{gathered}
-u'' = \alpha u^- + \beta u^+ + \frac{v(x, \sigma_1 \tau u)}{\sigma_1 \tau} \\
u'(0) = 1, \qquad u'(\pi) = \frac{\sigma_2}{\sigma_1}
\end{gathered}
\tag{10}
$$

has at least three solutions $u_{\sigma_1\tau}(\cdot\,; \tilde{\theta}_i)$, $i = 1, 2, 3$. Furthermore, if $\varepsilon > 0$, then Lemma 3.2 and continuity of solutions of system 5 with respect to the shooting parameter θ shows that

$$
\|u_{\sigma_1\tau}(\cdot\,; \tilde{\theta}_i) - u_0(\cdot\,; \theta_i)\| < \varepsilon, \qquad i = 1, 2, 3
$$

for all $\tau \geqslant (1/\sigma_1)T$ if $T>0$ sufficiently large.

To show that these are actually the *only* solutions of (10) for τ sufficiently large we argue by contradiction. So assume that for each large $\tau > 0$ there is another solution, $u_{\sigma_1\tau}(\cdot\,; \theta)$, of (10) different from the solutions $u_{\sigma_1\tau}(\cdot\,; \tilde{\theta}_i)$. Since $\sigma_{\alpha\beta}$ can be uniformly approximated on the interval $[\phi_1, \phi'_3] \subset (-\pi/2, \pi/2)$ by the map $\theta \mapsto v_{\sigma_1\tau}(\pi; \theta)$ for large τ, then it must be that

$$
u_{\sigma_1\tau}(0; \theta) = \frac{1}{c} \tan \theta, \qquad \text{for some} \quad \theta \in \left(-\frac{\pi}{2}, \phi_1\right) \cup \left(\phi'_3, \frac{\pi}{2}\right).
$$

Since ϕ_1 and ϕ'_3 can be chosen arbitrarily close to $-\pi/2$ and $\pi/2$, respectively, then these solutions are of large norm; i.e., $\|u_{\sigma_1\tau}(\cdot, \theta)\| > \max 1/c\{\tan \phi_1, \tan \phi'_3\}$. By setting $\tau = n/\sigma_1$, we can generate a sequence of solutions, u_n, to problem (10) with $\|u_n\| \to \infty$ as $n \to \infty$. Put $w_n := u_n/\|u_n\|$; then $\|w_n\| = 1$ and

$$
\begin{gathered}
-w''_n = \alpha w_n^- + \beta w_n^+ + \frac{v(x, n u_n)}{n\, \|u_n\|} \\
w'_n(0) = \frac{1}{\|u_n\|}, \qquad w'_n(\pi) = \frac{\sigma_2}{\sigma_1\, \|u_n\|}.
\end{gathered}
$$

By a standard compactness argument we see that, passing to a subsequence if necessary, $w_n \to w$ uniformly on $[0, \pi]$, where w is nontrivial, i.e., $\|w\| = 1$, and satisfies

$$-w'' = \alpha w^- + \beta w^+$$

$$w'(0) = 0, \qquad w'(\pi) = 0.$$

This is a contradiction since the choice of α and β prohibit this problem from having nontrivial solutions, cf. the remarks at the end of Section 2.

If $\sigma_1 < 0$ then, interchanging the roles of α and β, Theorem 2.2 tells us that the map $\tilde{\sigma}_{\alpha\beta}$ is a homeomorphism onto $\mathbb{R}$ with a derivative that is every positive; that is, the left and right hand limits $\sigma'_{\alpha\beta}(\theta^-)$ and $\sigma'_{\alpha\beta}(\theta^+)$ are always positive. From Theorem (2.2) for each such pair of boundary values σ_1 and σ_2 there is a unique shooting value $\theta \in (-\pi/2, \pi/2)$ with $\sigma_{\beta\alpha}(\theta) = \sigma_2/\sigma_1$, i.e., with $\tilde{\sigma}_{\alpha\beta}(\theta) = -\sigma_2/\sigma_1$. We use Theorem 3.5 again to prove that for any ϕ and ϕ' such that

$$-\frac{\pi}{2} < \phi < \theta < \phi' < \frac{\pi}{2}$$

there exists a $T > 0$ such that for each $\tau \geqslant (1/-\sigma_1)T$ there is a unique $\tilde{\theta} \in (\phi, \phi')$ such that $v_{-\sigma_1\tau}(\pi; \left[\begin{smallmatrix} -(1/c)\tan\tilde{\theta} \\ -1 \end{smallmatrix}\right]) = -\sigma_2/\sigma_1$ with $\tilde{\theta} \to \theta$ as $\tau \to \infty$. Thus, for τ sufficiently large, problem (10) has a solution $u_{-\sigma_1\tau}$ and, as above, Lemma 3.2 implies that $u_{-\sigma_1\tau}(\cdot; \tilde{\theta}) \to u_0(\cdot; \theta)$ uniformly as $\tau \to \infty$. The argument showing that this can be the only solution for τ sufficiently large is similar to the one above.

If $\sigma_1 > 0$ with $\sigma_2 > \sum_{\max}^{+} \sigma_1$ or $\sigma_2 < \sum_{\min}^{+} \sigma_1$ then the ratio $\sigma = \sigma_2/\sigma_1$ is a value of σ for which the equation $\sigma_{\alpha\beta}(\theta) = \sigma$ has a unique solution and we can proceed as before.

When $\sigma_1 = 0$ we need a separate shooting argument as the map $\sigma_{\alpha\beta}$ is useful only in investigating the boundary condition $\sigma_1 \neq 0$. We need to know the behavior of the solutions of (5) when starting with zero slope. This is easy to analyze and we outline the procedure below. If, for example, (α, β) satisfies the first inequality condition,

$$\frac{k}{\sqrt{\alpha}} + \frac{k+1}{\sqrt{\beta}} < 1 < \frac{k + (1/2)}{\sqrt{\alpha}} + \frac{k + (1/2)}{\sqrt{\beta}}.$$

Now let $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](t; \theta)$ denote the solution of the system (5) with initial condition $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](0; \theta) = \left[\begin{smallmatrix} \tan\theta \\ 0 \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](0; -\pi/4) = \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right](0; \pi/4) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, and it is easy to check that the conditions on the two parameters α and β imply that $\alpha < \beta$, $v_0(\pi; \theta) > 0$ if $\theta < 0$, $v_0(\pi; 0) = 0$, and that $v_0(\pi; \theta) < 0$ if $\theta > 0$. In fact, by scaling, we have

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} (t; \theta) = \tan\theta \begin{cases} -\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\left(t; -\dfrac{\pi}{4}\right) & \text{if } \theta \leqslant 0 \\ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\left(t; \dfrac{\pi}{4}\right) & \text{if } \theta > 0. \end{cases}$$

It follows that

$$\frac{d}{d\theta} v_0(\pi; \theta) = \sec^2\theta \begin{cases} -v_0\left(\pi; -\dfrac{\pi}{4}\right) & \text{if } \theta < 0 \\ v_0\left(\pi; \dfrac{\pi}{4}\right) & \text{if } \theta > 0. \end{cases}$$

Thus, the map we are interested in, $\theta \mapsto v_0(\pi; \theta)$, is a piecewise C^1 homeomorphism of the interval $(-\pi/2, \pi/2)$ onto $\mathbb{R}$ with $(d/d\theta)\, v_0(\pi; \theta) < 0$ for all $\theta \neq 0$, $(\partial/d\theta)\, v_0(\pi; 0^-) < 0$, and $(d/d\theta)\, v_0(\pi; 0^+) < 0$.

The estimates in Section in Section 3 are easily modified to show that $\theta \mapsto v_\tau(\pi; \theta)$, where now $[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}](t; \theta)$ is the solution of the boundary value problem (3) with $[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}](0; \theta) = [\begin{smallmatrix} \tan\theta \\ 0 \end{smallmatrix}]$, approximate on compact subsets of $(-\pi/2, \pi/2)$ the shooting map $\theta \mapsto v_0(\pi; \theta)$. The remainder of the analysis parallels whas has been done before. ▮

In a brief closing remark we summarize that the key idea in this paper is that solutions of the boundary value problem (9), for large τ, approximate those of the *model* or piecewise linear differential equation (5) $-w'' = \alpha w^- + \beta w^+$ in a C^1 sense. If the maps $\sigma_{\alpha\beta}$, called shooting maps in this paper (we referred to them so much we thought it would be easier if we called them something), are known then the number of solutions of this differential equation for a given set of boundary conditions can be obtained. In [2] the shooting maps for Dirichlet boundary conditions have been analyzed and in [3] can be found the analysis of the shooting maps for Neumann conditions that has been summarized in Theorem 2.2. The estimates of Section 3 can clearly be adapted to more general kinds of boundary conditions and the C^1 type of convergence of solutions of the nonlinear problem so obtained to solutions of the oscillator yields more information than just their number: e.g., frequencies of oscillation, the number of zeros of these solutions, estimates on their location, etc.

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