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Exponential formulas for the Jacobians and Jacobian matrices of analytic maps

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Abstract

Let $F=(F_1,F_2,\ldots,F_n)$ be an n-tuple of formal power series in n variables of the form $F(z)=z+O(|z|^2)$. It is known that there exists a unique formal differential operator $A=\sum_{i=1}^n a_i(z)\partial/\partial z_i$ such that $F(z)=\exp(A)z$ as formal series. In this article, we show the Jacobian $\mathscr{J}(F)$ and the Jacobian matrix $J(F)$ of F can also be given by some exponential formulas. Namely, $\mathscr{J}(F)=\exp(A+\nabla A)\cdot 1$, where $\nabla A(z)=\sum_{i=1}^n(\partial a_i/\partial z_i)(z)$, and $J(F)=\exp(A+R_{Ja})\cdot I_{n\times n}$, where $I_{n\times n}$ is the identity matrix and R_{Ja} is the multiplication operator by Ja for the right. As an immediate consequence, we get an elementary proof for the known result that $\mathscr{J}(F)\equiv 1$ if and only if $\nabla A=0$. Some consequences and applications of the exponential formulas as well as their relations with the well-known Jacobian Conjecture are also discussed. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

This research work mainly motivated by the well-known Jacobian Conjecture and inspired by an exponential formula in Conformal Field theory. First let us recall:

Jacobian Conjecture. *Let k be a field of characteristic* 0 *and $F{:}k^n\to k^n$ be a polynomial map. If Jacobian $j(F)=Det(\partial F_i/\partial z_j)=1$, then F is an automorphisms whose inverse is also a polynomial map.*

This conjecture was first proposed by Keller in 1939. For the history of this conjecture (see [2,7,4] and references there). Since then, it has been attracting enormous

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efforts from mathematicians. But unfortunately, this conjecture remains widely open at the present time. Nevertheless, many important results have been obtained in last six decades from the efforts of mathematicians trying to solve Jacobian Conjecture. Some of these results are not only crucial to the Jacobian Conjecture, they also play very important roles in other mathematical research areas.

One of the effective approaches to the Jacobian Conjecture is to develop nice formulas for the formal inverse G of the polynomial map F and to see if it is also a polynomial map. Several important formulas have been found and well studied, among which the most well known are Abhyankar's inversion formula (see [1]) and the tree expansion formula for the formal inverse G of F (see [2,8]).

Interestingly, an exponential formula for the formal power series or holomorphic functions in one variable, which plays a crucial role in two-dimensional Conformal Field Theory, seems closely related with the Jacobian Conjecture. To be more precise, let $F(x) = x + O(x^2)$ be a formal power series in one variable x. Then there exists a unique formal differential operator $A(x) = a(x)\partial/\partial x$ with $o(a) \geqslant 2$ such that $F(x) = \mathrm{e}^A x$. (Note that the exponential formula we quote here is a little different from the one used in [6]). The main reason that the exponential formula above is so important in two-dimensional Conformal Field Theory is that it gives the Virasoro algebra structure, which is the most fundamental algebraic structure to the whole theory. For more detail, see [6,3,9].

One of the advantages of the exponential formula $F(x) = \mathrm{e}^A x$ for the formal power series $F(x)$ is that e^A is an automorphism of the algebra $\mathbb{C}[[x]]$ of formal power series in one variable. This is because that A itself is a derivation fof the algebra $\mathbb{C}[[x]]$ and it is well known in Lie algebra theory that the exponential of any derivation of an algebra is an automorphism of the algebra. As an immediate consequence of this observation, the formal inverse G of F is given by the exponential formula $G(x) = \mathrm{e}^{-A}x$. Regarding the Jacobian Conjecture, it is certainly very interesting to see that the formal inverse G of F is given in such a simple way. Actually, for the formal power series in several variables, we also have similar exponential formulas (see [5] and also Proposition 2.1). Namely, let $F = (F_1, F_2, \ldots, F_n)$ be an n-tuple of formal power series in n variables of the form $F(z) = z + O(|z|^2)$. Let $G = (G_1, G_2, \ldots, G_n)$ be the formal inverse of F, i.e. $F(G) = G(F) = z$, where $z = (z_1, z_2, \ldots, z_n)$. Then there exists a unique formal differential operator $A = \sum_{i=1}^n a_i(z)\partial/\partial z_i$ with $o(a_i(z)) \geqslant 2$ such that $F_i(z) = \exp(A)z_i$ ($i = 1, 2, \ldots, n$). By the similar reason, e^A is an automorphism of the algebra $\mathbb{C}[[z]]$ of formal power series in z and $G_i(z) = \mathrm{e}^{-A}z_i$ for $i = 1, 2, \ldots, n$.

Since the formal power series F as well as its formal inverse G are completely determined by a unique formal differential operator A, naturally one may ask: how does the formal differential operator A determine the Jacobian $\mathscr{J}(F)$ and Jacobian matrix $J(F)$ of F? or in other words, are there any formulas via which the differential operator A also completely determines $\mathscr{J}(F)$ and $J(F)$? In this article, we show that the answer to the question above is "yes". In Section 2, we give two exponential formulas for the Jacobian $\mathscr{J}(F)$ and Jacobian matrix $J(F)$ of F, respectively. To be more precise, in Theorem 2.8, we show that $\mathscr{J}(F) = \exp(A + \nabla A) \cdot 1$, where

$\nabla A(z) = \sum_{i=1}^{n}(\partial a_i/\partial z_i)(z)$ is the *divergence* of the operator A. In Theorem 2.9, we show that $J(F) = \exp(A + R_{Ja}) \cdot I_{n\times n}$, where $I_{n\times n}$ is the identity matrix and R_{Ja} is the multiplication operator by Ja for the right. As an immediate consequence, we also give an elementary proof for the known result that $\mathscr{J}(F) \equiv 1$ if and only if $\nabla A = 0$. (See Corollary 2.12.) Various interesting properties of the differential operator A and the formal deformation $F_t(z) = e^{tA}z$ are also derived in this section.

In Section 3, we first give some explanations about the exponential formulas derived in Theorems 2.8 and 2.9 by relating them with some well-known formula in linear algebra. Then, we study the consequences of these exponential formulas to the Jacobian Conjecture, especially, we give a new proof to a theorem of Bass et al. in [2] (see Theorem 3.5).

In Section 4, we discuss some open problems related with these exponential formulas and the Jacobian Conjecture.

2. Exponential formulas

Notations. (1) Let $z_1, z_2, \ldots, z_n$ be n commutative variables and $z = (z_1, z_2, \ldots, z_n)$. Let $\mathbb{C}[[z]] = \mathbb{C}[z_1, z_2, \ldots, z_n]$ be the algebra of polynomials in n variables, $\mathbb{C}[[z]]$ be the algebra of formal power series. For any $k \geqslant 0$, set $\mathbb{C}_k[[z]] = \mathbb{C}_k[[z_1, z_2, \ldots, z_n]]$ be the subalgebra consisting of the elements of $\mathbb{C}[[z]]$ whose lowest degree is greater or equal to k.

(2) For any $F = (F_1, F_2, \ldots, F_n) \in \mathbb{C}[[z]]^n$, set

$$JF(z) = \left(\frac{\partial F_i}{\partial z_j}\right)_{1 \leqslant i,j \leqslant n}, \tag{2.1}$$

$$\mathscr{J}F(z) = Det\left(\frac{\partial F_i}{\partial z_j}\right)_{1 \leqslant i,j \leqslant n}. \tag{2.2}$$

We call JF the *Jacobian matrix* fand $\mathscr{J}F(z)$ the *Jacobian* of F.

Let $\mathscr{F}_1$ be the set of the elements $F = (F_1, F_2, \ldots, F_n) \in \mathbb{C}[[z]]^n$ such that $F_i(z) = z_i +$ high degree terms, for $i = 1, 2, \ldots, n$. Note that for any analytic map $F: U \to \mathbb{C}^n$ with Jacobian $\mathscr{J}(F)(0) \neq 0$ for the some open neighborhood U of $0 \in \mathbb{C}^n$, composing with some line isomorphism if necessary, the formal series of F will be in $\mathscr{F}_1$. Another observation is that, any $F \in \mathscr{F}_1$ gives an automorphism of the algebra $\mathbb{C}[[z]]$, which sends z_i to F_i. The inverse of this automorphism is the automorphism induced by the formal inverse of F.

One remark is that all the proofs and results in this paper work equally well for any field of characteristic 0, not necessarily algebraic closed. But for convenience, we will always take $\mathbb{C}$ to be the ground field.

The following proposition is known. For example, see [5]. Here we give an elementary proof.

Proposition 2.1. *For any* $F = (F_1, F_2, \ldots, F_n) \in \mathscr{F}_1$, *there exists a unique* $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}_2[[z]]^n$ *such that*

$$F_i(z) = \exp\left(a(z)\frac{\mathrm{d}}{\mathrm{dz}}\right) z_i = \exp(A) z_i, \tag{2.3}$$

where

$$A(z) = a(z)\frac{\mathrm{d}}{\mathrm{d}z} = \sum_{i=1}^{n} a_i(z)\frac{\partial}{\partial z_i}, \tag{2.4}$$

$$\exp(A) = \exp\left(a(z)\frac{\mathrm{d}}{\mathrm{d}z}\right) = \sum_{k=0}^{\infty}\frac{(a(z)\mathrm{d}/\mathrm{d}z)^k}{k!}. \tag{2.5}$$

Proof. This can be checked directly by solving the formal equation (2.3) incursively as follows.

For $i = 1, 2, \ldots, n$, we write

$$F_i(z) = z_i + b_i^{(2)}(z) + b_i^{(3)}(z) + \cdots + b_i^{(k)}(z) + \cdots, \tag{2.6}$$

$$a_i(z) = a_i^{(2)}(z) + a_i^{(3)}(z)+, \ldots + a_i^{(k)}(z) + \cdots, \tag{2.7}$$

where $a_i^{(k)}(z)$ and $b_i^{(k)}(z)$, for any $k \in \mathbb{N}$, fare homogeneous polynomials of degree k. We also write $F^{(k)} = (F_1^{(k)}, F_2^{(k)}, \ldots, F_n^{(k)})$, $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ and $A^{(k)} = a^{(k)}\partial/\partial z = \sum_{i=1}^n a_i^{(k)}\partial/\partial z_i$. Notice that the operator $A^{(k)}$ increase degree by $k-1$.

From Eqs. (2.3), we get

$$z_i + \sum_{k=1}^{\infty}\frac{(a(z)\mathrm{d}/\mathrm{d}z)^k}{k!} z_i = z_i + b_i^{(2)}(z) + b_i^{(3)}(z) + \cdots + b_i^{(k)}(z) + \cdots. \tag{2.8}$$

Comparing the homogeneous parts of both sides of (2.8), we get

$$\begin{aligned}
a_i^{(2)} &= b_i^{(2)}, \\
a_i^{(3)} &= b_i^{(3)} - \sum_{k=1}^{n} a_k^{(2)}\frac{\partial a_i^{(2)}}{\partial z_k}, \\
&\cdots \\
a_i^{(m)} &= b_i^{(m)} - \sum_{1 \leqslant r < m}\ \sum_{\substack{k_1+k_2+\cdots k_r = m+r \\ k_1, k_2, \ldots, k_r \geqslant 2}} \frac{A^{(k_1)}A^{(k_2)}\cdots A^{(k_r)}}{k_1!k_2!\cdots k_r!} z_i.
\end{aligned} \tag{2.9}$$

Hence $a(z)$ is completely determined by the equations above. □

One easy corollary of the calculation above is the following:

Corollary 2.2. *F is odd if and only if $a(z)$ is odd.*

This can also be proved by the similar arguments for Proposition 3.3.

Definition 2.3. We call the formal differential operator A in Proposition 2.1 the associated differential operator of F. We also define

$$(\nabla A) = (\nabla a)(z) = \sum_{i=1}^{n} \frac{\partial a_i}{\partial z_i}(z) \tag{2.10}$$

and call it the divergence of the differential operator A.

One of the advantages of formula (2.3) is that the operator $\exp(A)$ or $\exp(a(z)\partial/\partial z)$ is an automorphism of the $\mathbb{C}$-algebra $\mathbb{C}[[z]]$ which maps z_i to F_i. This follows from the well-known fact that the exponential of any derivative of any algebra, when it is well defined, is an automorphism of that algebra. It is because this remarkable property that formula (2.3) in the case of one variable plays a very important role in conformal field theory see [3] and [6]. (The formula used in [6] is a little different from (2.3).) The following are some immediate consequences of the property above.

Lemma 2.4. (a) *Let* $F^{-1} = (F_1^{-1}, F_2^{-1}, \ldots, F_n^{-1})$ *be the formal inverse of* F, *i.e. the composition* $F \circ F^{-1} = F^{-1} \circ F$ *is identity map of* $\mathbb{C}[[z]]$. *Then*

$$F^{-1}(z) = \exp(-A(z))z = \exp\left(-a(z)\frac{\partial}{\partial z}\right) z. \tag{2.11}$$

(b) *For any element* $g(z) \in \mathbb{C}[[z]]$, *we have*

$$g(F(z)) = \exp\left(a(z)\frac{\partial}{\partial z}\right) g(z). \tag{2.12}$$

In particular, *for any* $k \geqslant 0$, *we have*

$$F^{[k]}(z) = \exp(kA(z))z = \exp\left(ka(z)\frac{\partial}{\partial z}\right) z, \tag{2.13}$$

where

$$F^{[k]}(z) = \underbrace{F \circ F \circ \cdots F}_{k\ copies} \tag{2.14}$$

is the k*th-power of the automorphism of* $\mathbb{C}[[z]]$ *defined by* F *which sends* z_i *to* F_i.

Another advantage of formula (2.3) is that it allows us to deform the formal power series F in a very natural way. Introduce another variable t which commutes with z_i and define

$$F_t(z) = F(z;t) = (F_1(z;t), F_2(z;t), \ldots, F_n(z;t))$$

by setting

$$F_i(z;t) = \exp(tA(z))z_i = \exp\left(ta(z)\frac{\partial}{\partial z}\right) z_i. \tag{2.15}$$

Note that $F_i(z;t) \in \mathbb{C}[t][[z]]$, i.e. it is a formal power series in $\{z_i\}$ with coefficients in $\mathbb{C}[t]$. In particular, for any $t_0 \in \mathbb{C}$, $F(z;t_0) \in \mathscr{F}_1$ and when $t = k \in \mathbb{N}$, $F(z;k)$ is just the kth-power $F^{[k]}$ of the isomorphism F. This deformation will play the key role in our later arguments.

Lemma 2.5. *For any* $g(z;t) \in \mathbb{C}[t][[z]]$,

$$\frac{\partial}{\partial t} g(z;t) = Ag(z;t) \tag{2.16}$$

if and only if $g(z;t) = u(F(z;t)) = \exp(tA)u(z)$ *for some* $u \in \mathbb{C}[[z]]$.

Proof. First let $g(z;t) = \exp(tA)u(z)$, then

$$\begin{aligned} \frac{\partial}{\partial t} g(z;t) &= \frac{\partial}{\partial t} \exp(tA)u(z) = A\exp(tA)u(z) \\ &= Ag(z;t). \end{aligned}$$

Conversely, suppose that $g(z;t)$ satisfies (2.16). Set $u = \exp(-tA)g(z;t) \in \mathbb{C}[t][[z]]$, then by chain rule,

$$\begin{aligned} \frac{\partial}{\partial t} \exp(-tA)g(z;t) &= -A\exp(-tA)g(z;t) + \exp(-tA)\frac{\partial}{\partial t} g(z;t) \\ &= -A\exp(-tA)g(z;t) + A\exp(-tA)g(z;t) \\ &= 0. \end{aligned}$$

So $u = \exp(-tA)g(z;t)$ does not depend on t, therefore $u(z) \in \mathbb{C}[[z]]$ and $g(z;t) = \exp(tA)u(z)$. □

The following property is a little bit strange.

Proposition 2.6.

$$J(F)(z;t) \begin{pmatrix} a_1(z) \\ a_2(z) \\ \vdots \\ a_n(z) \end{pmatrix} = \begin{pmatrix} a_1(F(z;t)) \\ a_2(F(z;t)) \\ \vdots \\ a_n(F(z;t)) \end{pmatrix} \tag{2.17}$$

or in short notations

$$AF(z;t) = J(F)(z;t)a(z) = a(F(z;t)). \tag{2.18}$$

Proof. This follows from the following straightforward calculations. Consider

$$\begin{aligned} \frac{\partial}{\partial t} F_i(z;t) &= \frac{\partial}{\partial t} \exp\left(ta(z)\frac{\partial}{\partial z} \right) z_i \\ &= \sum_{k=1}^{n} a_k(z) \frac{\partial}{\partial z_k} \exp\left(ta(z)\frac{\partial}{\partial z} \right) z_i \\ &= \sum_{k=1}^{n} a_k(z) \frac{\partial}{\partial z_k} F_i(z;t) \\ &= \sum_{k=1}^{n} \frac{\partial F_i(z;t)}{\partial z_k} a_k(z). \end{aligned} \tag{2.19}$$

On the other hand, note that the operators $a(z)\partial/\partial z$ and $\exp(a(z)\partial/\partial z)$ commute with each other, so we also have

$$\begin{aligned}\frac{\partial}{\partial t}F_i(z;t) &= \exp\left(ta(z)\frac{\partial}{\partial z}\right)\left(\sum_{k=1}^{n} a_k(z)\frac{\partial}{\partial z_k}\right) z_i \\ &= \exp\left(ta(z)\frac{\partial}{\partial z}\right) a_i(z) \\ &= a_i\left(\exp\left(ta(z)\frac{\partial}{\partial z}\right) z\right) \\ &= a_i(F(z;t)). \end{aligned} \tag{2.20}$$

Comparing (2.19) and (2.20), we get (2.17). □

Unfortunately, Eq. (2.17) does not completely determine the operator $A(z)=a(z)\partial/\partial z$. Instead we have the following explicit formulas for $a(z)$ and the inverse $G = (G_1, G_2, \ldots, G_n)$ of F.

Proposition 2.7.

(a) $$a(z) = -\sum_{k=1}^{\infty}\frac{1}{k}(1-\mathrm{e}^A)^k z = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}F^{[j]}(z)\right). \tag{2.21}$$

(b) $$G(z) = z + \sum_{k=1}^{\infty}(1-\mathrm{e}^A)^k z = z + \sum_{k=1}^{\infty}\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}F^{[j]}(z)\right). \tag{2.22}$$

Notice that the operator $1-\mathrm{e}^A$ strictly increases the degree, so the infinite sums that appear in the lemma above all make sense.

Proof. (a) follows from the following formal identity:

$$A = \log \mathrm{e}^A = \log(1-(1-\mathrm{e}^A)) = -\sum_{k=1}^{\infty}\frac{1}{k}(1-\mathrm{e}^A)^k. \tag{2.23}$$

(b) Since the formal inverse of F exists and is unique, it is enough to check that the formal series G given by (2.22) is the inverse of F.

Consider

$$\begin{aligned}(G\circ F)(z) &= \mathrm{e}^A G(z) \\ &= \mathrm{e}^A z + \sum_{k=1}^{\infty}(1-\mathrm{e}^A)^k \mathrm{e}^A z \\ &= \mathrm{e}^A z + \sum_{k=1}^{\infty}(1-\mathrm{e}^A)^k z - \sum_{k=1}^{\infty}(1-\mathrm{e}^A)^{k+1} z \end{aligned}$$

$$= \mathrm{e}^{A} z + (1 - \mathrm{e}^{A}) z$$

$$= z. \qquad \square$$

Now we begin to prove our exponential formula for the Jacobian $\mathscr{J}(F_t)$.

Theorem 2.8. (*a*) *In the notations above, we have*

$$\mathscr{J}(F_t)(z) = \exp\left(t a(z) \frac{\mathrm{d}}{\mathrm{d}z} + t \nabla a(z) \right) \cdot 1, \tag{2.24}$$

where $F_t(z) = F(z;t) = (F_1(z;t), F_2(z;t), \ldots, F_n(z;t))$ *as before.*

(*b*) *For any* $u \in \mathbb{C}[[z]]$, *we have*

$$\exp(tA + t\nabla a(z))u = u(F(t,z)) \mathscr{J} F(t,z). \tag{2.25}$$

It is easy to see that (a) is an immediate consequence of (b), but here we need prove (a) first.

Proof. To keep notations simple, here we only give the proof for the case of two variables. For the general cases, the ideas are completely same.

Let $K(t) = \exp(ta(z)\mathrm{d}/\mathrm{d}z + t\nabla a(z)) \cdot 1$ and $H(t) = \mathscr{J}(F_t)$, i.e. the Jacobian of $F_t(z)$ with respect to the variables z_1, z_2. It is easy to see that

$$K(0) = 1, \tag{2.26}$$

$$\frac{\partial}{\partial t} K(t) = (A(z) + \nabla A(z)) K(t). \tag{2.27}$$

To show that $K(t) = H(t)$, it is enough to show that $H(t)$ also satisfies Eqs. (2.26) and (2.27) above. First when $t = 0$, $F_t(z) = (z_1, z_2)$ and $H(0) = \mathscr{J}(F)(z;0) = 1$. So it only remains to check (2.27) for $H(t)$.

$$\frac{\partial}{\partial t} H(t) = \frac{\partial}{\partial t} \begin{vmatrix} \frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial^2 F_1(z;t)}{\partial z_1 \partial t}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial^2 F_2(z;t)}{\partial z_1 \partial t}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial^2 F_1(z;t)}{\partial z_2 \partial t} \\ \frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial^2 F_2(z;t)}{\partial z_2 \partial t} \end{vmatrix}. \tag{2.28}$$

By Lemma 2.5, we calculate the first term of (2.28) as follows.

$$\begin{vmatrix} \frac{\partial^2 F_1(z;t)}{\partial z_1 \partial t}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial^2 F_2(z;t)}{\partial z_1 \partial t}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial z_1}(a_1(z)\frac{\partial}{\partial z_1} + a_2(z)\frac{\partial}{\partial z_2})F_1(z;t), & \frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial}{\partial z_1}(a_1(z)\frac{\partial}{\partial z_1} + a_2(z)\frac{\partial}{\partial z_2})F_2(z;t), & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix}$$

$$= \begin{vmatrix} A\frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ A\frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial a_1}{\partial z_1}\frac{\partial F_2(z;t)}{\partial z_1} + \frac{\partial a_2}{\partial z_1}\frac{\partial F_2(z;t)}{\partial z_2}, & \frac{\partial F_2(z;t)}{\partial z_2} \\ \frac{\partial a_1}{\partial z_1}\frac{\partial F_2(z;t)}{\partial z_1} + \frac{\partial a_2}{\partial z_1}\frac{\partial F_2(z;t)}{\partial z_2}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix}$$

$$= \begin{vmatrix} A\frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ A\frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix} + \left(\frac{\partial a_1}{\partial z_1}\right)\mathscr{J}(F_t). \tag{2.29}$$

Similarly, for the second term of (2.28), we have

$$\begin{vmatrix} \frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial^2 F_1(z;t)}{\partial z_2 \partial t} \\ \frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial^2 F_2(z;t)}{\partial z_2 \partial t} \end{vmatrix} = \begin{vmatrix} \frac{\partial F_1(z;t)}{\partial z_1}, & A\frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial F_2(z;t)}{\partial z_1}, & A\frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix} + \left(\frac{\partial a_2}{\partial z_2}\right)\mathscr{J}(F_t). \tag{2.30}$$

Combining (2.29) and (2.30), we get

$$\frac{\partial}{\partial t}H(t) = A\begin{vmatrix} \frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix} + \left(\frac{\partial a_1}{\partial z_1} + \frac{\partial a_2}{\partial z_2}\right)\begin{vmatrix} \frac{\partial F_1(z;t)}{\partial z_1}, & \frac{\partial F_1(z;t)}{\partial z_2} \\ \frac{\partial F_2(z;t)}{\partial z_1}, & \frac{\partial F_2(z;t)}{\partial z_2} \end{vmatrix}$$

$$= (A + \nabla A)\mathscr{J}(F_t). \tag{2.31}$$

(b) By formula (2.24) and Lemma 2.5, it is easy to check that both sides of (2.25) satisfy Eqs. (2.26) and (2.27). □

By the similar idea, we can also get an exponential formulas for the Jacobian matrix $JF(t,z)$ of $F(t,z)$. First, we fix the following notations: Let $Ja(z)$ be the Jacobian matrix of the n-tuple $(a_1(z), a_2(z), \dots, a_n(z))$. Let R_{Ja} be the operator over the algebra $M_{n\times n}(\mathbb{C}[[z]])$, i.e. the $n \times n$ matrices with entries lying in $\mathbb{C}[[z]]$, defined by multiplying the matrix $Ja(z)$ from the right-hand side. In the following theorem, we also view the differential operator $A(z) = a(z)\partial/\partial z$ as a differential operator of the algebra $M_{n\times n}(\mathbb{C}[[z]])$, which acts the matrices entry-wisely.

Theorem 2.9. *For any* $U(z) \in M_{n\times n}(\mathbb{C}[[z]])$, *we have*

$$\exp(tA + tR_{Ja})U = U(F_t(z))JF_t(z). \tag{2.32}$$

In particular, when U *is chosen to the identity matrix* Id, *we get*

$$JF_t(z) = \exp(tA + tR_{Ja}) \cdot Id. \tag{2.33}$$

Proof. For any $1 \leqslant i,j \leqslant n$, consider

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial F_i(t,z)}{\partial z_j} &= \frac{\partial}{\partial z_j}\frac{\partial F_i(t,z)}{\partial t}\\ &= \frac{\partial}{\partial z_j}\sum_{k=1}^{n} a_k(z)\frac{\partial F_i(t,z)}{\partial z_k}\\ &= \sum_{k=1}^{n}\frac{\partial a_k}{\partial z_j}\frac{\partial F_i(t,z)}{\partial z_k} + \left(\sum_{k=1}^{n} a_k\frac{\partial}{\partial z_k}\right)\frac{\partial F_i(t,z)}{\partial z_j}\\ &= \sum_{k=1}^{n}\frac{\partial F_i(t,z)}{\partial z_k}\frac{\partial a_k}{\partial z_j} + A\frac{\partial F_i}{\partial z_j}.\end{aligned}$$

Hence, we have

$$\frac{\partial}{\partial t}JF_t(z) = (A + R_{Ja})JF_t(z). \tag{2.34}$$

By Lemma 2.5, we also have $(\partial/\partial t)U(F_t(z)) = AU(F_t(z))$. So it is easy to see that the right-hand side of (2.32) satisfies the equations

$$\frac{\partial}{\partial t}(U(F_t(z))JF_t(z)) = (A + R_{Ja})(U(F_t(z))JF_t(z)), \tag{2.35}$$

$$U(F_0(z))JF_0(z) = Id. \tag{2.36}$$

Hence (2.32) holds. □

Remark 2.10. (a) Note that the proofs of Theorems 2.8 and 2.9 only need the condition $o(a(z)) \geqslant 1$ instead of $o(a(z)) \geqslant 2$. So for any $A(z) = a(z)\partial/\partial z$ with $o(a(z)) \geqslant 1$, set $F(t;z) = e^{tA(z)}z$, then the formulas in Theorems 2.8 and 2.9 still hold.

(b) In particular, over the complex field $\mathbb{C}$, it is straightforward to check that $F(z) = e^{A(z)}z$ is a well-defined formal power series and we can replace t by 1 in all the formulas in Theorems 2.8 and 2.9.

Next we will derive a little bit more information about $\mathscr{J}F_t$.

Proposition 2.11.

$$\frac{\partial}{\partial t}\mathscr{J}(F_t) = (A + (\nabla a)(z))\mathscr{J}(F_t) = (\nabla a)(F_t)\mathscr{J}(F_t). \tag{2.37}$$

In particular,

$$A\mathscr{J}(F_t) = ((\nabla a)(F_t) - (\nabla a)(z))\mathscr{J}(F_t). \tag{2.38}$$

Proof. From (2.31), we see that $(\partial/\partial t)\mathscr{J}(F_t) = (A + (\nabla a)(z))\mathscr{J}(F_t)$. Let $L(z;t) = (A + (\nabla a)(z))\mathscr{J}(F_t)$ and $R(z;t) = (\nabla a)(F_t)\mathscr{J}(F_t)$. Then by (2.24) and Lemma 2.5, it is easy to see that

$$\frac{\partial L(z;t)}{\partial t} = (A + (\bigtriangledown a)(z))L(z;t). \tag{2.39}$$

$$\frac{\partial R(z;t)}{\partial t} = (A + (\bigtriangledown a)(z))R(z;t). \tag{2.40}$$

While $L(z;0) = (\bigtriangledown a)(z) = R(z;0)$. Hence we must have $L(z;t) = R(z;t)$. $\square$

As an application of Theorem 2.8, we give a new proof to the following result, which was first proved by Pittaluga in [5] by using the theory of formal Lie groups and Lie algebras.

Corollary 2.12. *$\mathscr{J}(F) \equiv 1$ if and only if $\bigtriangledown A \equiv 0$.*

Proof. First from (2.24), it is easy to see that if $\bigtriangledown A \equiv 0$, then $\mathscr{J}(F) \equiv 1$. Conversely, suppose that $\mathscr{J}(F) \equiv 1$. Observe that $a(z) \in \mathbb{C}_2[[z]]$, or in other words, the least degree of a_i are at least 2, therefore the operators $A = a(z)\partial/\partial z$ and $A + \bigtriangledown A$ increase the degree at least by one. If $\bigtriangledown a(z) \neq 0$, say its lowest degree is m. Let M be it the homogeneous part of degree m. From (2.24) for $t = 1$, we have

$$\begin{aligned} 1 \equiv \mathscr{J}(F) &= \mathrm{e}^{(a(z)\partial/\partial z + \bigtriangledown a(z))} \cdot 1 \\ &= 1 + \left(a(z)\frac{\partial}{\partial z} + \bigtriangledown a(z)\right) \cdot 1 + \sum_{i \geqslant 2} \frac{1}{k!}\left(a(z)\frac{\partial}{\partial z} + \bigtriangledown a(z)\right)^{k-1} \bigtriangledown a(z) \\ &= 1 + M + \text{high degree terms}. \end{aligned} \tag{2.41}$$

Clearly $M = 0$, contradiction.

Another way to prove the result above is the following: Consider the “deformation” $F_t(z)$ of F as before. Notice the Jacobian $\mathscr{J}(F_t) \in \mathbb{C}[t][[z]]$ and $\mathscr{J}(F_t) = \mathscr{J}(F^{[k]})$ when $t=k$, for any $k \in \mathbb{N}$. Now since $\mathscr{J}(F)(z,1) \equiv 1$, then, by the chain rule, $\mathscr{J}(F^{[k]})(z) \equiv 1$. This implies that $\mathscr{J}(F_t) \equiv 1$, when $t = k$ for any $k \in \mathbb{N}$. Hence $\mathscr{J}(F_t)$ itself must be identically 1, for as a polynomial of t, the coefficient of any monomial of positive degree of F cannot have infinitely roots unless it is zero. In particular, $\mathscr{J}(F_t)$ does not depend on t. So we have

$$0 = \left.\frac{\partial}{\partial t}\right|_{t=0} \mathscr{J}F(z;t) = \left(a^{(z)}\frac{\partial}{\partial z} + \bigtriangledown a(z)\right)\mathscr{J}F(z;0) = \bigtriangledown a(z). \qquad \square \tag{2.42}$$

From the arguments in the proof for the Corollary above, or by the Corollary itself, we have:

Corollary 2.13. *For any $F \in \mathscr{F}_1$, if $\mathscr{J}(F) \equiv 1$. Then $\mathscr{J}(F_t) \equiv 1$.*

3. Some explanations and applications

At the first glance, the formulas we proved in Theorems 2.8 and 2.9 are a little mysterious. Here we try to give a little explanations to these two formulas.

First, the exponential formula (2.24) reminds us of the following elementary formula in linear algebra. Namely, for any $n \times n$ matrix $M \in M_{n\times n}(\mathbb{C})$, then

$$Det\, e^M = e^{TrM}. \tag{3.1}$$

Actually, we will see that formula (2.24) can be viewed as a generalization of the formula above.

First, we define the embedding

$$\Phi : M_{n\times n}(\mathbb{C}) \to \mathscr{D}(z), \tag{3.2}$$

$$M = (m_{ij}) \to \sum_{i,j=1}^{n} m_{ij} z_i \frac{\partial}{\partial z_j}, \tag{3.3}$$

where $\mathscr{D}(z)$ is the Lie algebra of the derivations of $\mathbb{C}[[z]]$. It is very easy to check that the linear map $\Phi : M_{n\times n} \to \mathscr{D}(z)$ is an injective homomorphism of Lie algebras.

Lemma 3.1. *Let $F(z) = \exp(\Phi(M))z$. Then*
(a) $J(F) = e^M$.
(b) $F(z) = e^M z$.
(c) $\mathscr{J}(F) = e^{TrM}$.

Proof. Note that $J\Phi(M) = M$ and $\nabla\Phi(M) = TrM$. By Remark 2.10, we can apply formula (2.33) to the map F, we get

$$\begin{aligned} J(F) &= e^{\Phi(M)+R_{J\Phi(M)}} I_{n\times n} \\ &= e^{R_M} e^{\Phi(M)} I_{n\times n} \\ &= e^{R_M} I_{n\times n} \\ &= e^M, \end{aligned}$$

where the second equality above follows from the fact that the operators $\Phi(M)$ and $R_{J\Phi(M)}$ commutes with each other. So we have proved (a). (b) follows immediately from (a).

To prove (c), we apply formula (2.24) to F, we get

$$\begin{aligned} \mathscr{J}(F) &= e^{\Phi(M)+\nabla\Phi(M)} \cdot 1 \\ &= e^{\nabla\Phi(M)} e^{\Phi(M)} \cdot 1 \\ &= e^{Tr(M)}. \qquad \square \end{aligned}$$

Combine (a) and (c) in the lemma above, we recover formula (3.1). Therefore, formula (2.24) and (2.33) can be viewed as some generalizations of formula (3.1).

One of the motivations for the present work is that we believe the exponential formulas (2.3), (2.24) and Corollary 2.12 are closely related with the well-known Jacobian Conjecture. In the rest of section, we will consider some applications to the Jacobian Conjecture.

From Proposition 2.1, Lemma 2.4 and Corollary 2.12, we see that the Jacobian Conjecture is equivalent to the following pure algebraic problem.

Conjecture 3.2. *Let* $a(z)\in\mathbb{C}_2[[z_1,z_2,\ldots,z_n]]$ *and* $\nabla a(z)=0$. *Then* $F(z)=\exp(a(z)\partial/\partial z)z\in(\mathbb{C}[z])^n$ *if and only if* $G(z)=\exp(-a(z)\partial/\partial z)z\in(\mathbb{C}[z])^n$.

In the case when $a(z)$ is even, we have a very simple answer to the conjecture above.

Proposition 3.3. (a) *For any* $F\in\mathscr{F}_1$, *let G be its formal inverse. Then* $G(z)=-F(-z)$ *if and only if* $a(z)$ *is even.*

(b) *If F satisfies the conditions in the Jacobian Conjecture and* $a(z)$ *is even, then G is also a polynomial map.*

Proof. Clearly (b) is an immediate consequence of (a). For (a), suppose $a(z)$ is even, then, replacing z by $-z$ in (2.3), we get

$$\begin{aligned}F(-z)&=\exp\left(a(-z)\frac{\partial}{\partial(-z)}\right)(-z)\\&=-\exp\left(-a(z)\frac{\partial}{\partial z}\right)z\\&=-G(z).\end{aligned}\tag{3.4}$$

Conversely, suppose the formal inverse $G(z)=-F(-z)$. Let $B=b(z)\partial/\partial z$ be the associated formal differential operator of G, i.e.

$$G(z)=\exp(B(z))z.\tag{3.5}$$

By the uniqueness of B, we have $B(z)=-A(z)$. On the other hand, from (3.4), we get

$$-F(-z)=\exp(A(-z))z.\tag{3.6}$$

Comparing (3.5) and (3.6), we have $A(-z)=B(z)=-A(z)$. Therefore $a(z)$ must be even. □

As an immediate consequence, we have the following:

Corollary 3.4. *With the same notation above, if* $a(z)$ *is even and* $F=e^Az$ *are polynomials, then* $\nabla a(z)=0$.

Note that this is not true for arbitrary formal power series $a(z)$.

Finally, we give a new proof for a theorem of Bass et al. in [2].

Theorem 3.5 (Bass et al. [2]). *Let $F(z) = z + H(z)$ be a polynomial map with $H(z)$ being homogeneous of degree $d \geqslant 2$. If $J(H)^2 = 0$, then the formal inverse map $G = z - H(z)$.*

Note that $J(H)^2 = 0$ implies that $\mathscr{J}(F) = 1$. Thus, the Jacobian Conjecture is true in this case.

Proof. First note that $J(H)z = dz$, since $H(z)$ is homogeneous of degree d. From $J(H)^2 = 0$, we have $0 = J(H)^2 z = dJ(H)H$, hence $J(H)H = 0$.

Now, let us calculate the formal differential operator $A(z) = a(z)\partial/\partial z$ by the incursive procedure in Proposition 2.1. Write $a(z) = \sum_{k=2}^{\infty} a_k(z)$, where $a_k(z)$ is homogeneous of degree k. By incursive formula (2.9), it is easy to see that $a_k(z) = 0$ if $k \neq m(d-1)+1$ for some $m > 0$. For $k = m(d-1)+1$ with $m > 0$, we have $a_d(z) = H(z)$ and

$$\begin{aligned} a_d(z) &= H(z), \\ a_{2d-1}(z) &= -\tfrac{1}{2}\left(H(z)\frac{\partial}{\partial z}\right)^2 z \\ &= -\tfrac{1}{2}\left(H(z)\frac{\partial}{\partial z}\right) H(z) \\ &= -\tfrac{1}{2} JH(z)\cdot H(z) \\ &= 0. \end{aligned}$$

By Mathematical Induction and incursive formula (2.9), it is easy to show that $a_{m(d-1)+1} = -(1/m!)(H(z)\partial/\partial z)^m z = 0$ for any $m \geqslant 2$. Therefore, we have $a(z) = H(z)$ and $A(z) = H(z)\partial/\partial z$. Note that $A^2(z) = 0$, so the formal inverse $G(z)$ of $F(z)$ is given by $G(z) = \mathrm{e}^{-A}z = z - H(z)$. $\square$

4. Some open problems

For the case of two variables, by using the residue and intersection theory in complex algebraic geometry, the author in [10] shows that, to prove the Jacobian Conjecture, it will be enough to consider the following special polynomial maps $F \in \mathscr{F}_1$.

Let $r(x)$ be a monic polynomial of degree $N+1 > 1$ with distinct roots and $\lambda(x)$ and $\mu(x)$ are unique polynomials satisfying

(a) $r(x)\mu(x) + r'(x)\lambda(x) = 1$.

(b) $\deg \mu(x) \leqslant N - 1$ and $\deg \lambda(x) \leqslant N$. (4.1)

Consider $F = (F_1, F_2)$, where

$$F_1(z_1, z_2) = r(z_1)H_1(z_1, z_2) - z_2\lambda(z_1)K_2(z_1, z_2), \tag{4.2}$$

$$F_2(z_1, z_2) = r(z_1)H_2(z_1, z_2) + z_2\lambda(z_1)K_1(z_1, z_2), \tag{4.3}$$

where H_i and K_i are polynomials in $z=(z_1,z_2)$ and satisfy $H_1K_1+H_2K_2=1$. Furthermore, without lose of generality, we also can assume that $F\in\mathscr{F}_1$. Then the Jacobian Conjecture is equivalent to the following

Conjecture 4.1. *Let $F=(F_1,F_2)$ as above, $A=a(z)\partial/\partial z$ be the associated formal differential operator of F, then $\nabla A\neq 0$.*

Finally, we ask the following very important and interesting question (this question for the case of one variable was first suggested to the author by Y.-Z. Huang), namely, if the analytic map F is well defined in an open neighborhood of $0\in\mathbb{C}^n$, is $a(z)$ convergent near $0\in\mathbb{C}^n$?

This is unknown both in the case of one variable and in the case F is a polynomial map with $\mathscr{J}(F)\equiv 1$. We believe the following conjecture is true, but we do not have much evidence.

Conjecture 4.2. *If F is convergent near $0\in\mathbb{C}^n$, then so is $a(z)$.*

The converse of the conjecture above is very easy to prove.

Proposition 4.3. *Suppose $a(z)\in\mathbb{C}_2[[z]]$ is convergent near point $0\in\mathbb{C}^n$, then so is the formal power series $F(z)=\mathrm{e}^{a(z)\partial/\partial z}z$.*

Proof. Consider the deformation $F(z;t)=\mathrm{e}^{ta(z)\partial/\partial z}z$, which satisfies the following differential equations:

$$\frac{\partial}{\partial t}F(z;t)=a(z)\frac{\partial}{\partial z}F(z;t), \tag{4.4}$$

$$F(z;0)=z. \tag{4.5}$$

It is well known in PDE that differential equation (4.4) with condition (4.5) has a unique analytic solution. Then as a formal power series solution of (4.4), F is convergent near $0\in\mathbb{C}^n$. □

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