Modeling the spread of fault in majority-based network systems: Dynamic monopolies in triangular grids

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\textbf{Abstract}

In a graph theoretical model of the spread of fault in distributed computing and communication networks, each element in the network is represented by a vertex of a graph where edges connect pairs of communicating elements, and each colored vertex corresponds to a faulty element at discrete time periods. Majority-based systems have been used to model the spread of fault to a certain vertex by checking for faults within a majority of its neighbors. Our focus is on \textit{irreversible majority processes} wherein a vertex becomes permanently colored in a certain time period if at least half of its neighbors were in the colored state in the previous time period. We study such processes on planar, cylindrical, and toroidal triangular grid graphs. More specifically, we provide bounds for the minimum number of vertices in a \textit{dynamic monopoly} defined as a set of vertices that, if initially colored, will result in the entire graph becoming colored in a finite number of time periods.

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1. Introduction

In distributed computing, crucial data are replicated and stored in multiple processors so that neighboring processors can compare such copies in an attempt to identify faults and prevent their spread. Recently, the spread of such faults has been modeled using a graph theoretical approach wherein each vertex of a graph $G$ represents a processor and a vertex is said to be \textit{colored} if the corresponding processor contains a faulty copy of the original data and \textit{not colored} otherwise [8,12,13]. Given an initial set of colored vertices of $G$, the faults might spread to the other vertices in the graph at discrete time periods according to different processes. For instance, this spreading might occur when a processor compares its data to that of its neighbors and converts to a permanently faulty state if a majority of its neighbors are in a faulty state. This spread can be modeled by \textit{irreversible majority processes} wherein a vertex becomes permanently colored in a certain time period if at least half of its neighbors were in the colored state in the previous time period. This model has also been used to study the spread of disease and opinion through social networks [6,9,10,12,13]. There is also a vast literature on other spread models in different types of networks following spreading rules other than the majority rule described above; for examples, we refer the reader to a few such recent articles [1–3,6,15]. A \textit{dynamic monopoly}, or \textit{dynamo}, is an initially colored vertex set of $G$ that will result in the full coloring of $G$ in a finite number of steps [4,5,7,8,10,11,14]. The minimum size of a dynamo of a graph $G$ will be denoted by $\text{min}_D(G)$ and a dynamo with exactly this number of vertices will be called \textit{optimal}.

Understanding dynamos of different families of graphs $G$ and being able to estimate $\text{min}_D(G)$ are potential key steps in the design of computer networks that resist fault propagation and in the design of immunization and containment strategies against the spread of diseases. For example, it is desirable to build a computer network topology that avoids...
small dynamos, as otherwise manufacturing defects or other malfunctions affecting a small number of units could result in total system failure. Similarly, one might seek to modify a given network topology to neutralize as many of its optimal dynamos as possible or to increase the $\min_D(G)$. In an analogous manner, dynamos can be used to build effective vaccination or quarantine tactics against the spread of disease. Alternatively, one might want to encourage the spread of a certain opinion starting from a small set of individuals. For example, a marketing company might want to recruit the people in an optimal dynamo within a Twitter network to spread positive reviews about a certain product to the entire group.

This paper has been inspired by the work of Flocchini et al. in determining bounds for the minimum size of dynamos for toroidal rectangular-like grids [8]. These authors were motivated by applications of such grids in modeling processors in a network, including the classical architecture for VLSI design. We propose that triangular grids could also be useful topologies for such networks, particularly when similar regularity is desired yet greater connectivity is required. In addition, triangular grids are widely used in computer graphics, three-dimensional geometric models, and geographic information systems (GISs) as they are convenient structures for computer hardware components. These commonly employed applications further motivated us to investigate dynamos of planar, cylindrical, and toroidal triangular grids. In Section 2, we study planar triangular grids and present an upper bound for the minimum size of their dynamos. In Sections 3 and 4, we study cylindrical and toroidal triangular grids, respectively, offering lower and upper bounds for the minimum size of dynamos for such grids. Table 1 summarizes the results in Sections 2–4 (formal definitions will follow). Our conclusions in Section 5 include suggestions for further research.

### Table 1
Summary of bounds on $\min_D(G)$ where $G$ is an $m$ by $n$ triangular grid.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Lower bound on $\min_D(G)$</th>
<th>Upper bound on $\min_D(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planar</td>
<td>NA</td>
<td>$\frac{\min(m, n)}{2}$ if $n$ and $m &gt; 1$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$\left\lceil \frac{n}{2} \right\rceil + 1$</td>
<td>$\left\lfloor \frac{2n}{3} \right\rfloor$ if $n \geq 2$ and $m \geq 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{n-m}{2} \right\rfloor$ if $n \geq m \geq 3$ and $n - 1$ is a multiple of 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-2}{2} \right\rfloor + \left\lfloor \frac{n-m}{2} \right\rfloor$ if $n \geq m \geq 3$ and $n - 1$ is not a multiple of 3</td>
</tr>
<tr>
<td>Toroidal</td>
<td>$\left\lceil \frac{\max(m, n)}{2} \right\rceil + 1$</td>
<td>$\left\lceil \frac{2m}{3} \right\rceil + 1$ if WLOG $m &gt; n &gt; 2$ and $n - 2$ is a multiple of 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left\lceil \frac{2n}{3} \right\rceil + 1$ if WLOG $m &gt; n &gt; 2$, $[n$ is a multiple of 3] and $[m - 2$ is not a multiple of 3]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left\lceil \frac{2m}{3} \right\rceil + 1$ if WLOG $m &gt; n &gt; 2$, $[n - 1$ is a multiple of 3] or $[n$ and $m - 2$ are multiples of 3]</td>
</tr>
</tbody>
</table>

**Fig. 1.** A 7 by 5 planar triangular grid with rows numbered 0–4 and columns 0–6.

Let $m$ and $n$ be two integers greater than 1. An $m$ by $n$ planar triangular grid $G$ consists of an array of $n$ rows of $m$ vertices $(x, y)$, with $0 \leq x \leq m - 1$, $0 \leq y \leq n - 1$, arranged on a standard Cartesian plane such that each vertex $(x, y)$ is adjacent to $(x, y + 1)$, $(x + 1, y + 1)$, and $(x + 1, y)$, provided that each coordinate is within its allowable range. Fig. 1 contains a 7 by 5 planar triangular grid where the row and column numbers are shown on the vertical and the horizontal axis, respectively.

In Theorem 1, we provide an upper bound for $\min_D(G)$ by constructing a dynamo with size equal to this upper bound.

**Theorem 1.** If $G$ is an $m$ by $n$ planar triangular grid, then $\min_D(G) \leq \left\lceil \frac{\min(m, n)}{2} \right\rceil$.

**Proof.** We may assume without loss of generality that $n \leq m$ (if $m \leq n$, perform a planar 90° counter clockwise rotation on $G$ followed by a reflection around the vertical axis, switch the roles of $m$ and $n$, and re-label vertices accordingly). In order to verify that $\min_D(G) \leq \left\lceil \frac{\min(m, n)}{2} \right\rceil$, it is enough to exhibit a dynamo of $G$ with $\left\lceil \frac{\min(m, n)}{2} \right\rceil$ vertices. Let $X$ be the set containing the following $\left\lfloor \frac{n}{2} \right\rfloor$ vertices: $(m - 1, 2k + 1)$ for $k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 2$, and $(m - 1, n - 1)$. Informally, $X$ contains the vertex in the top right corner of $G$ and every vertex in the last column of $G$ on an odd-numbered row. We will show that $X$ is a dynamo of $G$. 
Color the vertices of $X$ in time step 0. In time step 1, the uncolored vertices in column $m - 1$ will be colored since if vertex $(m - 1, 2k)$ for $k = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1$ is uncolored, it has degree 4 and is adjacent to the two colored vertices $(m - 1, 2k - 1)$ and $(m - 1, 2k + 1)$, while the uncolored vertex $(m - 1, 0)$ has degree 2 and is adjacent to the colored vertex $(m - 1, 1)$.

In the following time steps 2 through $n + 1$, the vertices in column $m - 2$ will be colored consecutively, one at a time from $(m - 2, 0)$ to $(m - 2, n - 1)$. To see that this is true, one must first note that $(m - 2, 0)$ will be colored in time step 2 because it has degree 4 and is adjacent to the two colored vertices $(m - 1, 0)$ and $(m - 1, 1)$. For each $k = 1, 2, \ldots, n - 2$, the vertex $(m - 2, k)$ will be colored in time step $k + 2$ since it has degree 6 and is adjacent to the three colored vertices $(m - 2, k - 1)$, $(m - 1, k)$ and $(m - 1, k + 1)$. Finally, $(m - 2, n - 1)$ is the last vertex in column $m - 2$ that will be colored in time step $n + 1$ since it has degree 4 and is adjacent to the two colored vertices $(m - 2, n - 2)$ and $(m - 1, n - 1)$.

In addition, whenever vertex $(m - 2, y)$ becomes colored in a certain time step, the vertices $(m - 2 - k, y - 2k)$ for $k = 1, 2, \ldots, \min\{m - 2, \left\lfloor \frac{n}{2} \right\rfloor\}$ will also be colored in the same time step for reasons similar to those used to justify the coloring of vertices in column $m - 2$. So, once the entire column $m - 2$ is colored, vertices $(m - 3, k)$ for $k = 0, 1, \ldots, n - 3$ will be already colored. Using similar arguments as above, it can be shown that the remaining two vertices in column $m - 3$, namely $(m - 3, n - 2)$ and $(m - 3, n - 1)$, will be colored in the next two time steps. This process repeats, one column at a time from column $m - 3$ to column 0, until all the vertices are colored and consequently $X$ is a dynamo of $G$. \hfill $\square$

The coloring progression described in the proof of Theorem 1 is illustrated in Figs. 2 and 3 for an odd and even $n$, respectively, where the vertices in the dynamo $X$ are highlighted and the vertex labels indicate the time step in which the vertices become colored. Fig. 2 shows the colored vertices in a 7 by 5 planar triangular grid immediately after time periods 0, 1, 2, 4, 6, and 15. Fig. 3 shows the colored vertices in a 7 by 6 triangular grid immediately after time periods 2, 7, and 16.

We conjecture that the dynamo constructed in Theorem 1 is actually an optimal dynamo and so $\min_D(G)$ would be exactly $\left\lfloor \frac{\min\{m,n\}}{2} \right\rfloor$.

**Note.** In subsequent sections, we will be constructing selected dynamos to obtain upper bounds for $\min_D(G)$. As seen in the proof of Theorem 1, the verification that coloring these sets of vertices at time zero results in certain color configurations at discrete time periods can be a straightforward but oftentimes tedious and lengthy exercise. For the sake of brevity, we will sometimes omit such details and favor informal descriptions and computer generated illustrations of the coloring progression on selected examples. The “∗” symbol will be used as a superscript for a sentence to indicate such omissions.
3. Cylindrical triangular grids

Let $m$ and $n$ be two integers with $m \geq 3$ and $n \geq 2$. An $m$ by $n$ cylindrical triangular grid $G$ consists of an array of $n$ rows of $m$ vertices $(x, y)$, with $0 \leq x \leq m - 1$, $0 \leq y \leq n - 1$, arranged on a standard Cartesian plane such that each vertex $(x, y)$ is adjacent to $(x, y + 1)$, $(x + 1, y + 1)$, and $(x + 1, y)$, where addition in the first coordinate is taken modulo $m$ and provided that the second coordinate is within its allowable range. Informally, we can build such a cylinder by taking an $m$ by $n$ planar triangular grid and adding edges connecting each vertex in the last column to the corresponding two vertices in the first column. In Fig. 4, we present a three-dimensional illustration of the 7 by 5 cylindrical triangular grid. In all subsequent figures in this section, we provide the underlying planar grid and omit the “wrapping around” edges in order to enhance clarity.

Theorems 2 and 3 provide a general lower and upper bound for $\text{min}_D(G)$, respectively. Theorem 4 provides another upper bound for $\text{min}_D(G)$ when $m \leq n$, lowering the upper bound of Theorem 3 in some cases.

**Theorem 2.** If $G$ is an $m$ by $n$ cylindrical triangular grid, then $\text{min}_D(G) \geq \left\lceil \frac{n}{2} \right\rceil + 1$.

**Proof.** In order to verify that $\text{min}_D(G) \geq \left\lceil \frac{n}{2} \right\rceil + 1$, we have to show that every dynamo of $G$ contains at least $\left\lceil \frac{n}{2} \right\rceil + 1$ vertices. Select an arbitrary proper dynamo $X$ of $G$, that is, $X$ is not the entire vertex set of $G$, and color its vertices at time period 0.

**Claim.** If $Y$ is the set of vertices in two consecutive rows of $G$, then $X \cap Y \neq \emptyset$. To verify this claim, let us assume by contradiction that $X \cap Y = \emptyset$, or equivalently, that no vertex in $Y$ is colored at time step 0. We must have $n \geq 3$, hence one or both rows of vertices in $Y$ must contain only vertices of degree 6, each with four neighbors in $Y$ and two neighbors outside $Y$. Thus, it is impossible for these degree 6 vertices to have three colored neighbors at any given time period since no vertex in $Y$ was initially colored, contradicting the fact that $X$ is a dynamo of $G$.

For $k = 0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil - 1$, if $Y_k$ is the set of vertices in rows $2k$ and $2k + 1$ of $G$, then the Claim implies $X \cap Y_k \neq \emptyset$. Therefore, $X$ must contain at least $\left\lceil \frac{n}{2} \right\rceil$ vertices. Assume by contradiction that $X$ contains exactly $\left\lceil \frac{n}{2} \right\rceil$ vertices. In this case, each $Y_k$ must contain exactly one colored vertex in $X$. Let $v$ be an arbitrary vertex of $G$ colored in time step 1. If $v$ has degree 4, we may assume without loss of generality that it belongs to the bottom row of $G$ (otherwise, perform a planar 180° rotation on $G$). The four neighbors of $v$ are in $Y_0$ which contains only one vertex of $X$ colored in time step 0, not enough to color $v$ in time step 1, a contradiction. If $v$ does not have degree 4, it must have degree 6 with four of its neighbors in $Y_0$ for some $0 \leq j \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and the other two neighbors both in $Y_{j-1}$ or both in $Y_{j+1}$; so at most two of the vertices adjacent to $v$ can be in $X$, not enough to color $v$ in time step 1, again a contradiction. Therefore, $X$ must contain at least $\left\lceil \frac{n}{2} \right\rceil + 1$ vertices. \(\square\)

The lower bound $\left\lceil \frac{n}{2} \right\rceil + 1$ for $\text{min}_D(G)$ provided in Theorem 2 is tight for $m$ by $n$ cylindrical triangular grids $G$ with $n = 3$ or 4 since in this case $\left\lceil \frac{n}{2} \right\rceil + 1 = \left\lceil \frac{3n}{4} \right\rceil$ which is the general upper bound for $\text{min}_D(G)$ as shown next in Theorem 3.

**Theorem 3.** If $G$ is an $m$ by $n$ cylindrical triangular grid, then $\text{min}_D(G) \leq \left\lceil \frac{2n}{3} \right\rceil$.

**Proof.** To verify that a set of vertices $X$ is a dynamo of $G$, it is enough to show that all the vertices in a single column will become colored by a certain time step. This is because a fully-colored column is the last column of the planar triangular grid obtained by removing the edges connecting the vertices in this colored column to the vertices in the next column (recall that the rows wrap around columns) and, as in the proof of Theorem 1, one can verify that all the remaining vertices will get colored.

Let $X$ be the set of vertices containing $(m - 2, 3k)$ and $(m - 3, 3k + 1)$ for $k = 0, 1, \ldots, \left\lfloor \frac{n-2}{3} \right\rfloor$, where the subtraction on the first coordinate is taken modulo $m$, plus, when $n - 1$ is a multiple of 3, one additional vertex $(m - 2, n - 1)$. Color the vertices in $X$ at time step 0. One can show that all the vertices in column $m - 2$ (for example) will get colored in the subsequent time steps*. Then our earlier discussion at the beginning of this proof implies that $G$ will be entirely colored. Therefore, $X$ is a dynamo with size $2 \left\lceil \frac{n-2}{3} \right\rceil + 1$, if $n - 1$ is not a multiple of 3, or $2 \left\lceil \frac{n-2}{3} \right\rceil + 1 + 1$, otherwise. (In Fig. 5, we provide an illustration of this coloring progression in a 7 by 5 cylindrical triangular grid immediately after time steps 0, 3, and 9.) Using properties of floor and ceiling functions, we can derive the more compact form $\left\lceil \frac{2n}{3} \right\rceil$ for the number of vertices in the dynamo $X$. \(\square\)
If G is an m by n cylindrical triangular grid with m ≤ n, then Theorem 4 provides alternative upper bounds for \( \text{min}_D(G) \) which are sometimes better than the general upper bound of Theorem 3.

**Theorem 4.** If G is an m by n cylindrical triangular grid with m ≤ n, then \( \text{min}_D(G) \leq \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil + \left\lceil \frac{n-m}{2} \right\rceil \), if \( n-1 \) is a multiple of 3, and \( \text{min}_D(G) \leq \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil + \left\lceil \frac{m-n}{2} \right\rceil \), otherwise.

**Proof.** We will first assume that \( m = n \). Let X be the set of vertices containing \( (3k, 3k) \) for \( k = 0, 1, \ldots, \left\lceil \frac{m-1}{3} \right\rceil \), and \( (3k + 2, 3k + 1) \) for \( k = 0, 1, \ldots, \left\lceil \frac{m-2}{3} \right\rceil \). Informally, X contains selected vertices in a diagonal band going from the lower left corner to the top two rows of G. Color the vertices in X in time step 0. One can show that vertex \((1,0)\) is colored in time step 1 and, in subsequent time steps, the coloring will spread diagonally from the lower-left to the upper-right corner, as well as from left to right along row 0 until all the vertices in G get colored*. (In Fig. 6, we provide illustrations of this coloring progression in m by n cylinders for \( m = 6, 7, \) and 8.) Therefore X is a dynamo of G with \( \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil + 2 \) vertices. Using properties of floor and ceiling functions, \( \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil + 2 = \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{n-2}{3} \right\rceil \).

Now suppose that \( m < n \). Let X be the set of vertices defined in the preceding paragraph. Let \((x, y)\) be the vertex in X with largest coordinates. So, \((x, y) = (3\left\lceil \frac{m-1}{3} \right\rceil + 2, 3\left\lceil \frac{m-1}{3} \right\rceil + 1) = (m - 1, m - 2)\) if m is a multiple of 3; \((x, y) = (3\left\lceil \frac{m-2}{3} \right\rceil + 2, 3\left\lceil \frac{m-2}{3} \right\rceil + 1) = (m - 1, m - 1)\) if m - 1 is a multiple of 3; and \((x, y) = (3\left\lceil \frac{m-1}{3} \right\rceil + 1, 3\left\lceil \frac{m-1}{3} \right\rceil) = (m - 2, m - 2)\) if \( m - 2 \) is a multiple of 3. Let Y be the set of vertices containing \((x, y + 2k)\) for \( k = 1, 2, \ldots, \left\lfloor \frac{m+n}{3} \right\rfloor \), if \( n - 1 \) is a multiple of 3, and for \( k = 1, 2, \ldots, \left\lfloor \frac{m+n}{3} \right\rfloor \), otherwise. Color the vertices in X ∪ Y at time step 0. One can show that the vertices in X cause the bottom m by m cylinder to be colored in subsequent time steps, advancing up the remainder of the cylinder using each initially colored vertex in Y as the third required colored neighbor to color an initial vertex on the row below it until all the vertices in G get colored*. (In Fig. 7, we provide illustrations of this coloring progression in the 8 by 12 cylindrical triangular grid, immediately after time steps 0, 20, and 31.) Therefore X ∪ Y is a dynamo of G with \( \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{n-2}{3} \right\rceil + \left\lceil \frac{n-m}{2} \right\rceil \) vertices, if \( n - 1 \) is a multiple of 3, and with \( \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{n-2}{3} \right\rceil + \left\lceil \frac{n-m}{2} \right\rceil \) vertices, otherwise. □

The upper bounds in Theorem 4 are smaller than the general upper bound provided in Theorem 3 for several infinite families of m by n cylindrical triangular grids with m ≤ n. For instance, when \( m = n \) and \( n - 2 \) is a multiple of 3; when \( m = n - 1 \) and \( n - 1 \) is a multiple of 3; when \( m = n - 2 \) and \( n \) is not multiple of 3; and when \( m = n - 3 \) and \( n \) is not multiple of 3, to mention a few.

### 4. Toroidal triangular grids

Let m and n be two integers greater than 2. An m by n toroidal triangular grid G consists of an array of n rows of m vertices \((x, y)\), with \( 0 \leq x \leq m - 1, 0 \leq y \leq n - 1 \) arranged on a standard Cartesian plane such that each vertex \((x, y)\) is adjacent to \((x, y + 1), (x + 1, y + 1), \) and \((x + 1, y)\), where addition in the first coordinate is taken modulo m and addition in the second coordinate is taken modulo n. Informally, we can build such a torus by taking an m by n planar triangular grid and adding edges connecting each vertex in the last column (resp., row) to the corresponding two vertices in the first column (resp., row). In Fig. 8, we present a three-dimensional illustration of the 7 by 5 toroidal triangular grid. In all subsequent figures in this section, we provide the underlying planar grid and omit all “wrapping around” edges in order to enhance clarity.

**Theorem 5.** If G is an m by n toroidal triangular grid, then \( \text{min}_D(G) \geq \left\lceil \frac{\max(n, m)}{2} \right\rceil + 1 \).

**Proof.** We may assume without loss of generality that \( n \geq m \) (if not, perform a planar 90° counter clockwise rotation on the planar representation of G followed by a reflection around the vertical axis, switch the roles of m and n, and re-label vertices accordingly). Select an arbitrary proper dynamo X of G, and color its vertices in time period 0. To verify the desired result, it is enough to show that X contains at least \( \left\lceil \frac{n}{2} \right\rceil + 1 \) vertices. We first need to state an auxiliary claim.

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*Fig. 5.* Coloring progression of the 7 by 5 cylindrical triangular grid as described in Theorem 3; the dynamo has size 4 = \( \left\lceil \frac{7+5}{2} \right\rceil \). Reminder: Henceforth the edges connecting the first and last columns are not shown.
Fig. 6. Coloring progression of the $m$ by $n$ cylindrical triangular grids with $m = n = 6, 7, \text{and } 8$ as described in Theorem 4; the dynamos have size $\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-2}{3} \right\rceil$.

Fig. 7. Coloring progression of the 8 by 12 cylindrical triangular grid as described in Theorem 4; the dynamo has size $7 = \left\lceil \frac{8}{3} \right\rceil + \left\lceil \frac{8-2}{3} \right\rceil + \left\lceil \frac{12-8}{3} \right\rceil$.

Claim. If $Y$ is the set of vertices in two consecutive rows of $G$ (rows $n - 1$ and 0 are considered consecutive), then $X \cap Y \neq \emptyset$. To check that this claim holds, let us assume by contradiction that $X \cap Y = \emptyset$, or equivalently, that no vertex in $Y$ is colored at time step 0. Every vertex in $Y$ must have degree 6 and have four neighbors in $Y$ and two neighbors outside $Y$; thus, it is impossible for a vertex in $Y$ to have three colored neighbors at any given time period since no vertex in $Y$ was initially colored, which contradicts the fact that $X$ is a dynamo of $G$.

Let $v$ be an arbitrary vertex of $G$ colored in time step 1 (such a vertex exists because $X$ is not the entire vertex set of $G$). We may assume without loss of generality that $v$ is in row $n - 1$ (if not, perform a rotation of $G$ around the horizontal axis.
and re-label vertices accordingly). For \( k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \), define \( Y_k \) as the set of vertices in rows \( 2k \) and \( 2k + 1 \) of \( G \). Using the previous claim, we conclude that \( X \cap Y_k \neq \emptyset \). So \( X \) must contain at least \( \left\lfloor \frac{n}{2} \right\rfloor \) vertices, one in each \( Y_k \).

Assume by contradiction that \( X \) contains exactly \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( n \) is even. Each \( Y_k \) must contain exactly one colored vertex in \( X \). Then vertex \( v \), which is in row \( n - 1 \) and is colored in time step 1, is in \( Y_j \) for \( j = \left\lfloor \frac{n}{2} \right\rfloor - 1 \). So \( v \) has four neighbors in \( Y_j \) and two neighbors in \( Y_0 \) with at most two of these neighbors in \( X \), which would not be enough to color \( v \) in time step 1, a contradiction. So \( X \) must have at least \( \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil + 1 \) vertices.

Alternatively, assume by contradiction that \( X \) contains exactly \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( n \) is odd. If \( 0 \) (resp., 1) contains a vertex in \( X \cap Y_0 \), then the only vertex in \( X \cap Y_k \) must be in row \( 2k \) (resp., \( 2k + 1 \)) for \( k = 0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil - 1 \), since otherwise there would exist two consecutive rows of \( G \) without a vertex in \( X \), going against the earlier Claim. Thus either row 0 or \( n - 2 \) would not contain a vertex in \( X \), which forces row \( n - 1 \) to contain a vertex in \( X \), otherwise the earlier Claim would be contradicted. Suppose row \( n - 1 \) and \( Y_k \) for \( k = 0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil - 1 \) contains exactly one vertex in \( X \), respectively. Then vertex \( v \) in row \( n - 1 \) has two neighbors in row 0 of \( Y_0 \), two neighbors in row \( n - 2 \) of \( Y_{\frac{n}{2} - 1} \), and two neighbors in row \( n - 1 \). But \( v \) is colored in step 1, so \( v \) must have at least three neighbors in \( X \). This could only be accomplished if row 0, \( n - 2 \), and \( n - 1 \) each contained a vertex in \( X \), a contradiction. Therefore row \( n - 1 \) or one of \( Y_k \) for \( k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \) will contain at least two vertices in \( X \). Hence, \( X \) has at least \( \left\lfloor \frac{n}{2} \right\rfloor + 2 = \left\lceil \frac{n}{2} \right\rceil + 1 \) vertices. \( \square \)

**Theorem 6.** If \( G \) is an \( m \times n \) toroidal triangular grid and \( m = n \), then \( \min_D(G) \leq m \).

**Proof.** Let \( X \) be the set of vertices containing \((0,0),(k,2k)\) and \((2k,k)\), for \( k = 1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \), and the additional vertex \((0,\frac{m}{2})\) if \( m \) is even. Informally, \( X \) contains selected vertices in two diagonal bands, one going from the lower left corner to roughly the middle of the top two rows, and the other from the lower left corner to roughly the middle of the last two columns. Color the vertices in \( X \) at time step 0. One can show that vertex \((1,1)\) is colored in time step 1 and, in subsequent time steps, the coloring will spread diagonally from the lower-left to the upper-right corner, as well as from left to right along row 0 and from bottom to top along column 0 until all the vertices in \( G \) get colored. (In Fig. 9, we provide illustrations of the coloring progression in \( m \) by \( n \) cylinders for \( m = n = 7 \) and 8.) Therefore \( X \) is a dynamo of \( G \) with \( 2 \left\lfloor \frac{m-1}{2} \right\rfloor + 1 \) vertices if \( m \) is odd, or with \( 2 \left\lceil \frac{m-1}{2} \right\rceil + 2 \) vertices if \( m \) is even. Using properties of floor and ceiling functions, one can verify that the preceding two expressions are exactly equal to \( m \), hence \( \min_D(G) \leq m \). \( \square \)

**Theorem 7.** If \( G \) is an \( m \times n \) toroidal triangular grid with \( m > n \), then

\[
\min_D(G) \leq \begin{cases} 
\frac{2m}{3} + 1, & \text{if } n - 2 \text{ is a multiple of } 3, \\
\frac{2m}{3}, & \text{if } n \text{ is a multiple of } 3 \text{ and } m - 2 \text{ is not a multiple of } 3, \\
\frac{2m}{3}, & \text{otherwise}.
\end{cases}
\]

**Proof.** To demonstrate the proposed upper bounds, we will construct dynamos which will be the union of two sets of vertices \( X \) and \( Y \). The set \( X \) will contain selected vertices in a diagonal band going from the lower left corner to the top two rows of \( G \), analogous to the diagonal band described in the proof of Theorem 4 but with a possible additional vertex. The set \( Y \) will contain selected vertices in a horizontal band in the last two rows of \( G \) starting from the right-most vertex in \( X \) and progressing to the top right corner of \( G \). We will say that \( X \cup Y \) has a “hockey stick” shape due to the obvious resemblance with the actual sports gear. We have three different cases to consider:

**Case 1:** \( n - 1 \) is a multiple of 3. Let \( X \) be the set of vertices containing \((3k, 3k)\) for \( k = 0, 1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor \) and \((3k + 2, 3k + 1)\) for \( k = 0, 1, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor \). Note that \((n - 1, n - 1)\) is the right-most vertex in \( X \). Let \( Y \) be the set of vertices containing
(n + 3k − 3, n − 2) for k = 1, 2, . . . , [m−n+2/3] and (n + 3k − 1, n − 1) for k = 1, 2, . . . , [m−n/3]. Therefore X ∪ Y has 
\((n-1)/3 + 1) + \left(\lfloor m-n+2/3 \rfloor + 1 \right) + \left\lfloor m-n/3 \right\rfloor \text{ vertices. Using properties of floor and ceiling functions, this expression can be simplified to} \left\lfloor m-1/3 \right\rfloor + \left\lfloor m+1/3 \right\rfloor + 1 = \left\lfloor 2m/3 \right\rfloor. (Fig. 10 shows X ∪ Y in m by 7 toroidal triangular grids for m = 10, 11, and 12.)

Case 2: n − 2 is a multiple of 3. Let X be the set of vertices containing (3k, 3k) and (3k − 2, 3k − 1) for k = 0, 1, . . . , [n−1/3], plus vertex (1, 2) if m − 1 is not a multiple of 3. Note that (n, n − 1) is the right-most vertex in X. Let Y be the set of vertices containing (n + 3k − 2, n − 2) for k = 1, 2, . . . , [m−n+1/3] and (n + 3k − 1, n − 1) for k = 1, 2, . . . , [m−n−1/3]. Therefore X ∪ Y has 
2 \left\lfloor (n-1)/3 + 1 \right\rfloor + \left\lfloor m-n+1/3 \right\rfloor + \left\lfloor m-n-1/3 \right\rfloor + 1 \text{ vertices, if m − 1 is not a multiple of 3, and one less than this value otherwise. Using properties of floor and ceiling functions, these two expressions can be simplified to} \left\lfloor m-1/3 \right\rfloor + \left\lfloor m+n/3 \right\rfloor + 2 \text{ if m − 1 is not a multiple of 3 and} \left\lfloor m-1/3 \right\rfloor + \left\lfloor m+n/3 \right\rfloor + 1 \text{ otherwise. These latter expressions are both equal to} \left\lfloor 2m/3 \right\rfloor + 1. (Fig. 11 shows X ∪ Y in m by 8 toroidal triangular grids for m = 13, 14, and 15.)

Case 3: n is a multiple of 3. Let X be the set of vertices containing (3k, 3k) and (3k + 2, 3k + 1) for k = 0, 1, . . . , [n−1/3], plus vertex (1, 2) if m − 2 is not a multiple of 3. Note that (n − 1, n − 2) is the vertex in X with largest coordinates. Let Y be the set of vertices containing (n + 3k − 2, n − 2) for k = 0, 1, . . . , [m−n+2/3] and (n + 3k + 1, n − 1) for k = 0, 1, . . . , [m−n−2/3]. Therefore X ∪ Y has 
2 \left\lfloor (n-1)/3 + 1 \right\rfloor + \left\lfloor m-n+1/3 \right\rfloor + \left\lfloor m-n-2/3 \right\rfloor + 1 \text{ vertices, if m − 2 is not a multiple of 3, and one less than this value otherwise. Using properties of floor and ceiling functions, these expressions can be simplified to} \left\lfloor m-1/3 \right\rfloor + \left\lfloor m+2/3 \right\rfloor + 3 = \left\lfloor 2m/3 \right\rfloor + 1 \text{ if m − 2 is not a multiple of 3, and} \left\lfloor m-1/3 \right\rfloor + \left\lfloor m+2/3 \right\rfloor + 2 = \left\lfloor 2m/3 \right\rfloor \text{ otherwise. (Fig. 12 shows X ∪ Y in m by 9 toroidal triangular grids for m = 13, 14, and 15.)}

Color the vertices in X ∪ Y at time step 0. It remains to be shown that all the vertices of G will become colored. In order to do so, we will first show that the set S consisting of vertices (k, k) for k = 0, 1, . . . , n − 1, and (k, n − 1) for k = n, n + 1, . . . , m − 1 is a dynamo of G. Define the set of vertices S_i for each i = 0, 1, . . . , n − 1, where S_0 = S and S_i is obtained from S_{i−1} by shifting it down one row, that is, S_i contains the vertex (x, y − i) (second coordinate subtraction is taken modulo n) for each (x, y) in S_{i−1}. Note that the vertex set of G is partitioned into the sets S_i for i = 0, 1, . . . , n − 1. Color the vertices in S_0 in time step 0. The set S_0 contains the vertices (k + 1, k) for k = 0, 1, . . . , n − 2, and (k, n − 2) for k = n − 1, n, . . . , m − 2. These vertices will become colored in subsequent time steps, beginning with (n − 1, n − 2) in time step 1 and spreading diagonally to the lower-left, and also horizontally to the right, one vertex per time step in each direction, as each vertex (x, y) following this order will be adjacent to the three vertices (x, y + 1), (x − 1, y), and (x + 1, y + 1) (first coordinate addition and subtraction are taken modulo m, and second coordinate addition is taken modulo n) previously colored in earlier time steps. A similar discussion shows that for each i = 2, 3, . . . , n − 1, if S_{i−1} is colored by a certain time step, then S_i will become colored in subsequent time steps. Therefore, the entire graph will become colored thus verifying that S is a dynamo of G.

For Case 1, where n − 1 is a multiple of 3, the vertex (n − 1, n − 2) becomes colored in time step 1 since its three neighbors (n − 1, n − 1), (n − 2, n − 3), and (n, n − 2) are colored. One can show that, in subsequent time steps, the coloring will spread from (n − 2, n − 2) through the main diagonal to (1, 1), as well as along row n − 1 from (n, n − 1) to (m − 1, n − 1)
Fig. 10. Dynamos $X \cup Y$ of size $\left\lfloor \frac{2m}{3} \right\rfloor$ in $m$ by 7 toroidal triangular grids for $m = 10, 11, 12$ as described in Theorem 7, Case 1.

Fig. 11. Dynamos $X \cup Y$ of size $\left\lfloor \frac{2m}{3} \right\rfloor + 1$ in $m$ by 8 toroidal triangular grids, $m = 13, 14, 15$ as described in Theorem 7, Case 2.

Fig. 12. Dynamos $X \cup Y$ in $m$ by 9 toroidal triangular grids for $m = 13, 14, 15$ as described in Theorem 7, Case 3; $X \cup Y$ has size $\left\lfloor \frac{2m}{3} \right\rfloor + 1$ if $m = 13, 15$, and size $\left\lfloor \frac{2m}{3} \right\rfloor$ if $m = 14$.

Fig. 13. Colored vertices immediately after all the vertices in $S$ become colored for a 12 by 7 toroidal triangular grid as described in Theorem 7, Case 1. until all the vertices in $S$ become colored*. (In Fig. 13, we show what vertices are colored immediately after all the vertices in $S$ become colored for a 12 by 7 toroidal triangular grid.)

For Cases 2 and 3, one can show that $S$ is colored by a similar process as in Case 1. For space considerations, we leave the details to the reader. (Full details are available from the authors, upon request.)

Since in all three cases it can be verified that $S$ becomes colored, from the earlier discussion we conclude that all the vertices in $G$ will become colored. Therefore, $X \cup Y$ is a dynamo of $G$. \hfill $\square$

The upper bound in Theorem 7 is not tight in certain instances. For example, consider the dynamo of a 12 by 6 toroidal triangular grid consisting of the 8 vertices $(0,0), (1,2), (3,3), (4,5), (5,4), (7,5), (8,4),$ and $(10,5)$. In Fig. 14 we show what vertices are colored immediately after all the vertices in $S$ become colored. So $\text{min}_G(G) \leq 8$, which is better than the upper bound $\left\lfloor \frac{2m}{3} \right\rfloor + 1 = 9$ provided in Theorem 7.

5. Conclusions

We have investigated irreversible majority processes in a fault propagation model as applied to planar, cylindrical, and toroidal triangular grids. These triangular grids, with their regular structure and high connectivity, are convenient topologies for computer hardware components, and they are widely used in computer graphics, three-dimensional geometric models,
and geographic information systems (GISs). We have presented upper bounds on the minimum size of dynamos for each class and lower bounds on the same for cylindrical and toroidal triangular grids. Our upper bounds are constructive, as we have demonstrated dynamos of the claimed size in each case.

We conjecture that the presented lower bound for planar triangular grids is the exact value for the minimum size of a dynamo. Establishing the exact size of optimal dynamos in general for any of the three types of triangular grids explored, or improving the presented upper/lower bounds, are possible paths for future investigation in the area.

We have also determined bounds for the minimum size of dynamos of planar, cylindrical, and toroidal hexagonal grids [3]. Another possibility for future work is to consider dynamos on grids formed by combinations of triangles, hexagons, and other polygons on various surfaces.

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