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Equivalence Relations on Stonian Spaces

Wolfgang Filter

Dipartimento di Matematica, Università di Palermo, I-90123 Palermo, Italy

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Quotient spaces of locally compact Stonian spaces which generalize in some sense the concept of Stone representation space of a Boolean algebra are investigated emphasizing the measure theoretical point of view, and a representation theorem for finitely additive measures is proved.

1. INTRODUCTION

1. Locally compact Stonian spaces play an outstanding role in representation theory of spaces of measures or, more generally, of Riesz spaces (see e.g. [4; Sect. 2], [17; Chap. 7] or [12]). It is the aim of this paper to study quotient spaces with respect to a natural equivalence relation on such locally compact Stonian spaces Y, thereby generalizing those quotient spaces arising from an equivalence relation which appears in representation theory of measure spaces [8; Sect. 2] (see Sect. 3 for the definition). It will turn out (Corollary 3.6) that the quotient spaces considered here have a close relation to Stone representation spaces of Boolean algebras.

The elementary facts will be presented in Sect. 3; in this section, it is only assumed that Y is a locally compact Hausdorff space. In Secs. 4 and 5, measures on Y and the quotient space are investigated. Finally, in Sect. 6, I present, as an application, a representation theorem for finitely additive measures which generalizes results going back to Halmos, Yosida–Hewitt, and Heider.

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2. Let me fix some notation.

For a set X, I denote by 1_A the characteristic function of a subset A of X; I write shortly $\{f < g\}$ for the set $\{x \in X: f(x) < g(x)\}$, provided $f, g \in \overline{\mathbb{R}}^X$, and use similar abbreviations.

Let $\mathscr{R}$ be a ring of subsets of X. The symbol $\mathscr{R}_\delta$ stands for the δ-ring generated by $\mathscr{R}$. I write $\mathscr{M}(\mathscr{R})$ for the Riesz space of all real-valued

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measures on $\mathscr{R}$ (a real-valued measure on $\mathscr{R}$ is a countably additive finite-valued set function with locally bounded variation).

Let $\mu \in \mathscr{M}(\mathscr{R})$. Then $|\mu|$ is defined as the map

$$\mathscr{R} \to \mathbb{R}^+, A \mapsto \sup_{\mathscr{D} \in \Delta(A)} \sum_{D \in \mathscr{D}} |\mu(D)|,$$

where $\Delta(A)$ denotes the set of all finite partitions of A in $\mathscr{R}$. I set

$$\mathscr{N}(\mu) := \{A \subset X : A \text{ is a } \mu\text{-null set}\},$$

$$\mathscr{L}(\mu) := \{A \subset X : 1_A \in \mathscr{L}^1(\mu)\},$$

$$\mathscr{L}^1_{\mathrm{loc}}(\mu) := \{f \in \overline{\mathbb{R}}^X : f1_A \in \mathscr{L}^1(\mu) \text{ for all } A \in \mathscr{R}\}.$$

The notion of integrability is the one used by Constantinescu–Weber [6] or by Ionescu-Tulceas for their investigations of localizable spaces [15; Chap. I, Sect. 8]; in locally compact Hausdorff spaces—which will be considered mainly in this paper—it coincides with Bourbaki's essential integrability [3; Sect. 2].

If $\mathscr{M}$ is a band of $\mathscr{M}(\mathscr{R})$, I write

$$\mathscr{M}_b := \{\mu \in \mathscr{M} : \mu \text{ is bounded}\},$$

$$\mathscr{M}_c := \{\mu \in \mathscr{M} : \exists A \in \mathscr{R} \text{ with } X \backslash A \in \mathscr{N}(\mu)\},$$

and, for $\mu \in \mathscr{M}$, I denote by $\mathscr{M}_\mu$ the band of $\mathscr{M}$ generated by μ.

By δ_x I always mean the Dirac measure at $x \in X$, i.e.

$$\delta_x : \mathscr{R} \to \mathbb{R}, A \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Now let Y be a Hausdorff space. I write

$$C(Y) := \{f \in \mathbb{R}^Y : f \text{ is continuous}\},$$

$$C_\infty(Y) := \{f \in \overline{\mathbb{R}}^Y : f \text{ is continuous}, \{|f| = \infty\} \text{ is nowhere dense}\},$$

$$\mathscr{K}(Y) := \{K \subset Y : K \text{ is compact}\},$$

$$\mathscr{B}_c(Y) := \{B \subset Y : B \text{ is a relatively compact Borel set}\}.$$

Then $\mathscr{B}_c(Y)$ is the δ-ring generated by $\mathscr{K}(Y)$. I denote by $\mathscr{M}_R(Y)$ the Riesz space of Radon measures on Y, i.e. the set of measures on Y which are interior regular with respect to the compact subsets of Y; I always consider $\mathscr{B}_c(Y)$ to be the natural domain of a Radon measure on Y. Furthermore I denote by $\mathscr{M}(Y)$ the band of $\mathscr{M}_R(Y)$ consisting of those $\mu \in \mathscr{M}_R(Y)$ which are also interior regular with respect to the open subsets of Y; the elements of $\mathscr{M}(Y)$ are called normal Radon measures.

A Stonian space is an extremally disconnected locally compact Hausdorff space. Let Y be Stonian; then $C_\infty(Y)$ is a Riesz space [17; 47.2], and for $\mu, \nu \in \mathscr{M}(Y)$ we have: Every nowhere dense set is μ-null, supp μ is open-closed, $\mu \perp \nu$ iff supp $\mu \cap$ supp $\nu = \emptyset$, $\mu \ll \nu$ iff supp $\mu \subset$ supp ν (cf. Dixmier [7]).

A Stonian space Y is called hyperstonian if $\bigcup_{\mu \in \mathscr{M}(Y)}$ supp μ is dense in Y.

For elementary Riesz space terminology, I refer to [1; Chap. I] or [17].

2. PRELIMINARIES

In this section, I collect some notions and results that will be used in the sequel.

Let X be a non-empty set, and let $\mathscr{D}$ be a set of subsets of X which is closed under finite intersections.

A filter $\mathscr{F}$ on X is called $\mathscr{D}$-filter if $\mathscr{F} \cap \mathscr{D}$ is a filter basis of $\mathscr{F}$.

A $\mathscr{D}$-filter $\mathscr{F}$ on X is called maximal if there exists no $\mathscr{D}$-filter properly containing $\mathscr{F}$. Using Zorn's Lemma, it is easy to see that each $\mathscr{D}$-filter is contained in a maximal $\mathscr{D}$-filter.

Analogously to a characterization of ultrafilters we have the following result, which can be verified by standard arguments:

PROPOSITION 2.1. *For a $\mathscr{D}$-filter $\mathscr{F}$ on X, the following are equivalent*:

(a) *$\mathscr{F}$ is maximal*;

(b) *$D \in \mathscr{D}$ and $F \cap D \neq \emptyset$ for all $F \in \mathscr{F}$ imply $D \in \mathscr{F}$*;

(c) *$D \in \mathscr{F}$ or $X \setminus D \in \mathscr{F}$, for each $D \in \mathscr{D}$.*

Let X_1 and X_2 be non-empty sets, let $\mathscr{R}_i$ be a ring of subsets of X_i $(i = 1, 2)$, let $\mu \in \mathscr{M}(\mathscr{R}_1)$, and let $\phi\colon X_1 \to X_2$ be a mapping such that $\phi^{-1}B \in \mathscr{L}(\mu)$ for all $B \in \mathscr{R}_2$. I denote by $\phi\mu$ the measure

$$\mathscr{R}_2 \to \mathbb{R},\ B \mapsto \int 1_{\phi^{-1}B}\, d\mu.$$

Then obviously $|\phi\mu| \leqslant \phi(|\mu|)$.

Let $\mathscr{S} \subset \mathscr{L}(\mu)$. I call μ $\mathscr{S}$-quasiregular in $A \in \mathscr{L}(\mu)$ if for every $\varepsilon > 0$, there exists $B \in \mathscr{S}$ such that $B \setminus A \in \mathscr{N}(\mu)$ and $\int 1_{A \setminus B}\, d|\mu| < \varepsilon$. If $\mathscr{R}$ is a δ-ring and μ is $\mathscr{S}$-quasiregular in all $A \in \mathscr{R}$, then obviously μ is $\mathscr{S}$-quasiregular in all $A \in \mathscr{L}(\mu)$.

PROPOSITION 2.2. *If* μ *is* $\phi^{-1}\mathscr{R}_2$*-quasiregular in all* $A \in \mathscr{L}(\mu)$, *then* $|\phi\mu| = \phi(|\mu|)$.

Proof. Only "$\geqslant$" has to be shown. Set $v := |\mu|$, and let $D \in \mathscr{R}_2$. Consider the equation

$$\int 1_B \, dv = \sup_{\mathscr{A} \in \Delta(B)} \sum_{A \in \mathscr{A}} \left| \int 1_A \, d\mu \right|, \tag{*}$$

where $\Delta(B)$ denotes the set of finite partitions of B in $\mathscr{L}(\mu)$.

(*) is obviously true for $B \in \mathscr{R}$, and thus also for $B \in \mathscr{R}_\delta$. From this and [6; 5.4.17] we conclude that (*) holds for all $B \in \mathscr{L}(\mu)$, and so in particular

$$(\phi v)(D) = \sup_{\mathscr{A} \in \Delta(\phi^{-1}D)} \sum_{A \in \mathscr{A}} \left| \int 1_A \, d\mu \right|.$$

Let $\varepsilon > 0$, let $\mathscr{A} \in \Delta(\phi^{-1}D)$, and let n be the number of elements of $\mathscr{A}$.

For each $A \in \mathscr{A}$ there exists $F_A \in \mathscr{R}_2$ with $\phi^{-1}F_A \backslash A \in \mathscr{N}(\mu)$ and $\int 1_{A \backslash \phi^{-1}F_A} \, dv < \varepsilon/n$, and we can assume $F_A \subset D$. Then

$$F := \bigcup_{\substack{(A, A') \in \mathscr{A} \times \mathscr{A} \\ A \neq A'}} F_A \cap F_{A'} \in \mathscr{R}_2$$

and $\phi^{-1}F \in \mathscr{N}(\mu)$. The sets $F_A \backslash F$ together with the set $D \backslash \bigcup_{A \in \mathscr{A}} (F_A \backslash F)$ form a partition of D in $\mathscr{R}_2$, and thus

$$|\phi\mu|(D) \geqslant \sum_{A \in \mathscr{A}} \left| \int 1_{\phi^{-1}(F_A \backslash F)} \, d\mu \right| \geqslant \sum_{A \in \mathscr{A}} \left| \int 1_A \, d\mu \right| - \varepsilon.$$

$\mathscr{A}$ and ε being arbitrary, we conclude $|\phi\mu|(D) \geqslant (\phi v)(D)$. ∎

Now let $\mathscr{R}$ be a ring of subsets of a set X, and let $\mathscr{M}$ be a band of $\mathscr{M}(\mathscr{R})$. For $f \in \mathbb{R}^X$, set

$$\mathscr{M}(f) := \{\mu \in \mathscr{M} : f \in \mathscr{L}^1(\mu)\}$$

and

$$\dot{f} \colon \mathscr{M}(f) \to \mathbb{R}, \mu \mapsto \int f \, d\mu.$$

Then, following [4], I denote by $\mathscr{L}_\infty := \mathscr{L}_\infty(\mathscr{M})$ the set of all $f \in \mathbb{R}^X$ for which the ideal $\mathscr{M}(f)$ is order dense in $\mathscr{M}$. By [10], $\mathscr{L}_\infty$ is exactly the set of all $f \in \mathbb{R}^X$ which are μ-measurable for all $\mu \in \mathscr{M}$ (for the notion of μ-measurability, see [6; 5.4.2]). Hence $\mathscr{L}_\infty$ is a unital subalgebra of $\mathbb{R}^X$ and a σ-ideal of $\mathbb{R}^X$ (cf. [4; 1.5.2]).

According to Constantinescu [4; 2.3.1], an ordered triple (Y, u, v) is called a representation of $(X, \mathscr{R}, \mathscr{M})$ if

(a) Y is a hyperstonian space;

(b) $u: \mathscr{L}_\infty \to C_\infty(Y)$ is a homomorphism of unital algebras;

(c) $u(\sup f_n) = \sup u(f_n)$ for each upper bounded sequence (f_n) in $\mathscr{L}_\infty$;

(d) $U_A := \operatorname{supp} u(1_A)$ is compact for all $A \in \mathscr{R}$, and $Y = \bigcup_{A \in \mathscr{R}} U_A$;

(e) $v: \mathscr{M} \to \mathscr{M}(Y)$ is a Riesz isomorphism;

(f) for all $f \in \mathscr{L}_\infty$ and all $\mu \in \mathscr{M}$ we have

$$f \in \mathscr{L}^1(\mu) \Leftrightarrow uf \in \mathscr{L}^1(v\mu), \text{ and in this case } \int f\, d\mu = \int (uf)\, d(v\mu),$$

$$f \in \mathscr{L}^1_{\text{loc}}(\mu) \Leftrightarrow uf \in \mathscr{L}^1_{\text{loc}}(v\mu), \text{ and in this case } v(f \cdot \mu) = (uf) \cdot (v\mu).$$

Since the map $\mathscr{M}(\mathscr{R}_\delta) \to \mathscr{M}(\mathscr{R})$, $\mu \mapsto \mu|_{\mathscr{R}}$ is a Riesz isomorphism, it is no loss of generality to consider only δ-rings (as is done in [4]).

By [4; 2.3.6, 2.3.8], there exists always a unique representation (Y, u, v) of $(X, \mathscr{R}, \mathscr{M})$.

Some conditions in the definition above can be weakened, as is shown in

Proposition 2.3 [9; 11.10, 11.12]. *(Y, u, v) is a representation of $(X, \mathscr{R}, \mathscr{M})$ iff the following assertions hold*:

(a) *Y is a hyperstonian space*;

(b) *$u: \mathscr{L}_\infty \to C_\infty(Y)$ is a map with $u(1_X) = 1_Y$*;

(c) *U_A is compact for all $A \in \mathscr{R}$, and $Y = \bigcup_{A \in \mathscr{R}} U_A$*;

(d) *$v: \mathscr{M} \to \mathscr{M}(Y)$ is a Riesz isomorphism*;

(e) *for all $f \in \mathscr{L}_\infty$ and all $\mu \in \mathscr{M}_c^+$, we have*

$$f \in \mathscr{L}^1(\mu) \Leftrightarrow uf \in \mathscr{L}^1(v\mu), \textit{ and in this case } \int f\, d\mu = \int (uf)\, d(v\mu).$$

In the context given above, I call Y a representation space for $(X, \mathscr{R}, \mathscr{M})$.

3. THE EQUIVALENCE RELATION ON Y, AND ELEMENTARY TOPOLOGICAL PROPERTIES OF THE QUOTIENT SPACE

Let Y be a locally compact Hausdorff space, and let $\mathscr{R}$ be a ring of open-compact subsets of Y with $Y = \bigcup_{A \in \mathscr{R}} A$. (As considered in several examples below, Y may be the representation space of a triple $(X, \mathscr{S}, \mathscr{M})$ and $\mathscr{R} = \{U_A: A \in \mathscr{S}\}$.)

I introduce an equivalence relation $\sim$ on Y by

$$y \sim z: \Leftrightarrow 1_A(y) = 1_A(z) \text{ for all } A \in \mathscr{R}.$$

I denote the equivalence class of y by $[y]$, and set $[B] := \{[y]: y \in B\}$ for all $B \subset Y$. We have $[y] = \bigcap_{A \in \mathscr{R}, y \in A} A$, and thus $[y]$ is a compact subset of Y.

Let $\mathscr{R}_a$ denote the algebra of sets generated by $\mathscr{R}$. Since $\{B \subset Y: B$ is open-compact$\}$ is an algebra of sets, each $A \in \mathscr{R}_a$ is open-compact, and since $\{B \subset Y: 1_B(y) = 1_B(z)\}$ is again an algebra of sets, we have

$$y \sim z \Leftrightarrow (y \sim z \text{ with respect to } \mathscr{R}_a).$$

PROPOSITION 3.1. *The following assertions hold*:

(a) $\bigcap_{A \in \mathscr{F}} A \neq \varnothing$ *for each* $\mathscr{R}$*-filter* $\mathscr{F}$.

(b) *For each maximal* $\mathscr{R}$*-filter* $\mathscr{F}$ *on* Y *there exists an equivalence class* $[y_F]$ *such that* $\bigcap_{A \in \mathscr{F}} A = [y_F]$; *if* $\mathscr{F}'$ *is a maximal* $\mathscr{R}$*-filter with* $\mathscr{F}' \neq \mathscr{F}$, *then* y_F *and* $y_{F'}$ *are not equivalent.*

(c) *If* $\mathscr{R}$ *is an algebra of sets, then the set of maximal* $\mathscr{R}$*-filters on* Y *and the set* $[Y]$ *are in bijection via* $\mathscr{F} \mapsto [y_F]$.

Proof. (a) Since all $A \in \mathscr{F} \cap \mathscr{R}$ are compact, we have $\bigcap_{A \in \mathscr{F} \cap \mathscr{R}} A \neq \varnothing$ which implies the assertion.

(b) Set $F := \bigcap_{A \in \mathscr{F}} A$, and fix $y_F \in F$. Let $z \in F$. By Proposition 2.1(c) we have either $y, z \in B$ or $y, z \notin B$ for each $B \in \mathscr{R}$; hence $z \sim y$, and thus $F \subset [y]$. Now let $z \in [y]$. For each $A \in \mathscr{F}$ there exists $B \in \mathscr{F} \cap \mathscr{R}$ with $B \subset A$, hence $z \in B \subset A$; thus $z \in F$, which implies $[y] \subset F$.

Now let $\mathscr{F}'$ be a maximal $\mathscr{R}$-filter with $\mathscr{F}' \neq \mathscr{F}$. Then there exists, say, $A \in \mathscr{F} \cap \mathscr{R} \backslash \mathscr{F}'$, hence, by Proposition 2.1(c), $X \backslash A \in \mathscr{F}'$. Since $y_F \in A$ and $y_{F'} \in X \backslash A$, they cannot be equivalent.

(c) Let $y \in Y$. Let $\mathscr{F}$ be the filter generated by the filter base $\mathscr{G} := \{A \in \mathscr{R}: y \in A\}$. We have $[y] = \bigcap_{A \in \mathscr{G}} A$, and since $Y \backslash B \in \mathscr{R}$ for all $B \in \mathscr{R}$, Proposition 2.1(c) shows that $\mathscr{F}$ is a maximal $\mathscr{R}$-filter. In view of (b), all is proved. ∎

$[Y]$ is endowed with the quotient topology, i.e. the finest topology making the map

$$\pi: Y \to [Y], y \mapsto [y]$$

continuous. I set

$$\mathscr{F} := \{ f \in C_\infty(Y) \colon f|_{[y]} = \text{const. for all } y \in Y \};$$
$$\tilde{f} \colon [Y] \to \overline{\mathbb{R}}, [y] \mapsto f(y) \quad \text{for } f \in \mathscr{F};$$
$$\mathscr{T} := \{ U \subset Y \colon 1_U \in \mathscr{F} \}.$$

The next observations are easily verified:

PROPOSITION 3.2. *The following assertions hold*:

(a) $\tilde{f} \in C_\infty([Y])$ *and* $f = \tilde{f} \circ \pi$ *for all* $f \in \mathscr{F}$.

(b) $\mathscr{F} \to C_\infty([Y])$, $f \mapsto \tilde{f}$ *is injective.*

(c) $\mathscr{F}$ *is a sublattice of* $C_\infty([Y])$, *and* $\mathscr{F} \cap C(Y)$ *is a Riesz space and a unital subalgebra of* $C_\infty(Y)$.

(d) *The restriction of the map* $f \mapsto \tilde{f}$ *to* $\mathscr{F} \cap C(Y)$ *is a homomorphism of Riesz spaces and of unital algebras.*

(e) $\mathscr{T}$ *is an algebra of sets containing* $\mathscr{R}$.

(f) $\pi U = \{ \widetilde{1_U} = 1 \}$ *is open-closed and* $\pi^{-1}(\pi U) = U$ *for all* $U \in \mathscr{T}$.

That $\mathscr{T}$ may contain $\mathscr{R}_a$ properly, can be seen by considering $Y := \mathbb{N}$ and $\mathscr{R} := \mathscr{K}(Y)$.

Some beautiful properties are lost by passing from Y to $[Y]$ even if Y is Stonian, as is shown in

EXAMPLE 3.3. Let $Y := \beta\mathbb{N}$ and $\mathscr{R} := \{ A \subset \mathbb{N} \colon A \text{ finite} \} \cup \{ \beta\mathbb{N} \setminus A \colon A \subset \mathbb{N}, A \text{ finite} \}$.

Then $[Y]$ is the Alexandrov compactification of $\mathbb{N}$, and we have

$$\mathscr{F} = \{ f \in \overline{\mathbb{R}}^Y \colon \exists \alpha := \lim_{n \to \infty} f(n) \in \overline{\mathbb{R}}, f|_{\beta\mathbb{N} \setminus \mathbb{N}} = \alpha \}.$$

By the definition

$$\left. \begin{aligned} &f(2n) := f(2n-1) := n+1, \\ &g(2n) := n+1, \ g(2n-1) := n \end{aligned} \right\} \quad \text{for all} \quad n \in \mathbb{N},$$

there are defined functions $f, g \in \mathscr{F}$. But $f - g \notin \mathscr{F}$ since $(f-g)(2n) = 0$, $(f-g)(2n-1) = 1$; hence $\mathscr{F}$ is not a vector space.

Also, $\tilde{f} - \tilde{g} \notin C_\infty([Y])$; hence $C_\infty([Y])$ is not a vector space (and thus $[Y]$ is not Stonian).

Moreover, $\mathscr{F}$ is not closed under forming countable suprema; for this claim, consider e.g. the sequence of functions $f_n := \sum_{k=1}^{n} 1_{\{2k\}}$.

Observe that there exists a triple $(X, \mathscr{S}, \mathscr{M})$ such that Y is a representation space for it and $\mathscr{R} = \{U_A : A \in \mathscr{S}\}$: Indeed, let $X := \mathbb{N}$, $\mathscr{S} := \{A \subset \mathbb{N} : A$ or $\mathbb{N} \setminus A$ is finite$\}$, $\mathscr{M}$ the band of $\mathscr{M}(\mathscr{S})$ generated by all Dirac measures on X.

PROPOSITION 3.4. *Let $K \in \mathscr{K}(Y)$, and set $L := \bigcup_{y \in K} [y]$. Then*:

(a) *If F is a closed subset of Y with $F \cap L = \emptyset$, then there exists $A \in \mathscr{R}$ with $L \subset A$ and $F \cap A = \emptyset$.*

(b) *L is compact.*

Proof. (a) For each $y \in K$, we have $\emptyset = [y] \cap F = \bigcap_{A \in \mathscr{R}, y \in A} (A \cap F)$; hence there exists $A_y \in \mathscr{R}$ with $y \in A_y$ and $A_y \cap F = \emptyset$. There are $y_1, ..., y_n \in K$ with $K \subset \bigcup_{k=1}^{n} A_{y_k} =: A$.

(b) Let (U_ι) be an open cover of L, and set $F := Y \setminus \bigcup U_\iota$. By (a), there exists $A \in \mathscr{R}$ with $L \subset A$ and $F \cap A = \emptyset$. Then $A \subset \bigcup U_\iota$, and thus A (hence also L) is covered by finitely many of the U_ι's. ∎

We can now collect the main properties of $[Y]$ and π:

THEOREM 3.5. *The following assertions hold*:

(a) *$[Y]$ is a totally disconnected locally compact Hausdorff space, and $\{\pi A : A \in \mathscr{R}\}$ is a base for the topology of $[Y]$.*

(b) *$\pi^{-1} K \in \mathscr{K}(Y)$ for all $K \in \mathscr{K}([Y])$.*

(c) *πF is closed for each closed $F \subset Y$.*

(d) *If K is a compact subset of $[Y]$ and F is a closed subset of $[Y]$ with $K \cap F = \emptyset$, then there exists $A \in \mathscr{R}$ with $K \subset \pi A$ and $F \cap \pi A = \emptyset$.*

(e) *For all $K \in \mathscr{K}([Y])$ we have*

$$K = \bigcap_{\substack{A \in \mathscr{R} \\ \pi^{-1} K \subset A}} \pi A \text{ and } \pi^{-1} K = \bigcap_{\substack{A \in \mathscr{R} \\ \pi^{-1} K \subset A}} A.$$

(f) *The map*

$$\mathscr{R} \to \{U \subset [Y] : U \text{ open-compact}\}, \; A \mapsto \pi A$$

is an order isomorphism onto (the order given by the inclusion relation).

Proof. Using Proposition 3.4(b), assertions (a),(b),(c) follow from [2; Sect. 10, Prop. 17], except for the total disconnectedness. To prove (d), observe that $\pi^{-1} K$ is compact by (c), and apply Proposition 3.4(a) for $\pi^{-1} K$ and $\pi^{-1} F$. Now let $[y] \subset U \subset [Y]$, with U open. By (d), there exists $A \in \mathscr{R}$ with $[y] \in \pi A$ and $([Y] \setminus U) \cap \pi A = \emptyset$; since πA is open-closed by

Proposition 3.2(f) and $[y] \in \pi A \subset U$, it follows that $[Y]$ is totally disconnected and that $\{\pi A : A \in \mathscr{R}\}$ is a base for the topology of $[Y]$. The first assertion of (e) follows from (d), while the second one is derived from the first one, observing Proposition 3.2(f). Finally, (f) is a consequence of (d) and Proposition 3.2(f). ∎

That π is also for Stonian Y in general not open, can be seen by modifying Example 3.3: Let $Y := \beta\mathbb{N} \cup \{0\}$ (I assume $0 \notin \mathbb{N}$) and $\mathscr{R} := \{A \subset \mathbb{N} : A \text{ finite}\} \cup \{\beta\mathbb{N} \cup \{0\} \setminus A : A \subset \mathbb{N}, A \text{ finite}\}$. Then $[Y] = \mathbb{N} \cup \{[0]\}$ is the Alexandrov compactification of $\mathbb{N}$, and $\pi(\{0\}) = \{[0]\}$ is not open.

In the case of algebras of sets, the space $[Y]$ is very familiar:

COROLLARY 3.6. *If $\mathscr{R}$ is an algebra of sets, then $[Y]$ is the Stone representation space of the Boolean algebra $\mathscr{R}$.*

Proof. Immediate from Theorem 3.5(a), (f). ∎

The topology on $[Y]$ can now be described in the following way:

COROLLARY 3.7. *The topology on $[Y]$ is the coarsest for which all maps $\widetilde{1_U}$ are continuous ($U \in \mathscr{T}$), and the coarsest for which all maps $\tilde{f}$ are continuous ($f \in \mathscr{F}$).*

Proof. Let τ be the quotient topology on $[Y]$, and let σ_1 (σ_2, resp.) be the coarsest topology for which all $\widetilde{1_U}$ (all $\tilde{f}$, resp.) are continuous.

To show that $\tau \subset \sigma_1$ holds, let $y \in Y$, and let W be a τ-open neighbourhood of $[y]$. By Proposition 3.4(a), there exists $A \in \mathscr{R}$ with $y \in A \subset \pi^{-1} W$. From $\{\widetilde{1_A} > 1/2\} \subset W$ it follows that W is a σ_1-neighbourhood of $[y]$. The inclusions $\sigma_1 \subset \sigma_2 \subset \tau$ are obvious. ∎

The following corollary describes the restriction of $\sim$ to an open-closed subset of Y.

COROLLARY 3.8. *Let Y_1 be an open-closed subset of Y, and let $\mathscr{R}_1 := \{A \cap Y_1 : A \in \mathscr{R}\}$. For all $y \in Y_1$ let (y) be the equivalence class with respect to $\mathscr{R}_1$, and endow $(Y_1) := \{(y) : y \in Y_1\}$ with the quotient topology with respect to $\pi_1 : Y_1 \to (Y_1)$, $y \mapsto (y)$. Then the well-defined map*

$$\phi : (Y_1) \to \pi Y_1, (y) \mapsto [y] \quad (\textit{where } y \in (y))$$

is a homeomorphism.

Proof. It is easy to see that ϕ is bijective and continuous. To show that ϕ^{-1} is continuous, take a closed set F of (Y_1). By Theorem 3.5(c), $G := \pi(\pi_1^{-1} F)$ is closed in $[Y]$, hence also in πY_1; moreover $\phi F = G$. ∎

4. NORMAL POINTS AND MEASURES

From now on, let Y be a Stonian space.

In this section I assume that the Hahn decomposition property for $\mathcal{M}(Y)$ and $\mathcal{R}$ is satisfied, i.e. for all $A \in \mathcal{R}$ and all $\mu, \nu \in \mathcal{M}(Y)^+$, $\mu \perp \nu$, there exists $B \in \mathcal{R}$ with $\mu(B) = 0 = \nu(A \backslash B)$.

Some of the results of this section are only minor generalizations of results obtained in [8]; in these cases I refer to the proofs given there, which can be adopted with only slight modifications.

I call $y \in Y$ normal (or $\mathcal{R}$-normal if it is necessary to specify the underlying ring of sets $\mathcal{R}$), if $[y] = \{y\}$. Moreover I set

$$Y_0 := \bigcup_{\mu \in \mathcal{M}(Y)} \operatorname{supp} \mu.$$

Since Y is Stonian, Y_0 is open.

PROPOSITION 4.1. *If y and z are two different points of Y_0, then y and z are not equivalent.*

Proof. See [8; 2.5]. Let me remark that also in this proof the assumption is used that Y be Stonian. ∎

Thus, if $Y = Y_0$, then all points of Y are normal. The converse is not true, as the following example shows.

EXAMPLE 4.2. Let X be an uncountable set, endowed with the discrete topology. Set $Y := \beta X$ and $\mathcal{R} := \{A \subset Y : A \text{ open-compact}\}$. All points of Y are normal, but $Y_0 = \bigcup_{A \subset X,\, A \text{ countable}} \overline{A}$. Thus Y_0 is not compact, whence $Y_0 \neq Y$.

Using an indirect argument, we get as an easy consequence of Proposition 4.1 the following

COROLLARY 4.3. *For each subset A of Y_0, the set $\pi^{-1}(\pi A) \backslash A$ is a subset of $Y \backslash Y_0$ (and hence a μ-null set for all $\mu \in \mathcal{M}(Y)$.*

But $\pi^{-1}(\pi(Y \backslash Y_0)) \backslash (Y \backslash Y_0)$ need not be a μ-null set, as the next example shows:

EXAMPLE 4.4. Let X be an uncountable set, put $\mathcal{S} := \{A \subset X : A \text{ or } X \backslash A \text{ is countable}\}$, and

$$\mu : \mathcal{S} \to \mathbb{R},\ A \mapsto \begin{cases} 0 & \text{if } A \text{ is countable}, \\ 1 & \text{if } X \backslash A \text{ is countable}. \end{cases}$$

Then

$$\mathscr{M}(\mathscr{S})=\left\{\sum_{x\in X}\alpha_x\delta_x\colon(\alpha_x)_{x\in X}\in l^1(X)\right\}\oplus\mathscr{M}_\mu.$$

Set $Y:=\beta X\cup\{y\}$, where X is endowed with the discrete topology and y is a point not belonging to βX. Then Y is a representation space for $(X,\mathscr{S},\mathscr{M}(\mathscr{S}))$ such that

$$\mathscr{R}:=\{U_A\colon A\in\mathscr{S}\}=\{\bar{A}\colon A\subset X, A \text{ countable}\}$$
$$\cup\{\bar{A}\cup\{y\}: A\subset X, X\backslash A \text{ countable}\},$$

$$\mathscr{M}(Y)=\left\{\sum_{x\in X}\alpha_x\delta_x\colon(\alpha_x)_{x\in X}\in l^1(X)\right\}\oplus\mathscr{M}_{\delta_y},$$

and, with $A_0:=\bigcup_{A\subset X, A\text{ countable}}\bar{A}$, we have $Y_0=A_0\cup\{y\}$.

Each point of A_0 is normal, while $[y]=Y\backslash A_0$. Thus $[Y]=A_0\cup\{[y]\}$ is the Alexandrov compactification of A_0.

We have $\pi^{-1}(\pi(Y\backslash Y_0))\backslash(Y\backslash Y_0)=\{y\}$, and this set is not a null set for $\delta_y\in\mathscr{M}(Y)$.

The last example can be generalized as follows:

PROPOSITION 4.5. *Let $\mathscr{S}$ be a δ-ring of subsets of a set X such that $\{x\}\in\mathscr{S}$ for all $x\in X$. Let $\mathscr{M}$ be a band of $\mathscr{M}(\mathscr{S})$ containing the band $\mathscr{M}_D:=\bigoplus_{x\in X}\mathscr{M}_{\delta_x}$ generated by the Dirac measures. Furthermore let (Y,u,v) be a representation of $(X,\mathscr{S},\mathscr{M})$. We set $\mathscr{R}:=\{U_A\colon A\in\mathscr{S}\}$, $Y':=\overline{\bigcup_{\mu\in\mathscr{M}_D}\operatorname{supp} v\mu}$, $Y'':=Y\backslash Y'$.*

Then we have:

(a) *All points of $Y'\cap Y_0$ are normal.*

(b) *$\varnothing\neq[y]\cap Y'\subset Y'\backslash Y_0$ for all $y\in Y''$.*

(c) *$\pi Y=\pi Y'$ and $\pi Y''\subset\pi(Y'\backslash Y_0)$.*

(d) *$[Y]$ and the quotient space constructed from a representation of $\mathscr{M}_D$ coincide.*

Proof. (a) follows from [8; 3.2,1.4].

(b) Obviously we have $U_A\cap Y'\neq\varnothing$ for each $A\in\mathscr{S}$ with $y\in U_A$, which implies $[y]\cap Y'=\bigcap_{A\in\mathscr{S}, y\in U_A}(U_A\cap Y')\neq\varnothing$. Moreover, by (a), $[y]\cap[Y'\cap Y_0]=\varnothing$.

(c) follows from (b).

(d) follows from Corollary 3.8. ∎

One might suspect that *all* points of Y be normal if $\mathscr{M} = \mathscr{M}_D$ in the preceding proposition. I give a counterexample to this conjecture:

EXAMPLE 4.6. Assume ($\neg$ ch).

Let $X := [0, 1]$, and let $\mathscr{B}$ denote the set of Borel sets of X. Set $Y := \beta X$ (here X is considered with the discrete topology). Then Y is a representation space for $(X, \mathscr{B}, \mathscr{M}_D)$, and $U_B = \bar{B}$ (the closure in βX) for all $B \in \mathscr{B}$. By Proposition 4.5(a), all points of Y_0 are normal.

Now let A be a subset of X with $\aleph_0 < \text{card } A < 2^{\aleph_0}$. Let $\mathscr{G}$ be an ultrafilter on X containing all subsets B of A for which $A \backslash B$ is countable.

Let $B \in \mathscr{G} \cap \mathscr{B}$. The assumption $B \subset A$ implies card $B \leqslant \aleph_0$ [16; Sect. 33, Part I, Th. 3] which yields the contradiction $A \backslash B \in \mathscr{G}$. Thus $\{B \backslash A : B \in \mathscr{G} \cap \mathscr{B}\}$ is a filter basis on X; let $\mathscr{H}$ be a finer ultrafilter.

For $B \in \mathscr{B}$, we have obviously: $B \in \mathscr{G}$ iff $B \in \mathscr{H}$. Let $\mathscr{G}'$ and $\mathscr{H}'$ be the extensions of $\mathscr{G}$ and $\mathscr{H}$ to ultrafilters on βX. Then $\mathscr{G}'$ converges to some y satisfying $\{y\} = \bigcap_{C \in \mathscr{G}} \bar{C}$, and likewise $\mathscr{H}' \to z$ with $\{z\} = \bigcap_{D \in \mathscr{H}} \bar{D}$. Since $A \in \mathscr{G}$ and $X \backslash A \in \mathscr{H}$, we have $y \neq z$. But $y \sim z$: Indeed let $B \in \mathscr{B}$ with $y \in U_B$. Then $U_B \in \mathscr{G}'$ and thus $B = U_B \cap X \in \mathscr{G}$. Hence $B \in \mathscr{H}$ and thus $z \in U_B$. That $z \in U_B$ for $B \in \mathscr{B}$ implies $y \in U_B$, is shown analogously.

I set

$$Z_0 := \{y \in Y_0 : y \text{ is normal}\}.$$

PROPOSITION 4.7. *We have* $Z_0 = \bigcup_{A \in \mathscr{R},\, A \subset Y_0} A$; *in particular,* Z_0 *is open.*

Proof. See [8; 2.7]. ∎

Let me denote by (*) the following property of $\mathscr{R}$: For each sequence (A_n) from $\mathscr{R}$ whose union is contained in some $A \in \mathscr{R}$, we have $\overline{\bigcup_{n \in \mathbb{N}} A_n} \in \mathscr{R}$.

Observe that in the case $\mathscr{R} = \{U_A : A \in \mathscr{S}\}$ for some representation of a triple $(X, \mathscr{S}, \mathscr{M})$, property (*) is just the "translation" of the assumption that $\mathscr{S}$ be a δ-ring.

PROPOSITION 4.8. *Let* (K_n) *be a sequence of open-compact subsets of* Y. *Then we have*:

(a) $\bigcup_{n \in \mathbb{N}} K_n \subset Y_0$ *implies* $\overline{\bigcup_{n \in \mathbb{N}} K_n} \subset Y_0$.

(b) (*) *and* $\bigcup_{n \in \mathbb{N}} K_n \subset Z_0$ *imply* $\overline{\bigcup_{n \in \mathbb{N}} K_n} \subset Z_0$.

Proof. See [8; 2.8]. ∎

I call $\mu \in \mathscr{M}(Y)$ $\mathscr{R}$-normal ($\mathscr{R}$-anomalous, resp.) if all points of supp μ are $\mathscr{R}$-normal (if no point of supp μ is $\mathscr{R}$-normal, resp.), and I set

$$\mathscr{M}_{\text{no}, \mathscr{R}}(Y) := \{\mu \in \mathscr{M}(Y) : \mu \text{ is } \mathscr{R}\text{-normal}\},$$

$$\mathscr{M}_{\text{an}, \mathscr{R}}(Y) := \{\mu \in \mathscr{M}(Y) : \mu \text{ is } \mathscr{R}\text{-anomalous}\}.$$

PROPOSITION 4.9. *For $\mu \in \mathscr{M}(Y)$ we have*:

$$\mu \in \mathscr{M}_{\text{no}, \mathscr{R}}(Y) \Leftrightarrow \operatorname{supp} \mu = \bigcup_{\substack{A \in \mathscr{R} \\ A \subset \operatorname{supp} \mu}} A.$$

Proof. "$\Rightarrow$" follows from the fact that $A \cap \operatorname{supp} \mu \in \mathscr{R}$ for each $A \in \mathscr{R}$ (Proposition 3.4(a)). "$\Leftarrow$" follows from Proposition 4.7. ∎

The most important result of this section is

THEOREM 4.10. *The following assertions hold*:

(a) $\mathscr{M}_{\text{an}, \mathscr{R}}(Y)$ *is a band of* $\mathscr{M}(Y)$.

(b) $\mathscr{M}_{\text{no}, \mathscr{R}}(Y)$ *is an order dense ideal of* $(\mathscr{M}_{\text{an}, \mathscr{R}}(Y))^d$.

(c) *If* (*) *holds, then* $\mathscr{M}_{\text{no}, \mathscr{R}}(Y)$ *is a band of* $\mathscr{M}(Y)$, *and we have*

$$\mathscr{M}(Y) = \mathscr{M}_{\text{no}, \mathscr{R}}(Y) \oplus \mathscr{M}_{\text{an}, \mathscr{R}}(Y).$$

Proof. See [8; 2.9]. ∎

To see that (b) cannot be improved, consider again Example 3.3: Then (*) is not satisfied, and we have $\mathscr{M}(Y) = \bigoplus_{n \in \mathbb{N}} \mathscr{M}_{\delta_n}$, $\mathscr{M}_{\text{an}, \mathscr{R}}(Y) = \{0\}$ and

$$\mathscr{M}_{\text{no}, \mathscr{R}}(Y) = \left\{ \sum_{n \in M} \alpha_n \delta_n : M \subset \mathbb{N} \text{ finite}, \alpha_n \in \mathbb{R} \right\}.$$

The proof of the last observation in this section is analogous to [8; 2.10]:

COROLLARY 4.11. *If* (*) *holds, then* $(\overline{Z_0} \backslash Z_0) \cap Y_0 = \varnothing$.

5. MEASURES ON THE QUOTIENT SPACE

Again, as in Sect. 4, $\mathscr{R}$ is assumed to possess the Hahn decomposition property with respect to $\mathscr{M}(Y)$.

In the sequel, I denote by $\mathscr{B}_1$($\mathscr{B}_2$, resp.) the set of relatively compact Borel sets of Y (of $[Y]$, resp.).

PROPOSITION 5.1. *For each $\mu \in \mathscr{M}(Y)$, we have:*

(a) μ *is $\pi^{-1}\mathscr{B}_2$-quasiregular in all $A \in \mathscr{L}(\mu)$.*

(b) $\pi\mu \in \mathscr{M}_R([Y])$.

(c) *If $A \in \mathscr{L}(\mu)$ satisfies $\bar{A} \subset Y_0$, then $\pi A \in \mathscr{L}(\pi\mu)$ and $\int 1_{\pi A}\, d(\pi\mu) = \int 1_A\, d\mu$.*

(d) $[Y]\backslash\pi(\operatorname{supp}\mu) \in \mathscr{N}(\pi\mu)$.

Proof. (a) Let $A \in \mathscr{B}_1$. Then $B := \pi(\overline{A \cap \operatorname{supp}\mu}) \in \mathscr{K}([Y])$. We have $A\backslash\pi^{-1}B \subset \bar{A}\backslash\overline{A \cap \operatorname{supp}\mu} \in \mathscr{N}(\mu)$ and, using Proposition 4.1 and Corollary 4.3,

$$\pi^{-1}B\backslash A \subset (\pi^{-1}B\backslash\bar{A}) \cup (\bar{A}\backslash A) \subset (Y\backslash Y_0) \cup (\bar{A}\backslash A) \in \mathscr{N}(\mu).$$

(b) is obvious.

(c) (i) If $A \in \mathscr{K}(Y)$, then $\pi A \in \mathscr{K}([Y])$, and Corollary 4.3 gives the assertion.

(ii) Let $A \in \mathscr{N}(\mu)$. Then for each $B \in \mathscr{K}([Y])$, we have $\bar{A} \cap \pi^{-1}B \in \mathscr{N}(\mu) \cap \mathscr{K}(Y)$, hence by (i) $\pi\bar{A} \cap B \subset \pi(\bar{A} \cap \pi^{-1}B) \in \mathscr{N}(\pi\mu)$. It follows $\pi\bar{A} \in \mathscr{N}(\pi\mu)$.

(iii) In the general case, there exists a sequence (K_n) of compact subsets of A with $A\backslash\bigcup K_n \in \mathscr{N}(\mu)$. Case (i) for μ^+ yields $\sup \pi(\mu^+)(\pi K_n) = \int 1_A\, d\mu^+$, and thus $\pi(\bigcup K_n) \in \mathscr{L}(\pi(\mu^+))$ and $\pi(\mu^+)(\pi\bigcup K_n)) = \int 1_A\, \mu^+$. Case (ii) applied to μ^+ gives $\pi(A\backslash\bigcup K_n) \in \mathscr{N}(\pi(\mu^+))$, and thus $\pi A \in \mathscr{L}(\pi(\mu^+))$ and $\int 1_{\pi A}\, d\pi(\mu^+) = \int 1_A\, d\mu^+$. Similarly one proves the assertion for μ^-.

(d) Set $A := \operatorname{supp}\mu$. For each compact subset K of $[Y]\backslash\pi A$ we have $\pi^{-1}K \cap A = \varnothing$, hence $K \in \mathscr{N}(\pi\,|\mu|)$. By (b) and [6; 5.4.17], we get $\int_* 1_{[Y]\backslash\pi A}\, d\pi\,|\mu| = 0$. By Theorem 3.5(c), πA is a Borel set of $[Y]$; hence

$$\int^* 1_{[Y]\backslash\pi A}\, d\pi\,|\mu| = \int_* 1_{[Y]\backslash\pi A}\, d\pi\,|\mu| = 0. \quad \blacksquare$$

Example 4.4 shows that the assumption "$\bar{A} \subset Y_0$" in (c) cannot be omitted.

Let us now consider integrable functions:

THEOREM 5.2. *For $\mu \in \mathscr{M}(Y)$ and $f \in \bar{\mathbb{R}}^{[Y]}$ we have:*

$$f \in \mathscr{L}^1(\pi\mu) \Leftrightarrow f \circ \pi \in \mathscr{L}^1(\mu), \text{ and in this case } \int f\, d(\pi\mu) = \int f \circ \pi\, d\mu;$$

$$f \in \mathscr{L}^1_{\mathrm{loc}}(\pi\mu) \Leftrightarrow f \circ \pi \in \mathscr{L}^1_{\mathrm{loc}}(\mu), \text{ and in this case } f \cdot (\pi\mu) = \pi((f \circ \pi) \cdot \mu).$$

Proof. Observing Theorem 3.5(b) and [3; Sect. 6, no. 1, Rem. 2)], the first line follows from [3; Sect. 6, no. 2, Th. 1]. To prove "$\Rightarrow$" in the second line, let $K \in \mathcal{K}(Y)$. Then $1_K = 1_{\pi K} \circ \pi$ μ-a.e., and hence $(f \circ \pi) 1_K = ((f 1_{\pi K}) \circ \pi)$ μ-a.e. The implication "$\Rightarrow$" of the first part shows now that $(f 1_{\pi K}) \circ \pi \in \mathcal{L}^1(\mu)$; hence $f \circ \pi \in \mathcal{L}^1_{\mathrm{loc}}(\mu)$. For each $K \in \mathcal{K}([Y])$ we have $\pi^{-1} K \in \mathcal{K}(Y)$ by Theorem 3.5(b), and since $(f 1_K) \circ \pi = (f \circ \pi) 1_{\pi^{-1} K}$, "$\Leftarrow$" of the first part shows that "$\Leftarrow$" holds also in the second line. The identity $f \cdot (\pi\mu) = \pi((f \circ \pi) \cdot \mu)$ again is a consequence of the corresponding identity for the integrals. ∎

I set

$$\psi : \mathcal{M}(Y) \to \mathcal{M}_R([Y]), \mu \mapsto \pi\mu.$$

THEOREM 5.3. *The map ψ is an injective Riesz homomorphism, and $\psi(\mathcal{M}(Y))$ is a band of $\mathcal{M}_R([Y])$; in particular, ψ preserves arbitrary suprema and infima.*

Proof. That ψ is a Riesz homomorphism, follows from Proposition 5.1(a) and Proposition 2.2.

Let $\mu, \nu \in \mathcal{M}(Y)$ with $\psi\mu = \psi\nu$. Using Proposition 5.1(c), we get for each $A \in \mathcal{B}_1$ (with $F := (\operatorname{supp} \mu) \cup (\operatorname{supp} \nu)$):

$$\mu(A) = \mu(\bar{A} \cap F) = (\psi\mu)(\pi(\bar{A} \cap F)) = (\psi\nu)(\pi(\bar{A} \cap F)) = \nu(\bar{A} \cap F) = \nu(A).$$

Hence ψ is injective.

Now let $\mu \in \mathcal{M}(Y)^+$ and $\nu \in \mathcal{M}_R([Y])$ with $0 \leqslant \nu \leqslant \psi\mu$. By Proposition 5.1(c), the map

$$\lambda : \mathcal{B}_1 \to \mathbb{R}, A \mapsto \int 1_{\pi(A \cap \operatorname{supp} \mu)} \, d\nu$$

is well-defined. If (A_n) is a disjoint sequence from $\mathcal{B}_1$ with $\bigcup A_n \in \mathcal{B}_1$, then by Proposition 4.1 $(\pi(A_n \cap \operatorname{supp} \mu))$ is a disjoint sequence, from which we conclude $\lambda(\bigcup A_n) = \sum \lambda(A_n)$; thus $\lambda \in \mathcal{M}(\mathcal{B}_1)$. For all $A \in \mathcal{B}_1$ we have $\lambda(A) \leqslant \int 1_{\pi(A \cap \operatorname{supp} \mu)} \, d(\pi\mu) = \mu(A)$, and hence $0 \leqslant \lambda \leqslant \mu$, which implies $\lambda \in \mathcal{M}(Y)$. To show that $\nu = \pi\lambda$ holds, let $B \in \mathcal{B}_2$. By Proposition 5.1(c), $B \backslash \pi(\pi^{-1} B \cap \operatorname{supp} \mu) \in \mathcal{N}(\pi\mu) \subset \mathcal{N}(\nu)$, and thus $\nu(B) = \lambda(\pi^{-1} B) = (\pi\lambda)(B)$. We get $\nu \in \psi(\mathcal{M}(Y))$, and so $\psi(\mathcal{M}(Y))$ is an ideal of $\mathcal{M}_R([Y])$.

Finally, let $0 \leqslant \psi\mu_\iota \uparrow \nu \in \mathcal{M}_R([Y])$, with $\mu_\iota \in \mathcal{M}(Y)$. Since ψ is injective, we conclude $0 \leqslant \mu_\iota \uparrow$. For all $A \in \mathcal{B}_1$ and all ι we have $\mu_\iota(A) \leqslant (\psi\mu_\iota)(\pi\bar{A}) \leqslant \nu(\pi\bar{A})$, and thus $\mu := \sup \mu_\iota$ exists in $\mathcal{M}(Y)$. Then $\psi\mu_\iota \leqslant \psi\mu$ implies $\nu \leqslant \psi\mu$, and thus, by what has been proved above, $\nu = \psi\mu \in \psi(\mathcal{M}(Y))$. ∎

That ψ is in general not onto $\mathscr{M}_R([Y])$, even if Y is hyperstonian, can easily be seen using the characterization of elements of $\psi(\mathscr{M}(Y))$ given in Theorem 5.10.

The easy proof of the following proposition, which describes the (very natural) behaviour of atomical and atomfree measures, is omitted.

PROPOSITION 5.4. *For $\mu \in \mathscr{M}(Y)$ we have:*

(a) *If $\{y\} \subset Y_0$ is a μ-atom, then $\pi(\{y\})$ is a $\pi\mu$-atom; if $[y]$ is a $\pi\mu$-atom, then $[y] \cap Y_0$ is a μ-atom.*

(b) *μ is atomical iff $\pi\mu$ is atomical.*

(c) *μ is atomfree iff $\pi\mu$ is atomfree.*

(d) *Denoting by v_a (v_f, resp.) the atomical (atomfree, resp.) component of a measure v, we have: $\pi(\mu_a) = (\pi\mu)_a$ and $\pi(\mu_f) = (\pi\mu)_f$.*

The following example shows that even if $[Y]$ is hyperstonian, the inclusion $\psi(\mathscr{M}(Y)) \subset \mathscr{M}([Y])$ need not hold:

EXAMPLE 5.5. Let X be an uncountable set, endowed with the discrete topology. We fix points $y \in \beta X \setminus \bigcup_{A \subset X,\, A \text{ countable}} \bar{A}$ and $z \notin \beta X$, and set $Y := \beta X \cup \{z\}$ and

$$\mathscr{R} := \{A \subset \beta X\colon A \text{ open-compact}, y \notin A\}$$
$$\cup \{A \subset Y\colon A \text{ open-compact}, y \in A, z \in A\}.$$

Then

$$\mathscr{M}(Y) = \left\{ \sum_{x \in X} \alpha_x \delta_x \colon (\alpha_x)_{x \in X} \in l^1(X) \right\} \oplus \mathscr{M}_{\delta_x}.$$

All points of $\beta X \setminus \{y\}$ are normal, and $[y] = [z] = \{y, z\}$. Since z is an isolated point of Y, $[Y]$ and βX are homeomorphic; thus $[Y]$ is hyperstonian. But we have $\pi\delta_z = \delta_{[z]} \notin \mathscr{M}([Y])$.

Let me remark that Y is again a representation space of a triple $(X, 2^X, \mathscr{M})$ with $\mathscr{R} = \{U_A\colon A \subset X\}$. Namely, let $\mathscr{G}$ be a free ultrafilter on X with the property "$A_n \in \mathscr{G}$ for all $n \in \mathbb{N} \Rightarrow \bigcap_{n \in \mathbb{N}} A_n \in \mathscr{G}$", such that the extension of $\mathscr{G}$ to βX converges to y, and set

$$\mathscr{M} := \left\{ \sum_{x \in X} \alpha_x \delta_x \colon (\alpha_x)_{x \in X} \in l^1(X) \right\} \oplus \mathscr{M}_\mu,$$

with

$$\mu: 2^X \to \mathbb{R},\ A \mapsto \begin{cases} 1 & \text{if } A \in \mathcal{G} \\ 0 & \text{if } A \notin \mathcal{G}. \end{cases}$$

Nevertheless the elements of $\psi(\mathcal{M}(Y))$ are not too far from being normal Radon measures, as Corollary 5.7 will show. First, I prove

PROPOSITION 5.6. $\pi(\operatorname{supp}\mu) = \operatorname{supp}(\pi\mu)$, *for each* $\mu \in \mathcal{M}(Y)$, *and hence*

$$\pi Y_0 = \bigcup_{\nu \in \psi(\mathcal{M}(Y))} \operatorname{supp} \nu.$$

Proof. $[Y] \backslash \pi(\operatorname{supp}\mu)$ is an open $\pi\mu$-null set (Theorem 3.5(c) and Proposition 5.1(d)), hence $\operatorname{supp}(\pi\mu) \subset \pi(\operatorname{supp}\mu)$. By Theorem 5.2, $Y \backslash \pi^{-1}(\operatorname{supp}(\pi\mu)) = \pi^{-1}([Y] \backslash \pi(\operatorname{supp}\mu))$ is an open μ-null set, which implies $\operatorname{supp}\mu \subset \pi^{-1}(\operatorname{supp}(\pi\mu))$ from which we conclude $\pi(\operatorname{supp}\mu) \subset \operatorname{supp}(\pi\mu)$. ▮

COROLLARY 5.7. *For* $\mu, \nu \in \psi(\mathcal{M}(Y))$ *we have*:

(a) $\mu \perp \nu \Leftrightarrow (\operatorname{supp}\mu) \cap (\operatorname{supp}\nu) = \varnothing$.

(b) $\mu \ll \nu \Leftrightarrow \operatorname{supp}\mu \subset \operatorname{supp}\nu$.

Proof. Since $\pi|_{Y_0}$ is injective (Proposition 4.1) and ψ is an injective Riesz homomorphism (Theorem 5.3), the assertions follow, using Proposition 5.6, from the corresponding assertions which hold in $\mathcal{M}(Y)$. ▮

COROLLARY 5.8. $\psi(\mathcal{M}_c(Y)) = (\psi(\mathcal{M}(Y)))_c$ *and* $\psi(\mathcal{M}_b(Y)) = (\psi(\mathcal{M}(Y)))_b$.

Proof. The first assertion follows by applying Proposition 5.6 and Theorem 3.5(b), while the second is a consequence of Theorem 5.2.

COROLLARY 5.9. *For* $\mu \in \mathcal{M}(Y)$, *the map*

$$\pi_\mu: \operatorname{supp}\mu \to \operatorname{supp}(\pi\mu),\ y \mapsto [y]$$

is a homeomorphism (*and hence* $\operatorname{supp}(\pi\mu)$ *is hyperstonian*).

Proof. By Propositions 5.6 and 4.1, π_μ is bijective. Furthermore, π_μ is obviously continuous, and π_μ^{-1} is continuous by Theorem 3.5(c). ▮

Now I can give a characterization of those Radon measures on $[Y]$ which occur as image of an element of $\mathcal{M}(Y)$:

THEOREM 5.10. *For* $\nu \in \mathcal{M}_R([Y])$, *the following are equivalent*:

(a) $\nu \in \psi(\mathcal{M}(Y))$;

(b) $\operatorname{supp} \nu \subset \pi Y_0$, $Y_0 \cap \pi^{-1}(\operatorname{supp} \nu)$ *is open-closed, and* $\nu|_{\operatorname{supp} \nu} \in \mathcal{M}(\operatorname{supp} \nu)$.

Proof. We can assume $\nu > 0$. Set $W := Y_0 \cap \pi^{-1}(\operatorname{supp} \nu)$.

(a) $\Rightarrow$ (b): Set $\mu := \psi^{-1}\nu$. Using Proposition 5.6, we get $\operatorname{supp} \nu \subset \pi Y_0$ and $W = \operatorname{supp} \mu$. Furthermore $\nu|_{\operatorname{supp} \nu} = \pi_\mu(\mu)$, which implies the third property (Corollary 5.9).

(b) $\Rightarrow$ (a): Since $\operatorname{supp} \nu \subset \pi Y_0$, there exists, by Proposition 4.1, for each $z \in \operatorname{supp} \nu$ a unique $y_z \in Y_0$ with $[y_z] = z$. Then

$$\rho: \operatorname{supp} \nu \to W, z \mapsto y_z$$

is a homeomorphism. Hence $\lambda := \rho(\nu|_{\operatorname{supp} \nu}) \in \mathcal{M}(W)$. Let μ be the natural extension of λ to Y (i.e. $Y \backslash W \in \mathcal{N}(\mu)$). Since W is open-closed, we have $\mu \in \mathcal{M}(Y)$, and we conclude $\nu = \pi\mu$. ∎

While in Y the set $\mathcal{M}(Y)$ of normal Radon measures plays the central role, Example 5.5 shows that in $[Y]$ all Radon measures are important. Therefore it is of interest to decide whether $[Y]$ is a Radon space. The following example disproves this conjecture:

EXAMPLE 5.11. Let X be an uncountable set, endowed with the discrete topology, set $Y := \beta X$ and

$$\mathcal{R} := \{A \subset X: A \text{ finite}\} \cup \{\beta X \backslash A: A \subset X, A \text{ finite}\}.$$

All $x \in X$ are normal, for each $y \in \beta X \backslash X$ we have $[y] = \beta X \backslash X$, and $\mathcal{B}_c([Y]) = 2^{[Y]}$ holds.

Let $\mathcal{G}$ be a free ultrafilter on $[Y]$ with the property "$A_n \in \mathcal{G}$ for all $n \in \mathbb{N} \Rightarrow \bigcap_{n \in \mathbb{N}} A_n \in \mathcal{G}$", and set

$$\mu: 2^{[Y]} \to \mathbb{R}, A \mapsto \begin{cases} 1 & \text{if } A \in \mathcal{G} \\ 0 & \text{if } A \notin \mathcal{G}. \end{cases}$$

Then μ is not a Radon measure, since $\mu(\pi X) = 1$.

To finish this section, I want to make concrete the natural observation that Y is a representation space for $([Y], \mathcal{B}_2, \psi(\mathcal{M}(Y)))$.

THEOREM 5.12. *Set* $\mathscr{M} := \psi(\mathscr{M}(Y))$. *Then the following assertions hold*:

(a) *For each* $h \in \mathscr{L}_\infty(\mathscr{M})$, *there exists a unique* $\bar{u}h \in C_\infty(Y)$ *such that* $\{\bar{u}h \neq h \circ \pi\}$ *is nowhere dense.*

(b) $(Y, \bar{u}, \bar{v})$ *is a representation of* $([Y], \mathscr{B}_2, \mathscr{M})$, *where* $\bar{v}\mu := \psi^{-1}\mu$ *for all* $\mu \in \mathscr{M}$.

(c) $\bar{u}1_{\pi A} = 1_A$ *for each open-closed subset* A *of* Y_0 *and for each* $A \in \mathscr{R}$.

Proof. (a) Using Theorems 5.3 and 5.2, we get for $h \in \mathbb{R}^{[Y]}$:

$$\begin{aligned} & h \in \mathscr{L}_\infty(\mathscr{M}) \\ & \Leftrightarrow \{\pi\mu : \mu \in \mathscr{M}(Y), h \in \mathscr{L}^1(\pi\mu)\} \text{ is an ideal of } \mathscr{M} \\ & \Leftrightarrow \{\mu \in \mathscr{M}(Y) : h \in \mathscr{L}^1(\pi\mu)\} \text{ is an ideal of } \mathscr{M}(Y) \\ & \Leftrightarrow \{\mu \in \mathscr{M}(Y) : h \circ \pi \in \mathscr{L}^1(\mu)\} \text{ is an ideal of } \mathscr{M}(Y) \\ & \Leftrightarrow h \circ \pi \in \mathscr{L}_\infty(\mathscr{M}(Y)). \end{aligned}$$

The claim now follows from [4; 2.3.9a)].

(b) We have to check (a)–(e) of Proposition 2.3. Conditions (a), (b), (d) are obvious, (e) follows from (a) of the present theorem and from Theorem 5.2. To verify (c), let $K \in \mathscr{K}([Y])$. Then $\bar{u}1_K$ is the characteristic function of the interior of $\pi^{-1}K$. Hence $\operatorname{supp}(\bar{u}1_B)$ is compact for each $B \in \mathscr{B}_2$ (Theorem 3.5(b)). For each open-compact $A \subset Y$ we have $1_A \leqslant \bar{u}1_{\pi A}$, which implies $Y = \bigcup_{B \in \mathscr{B}_2} \operatorname{supp}(\bar{u}1_B)$.

(c) Since the second assertion is obvious, let us consider an open-closed $A \subset Y_0$. For all $\nu \in \mathscr{M}_c(Y)^+$ we have, by (a) and Proposition 5.1(c):

$$\int \bar{u}1_{\pi A}\, d\nu = \int (1_{\pi A} \circ \pi)\, d\nu = \int 1_{\pi A}\, d(\pi\nu) = \int 1_A\, d\nu.$$

Since $\{\bar{u}1_{\pi A} = 1\} \setminus A$ is open-closed, the claim follows. ∎

Remark. With the terminology of Theorem 5.12, set $\tilde{\mathscr{R}} = \{\{\bar{u}1_B = 1\} : B \in \mathscr{B}_2\}$. By Theorem 5.12(c), we have $\mathscr{R} \subset \tilde{\mathscr{R}}$. That the partitioning of Y into equivalence classes defined by $\tilde{\mathscr{R}}$ is in general properly finer than that defined by $\mathscr{R}$, can be seen considering Example 3.3: All points of $\beta\mathbb{N} \setminus \mathbb{N}$ are $\mathscr{R}$-equivalent. But take $y, z \in \beta\mathbb{N} \setminus \mathbb{N}$, $y \neq z$. There exists $A \subset \mathbb{N}$ with $y \in \bar{A}$, $z \notin \bar{A}$. Then $\pi A \in \mathscr{B}_2$, $1_{\pi A} \circ \pi = 1_{A \cup (\beta\mathbb{N} \setminus \mathbb{N})}$, $\bar{u}1_{\pi A} = 1_{\bar{A}}$. Thus y and z are not $\tilde{\mathscr{R}}$-equivalent.

6. A REPRESENTATION THEOREM FOR FINITELY ADDITIVE MEASURES

As an application of the theory developed in the preceding sections, I want to give a representation theorem for the Riesz space of finitely additive measures.

Halmos remarked [13; Sect. 2] that if $\mathscr{S}$ is a σ-algebra and ν is an additive set function on $\mathscr{S}$ with values in $\mathbb{R}^+$, then ν can be extended in a unique way to a Baire measure on the Stone space of $\mathscr{S}$. (Observe that, by the Riesz Representation Theorem, such Baire measure can be extended uniquely to a Radon measure.) Yosida and Hewitt proved [19; 4.5] that this extension process generates an isomorphism between the space of bounded finitely additive measures on the σ-algebra $\mathscr{S}$ and the space of all Radon measures on the Stone space of $\mathscr{S}$. Heider generalized this result to the case of an algebra of sets [14; 3.1].

In Theorem 6.5, I will prove that if $\mathscr{S}$ is an arbitrary ring of sets, then the Riesz space $\mathscr{E}(\mathscr{S})$ of finitely additive real-valued measures with locally bounded variation (or equivalently: which are locally exhaustive; cf. [5; 4.1.8]) on $\mathscr{S}$ is Riesz isomorphic to the space $\mathscr{M}_R([Y])$ of all Radon measures on an appropriate space $[Y]$: namely, let (Y, u, v) be a representation of $(X, \mathscr{S}, \mathscr{M}(\mathscr{S}))$, set $\mathscr{R} := \{U_A : A \in \mathscr{S}\}$, and let $[Y]$ be the corresponding quotient space; let this setting be fixed for the rest of this section. By Corollary 3.6 (and Proposition 6.1), all results mentioned above are contained in Theorem 6.5.

PROPOSITION 6.1. *Let A and B be sets which are μ-measurable for all $\mu \in \mathscr{M}(\mathscr{S})$. Then $A \subset B$ iff $U_A \subset U_B$.*

Proof. Let $U_A \subset U_B$. Then, by [8; 2.3], $A \setminus B$ is a μ-null set for all $\mu \in \mathscr{M}(\mathscr{S})$. Since $\mathscr{M}(\mathscr{S})$ contains all Dirac measures, $A \setminus B$ must be empty. ∎

I denote by $\mathscr{P}$ the set of all increasing sequences (B_n) of open-compact subsets of $[Y]$ for which $\overline{\bigcup B_n}$ is open-compact.

PROPOSITION 6.2. *$(B_n) \in \mathscr{P}$ iff there exists an increasing sequence (A_n) in $\mathscr{S}$ with $A := \bigcup A_n \in \mathscr{S}$ such that $B_n = \pi U_{A_n}$ for all $n \in \mathbb{N}$ and $\overline{\bigcup B_n} = \pi U_A$.*

Proof. Let $(B_n) \in \mathscr{P}$. By Theorem 3.5(f) there exist $A \in \mathscr{S}$ and a sequence (A_n) in $\mathscr{S}$ such that $\pi U_A = \overline{\bigcup B_n}$, $\pi U_{A_n} = B_n$ for all n, (U_{A_n}) increases, and $\overline{\bigcup U_{A_n}} = U_A$. By Proposition 6.1, we conclude that (A_n) increases and that $A = \bigcup A_n$.

Conversely, let (A_n) be an increasing sequence from $\mathscr{S}$ with $A := \bigcup A_n \in \mathscr{S}$. Then $U_A = \overline{\bigcup U_{A_n}}$, and, by the continuity of π, $\pi U_A \subset \overline{\pi(\bigcup U_{A_n})} = \overline{\bigcup \pi U_{A_n}}$, which implies $\pi U_A = \overline{\bigcup \pi U_{A_n}}$. ∎

PROPOSITION 6.3. *Let* $v \in \mathscr{E}(\mathscr{S})^+$. *For the map*

$$\phi\colon \mathscr{K}([Y]) \to \mathbb{R}^+, K \mapsto \inf_{\substack{A \in \mathscr{S} \\ \pi^{-1}K \subset U_A}} v(A)$$

(*which is well-defined by Theorem* 3.5(e)), *we have*:

(a) $K, L \in \mathscr{K}([Y]) \Rightarrow \phi(K \cup L) \leqslant \phi(K) + \phi(L)$;

(b) $K, L \in \mathscr{K}([Y]), K \cap L = \varnothing \Rightarrow \phi(K \cup L) = \phi(K) + \phi(L)$;

(c) $K_\iota \in \mathscr{K}([Y]), K_\iota \downarrow \Rightarrow \phi(\bigcap K_\iota) = \inf \phi(K_\iota)$.

Proof. (a) is easy to see.

(b) Let $A \in \mathscr{S}$ with $\pi^{-1}(K \cup L) \subset U_A$. By Theorem 3.5(d), there is $B \in \mathscr{S}$ with $K \subset \pi U_B, L \subset [Y] \backslash \pi U_B$. Then $\pi^{-1}K \subset U_{A \cap B}, \pi^{-1}L \subset U_{A \backslash B}$ and thus $\phi(K) + \phi(L) \leqslant v(A \cap B) + v(A \backslash B) = v(A)$. We conclude $\phi(K \cup L) \leqslant \phi(K) + \phi(L)$.

(c) Let $A \in \mathscr{S}$ with $\pi^{-1}(\bigcap K_\iota) \subset U_A$. By compactness, there is an index λ with $\pi^{-1}K_\lambda \subset U_A$. It follows $\inf \phi(K_\iota) \leqslant \phi(K_\lambda) \leqslant v(A)$. Thus $\inf \phi(K_\iota) \leqslant \phi(\bigcap K_\iota)$. ∎

COROLLARY 6.4. *The following assertions hold.*

(a) *For each* $v \in \mathscr{E}(\mathscr{S})^+$, *there exists* $\tilde{v} \in \mathscr{M}_R([Y])^+$ *such that*

$$\tilde{v}(K) = \inf_{\substack{A \in \mathscr{S} \\ \pi^{-1}K \subset U_A}} v(A)$$

for all $K \in \mathscr{K}([Y])$.

(b) $\tilde{v}(\pi U_A) = v(A)$ *for each* $v \in \mathscr{E}(\mathscr{S})^+$ *and each* $A \in \mathscr{S}$.

(c) $\widetilde{v + \mu} = \tilde{v} + \tilde{\mu}$ *for all* $v, \mu \in \mathscr{E}(\mathscr{S})^+$.

Proof. (a) follows from Proposition 6.3 and [6; Exerc. 5.2.17]. (b) is a consequence of Proposition 6.1, while (c) is easy to see. ∎

THEOREM 6.5. *For a ring of sets* $\mathscr{S}$, *we have*:

(a) *There exists a unique positive linear operator* $\rho\colon \mathscr{E}(\mathscr{S}) \to \mathscr{M}_R([Y])$ *such that* $\rho v = \tilde{v}$ *for all* $v \in \mathscr{E}(\mathscr{S})^+$ (*where* $\tilde{v}$ *is as in Corollary* 6.4).

(b) ρ *is a Riesz isomorphism.*

(c) $\rho v(\pi U_A) = v(A)$ *for each* $v \in \mathscr{E}(\mathscr{S})$ *and each* $A \in \mathscr{S}$.

(d) $\rho|_{\mathscr{M}(\mathscr{S})} = \psi \circ v$.

(e) *ν is bounded iff $\rho\nu$ is bounded.*

(f) *There exists $A \in \mathscr{S}$ such that $\nu(B) = 0$ for all $B \in \mathscr{S}$ with $B \cap A = \emptyset$ iff* $\operatorname{supp}(\rho\nu) \in \mathscr{K}([Y])$.

(g) *$\nu \in \mathscr{M}(\mathscr{S})$ iff $\overline{\bigcup B_n} \setminus \bigcup B_n \in \mathscr{N}(\rho\nu)$ for each $(B_n) \in \mathscr{P}$.*

(h) *ν is purely finitely additive iff for each $\lambda \in \mathscr{E}(\mathscr{S})$ with $0 < \lambda \leqslant |\nu|$ there exists $(B_n) \in \mathscr{P}$ with $\lambda(\overline{\bigcup B_n} \setminus \bigcup B_n) > 0$.*

Proof. (a) follows from [20; 83.1] by observing Corollary 6.4(c).

(b) To show that ρ is a Riesz homomorphism, let $\nu, \mu \in \mathscr{E}(\mathscr{S})$ with $\inf(\nu, \mu) = 0$, and let $K \in \mathscr{K}([Y])$. Let $\varepsilon > 0$. There exists $C \in \mathscr{S}$ with $\pi^{-1}K \subset U_C$. Furthermore, there are $A, B \in \mathscr{S}$ such that $A \cap B = \emptyset$, $A \cup B = C$, and $\nu(A) + \mu(B) < \varepsilon$. Setting $L := K \cap \pi U_A$ and $J := K \cap \pi U_B$, we have $\pi^{-1}L \subset U_A$, $\pi^{-1}J \subset U_B$, $L \cap J = \emptyset$ and $L \cup J = K$. Thus

$$(\inf(\rho\nu, \rho\mu))(K) \leqslant \rho\nu(L) + \rho\mu(J) \leqslant \nu(A) + \mu(B) < \varepsilon.$$

We conclude $(\inf(\rho\nu, \rho\mu))(K) = 0$, hence $\inf(\rho\nu, \rho\mu) = 0$.

In order to prove that ρ is injective, let $\nu \in \mathscr{E}(\mathscr{S})$ with $\rho\nu = 0$. Then $\rho(\nu^+) = (\rho\nu)^+ = 0$, and we get $\nu^+(A) = 0$ for each $A \in \mathscr{S}$, by Corollary 6.4(b). Hence $\nu^+ = 0$, and analogously $\nu^- = 0$.

To prove that ρ is onto, let $\mu \in \mathscr{M}_R([Y])^+$. We set

$$\nu\colon \mathscr{S} \to \mathbb{R}^+,\ A \mapsto \mu(\pi U_A).$$

Obviously ν is finitely additive. To show that ν is also locally exhaustive, let (A_n) be a disjoint sequence from $\mathscr{S}$ with $A := \bigcup A_n \in \mathscr{S}$. Then (πU_{A_n}) is a disjoint sequence, and we get

$$\sum \nu(A_n) = \sum \mu(\pi U_{A_n}) \leqslant \mu(\pi U_A)$$

which implies $\nu(A_n) \to 0$. Hence $\nu \in \mathscr{E}(\mathscr{S})^+$. By Theorem 3.5(e) we get $\rho\nu(K) = \mu(K)$ for all $K \in \mathscr{K}([Y])$, and thus $\rho\nu = \mu$.

(c) follows from (b) and Corollary 6.4(b).

(d) is easy to see.

(e) Using (b),(c) and Theorem 3.5(e), we get

$$\sup_{B \in \mathscr{B}_2} |\rho\nu|\,(B) = \sup_{A \in \mathscr{S}} |\rho\nu|\,(\pi U_A) = \sup_{A \in \mathscr{S}} |\nu|\,(A).$$

(f) Assume that $A \in \mathscr{S}$ exists with $\nu(B) = 0$ for all $B \in \mathscr{S}$, $B \cap A = \emptyset$. Then $\operatorname{supp}(\rho\nu) \subset \pi U_A$: Indeed, let $K \in \mathscr{K}([Y])$ with $K \cap \pi U_A = \emptyset$.

There exists $B \in \mathscr{S}$ with $K \subset \pi U_B$. Set $C := B \setminus A$. Then $|\nu|(C) = \sup\{\nu(D) : D \in \mathscr{S}, D \subset C\} = 0$, and by (b) and (c) $|\rho\nu|(K) \leqslant |\rho\nu|(\pi U_C) = 0$. Hence $[Y] \setminus \pi U_A \in \mathscr{N}(\rho\nu)$. The converse implication follows from (c).

(g) is a consequence of Proposition 6.2.

(h) Let ν be purely finitely additive, and let $0 < \lambda \leqslant |\nu|$. If no $(B_n) \in \mathscr{P}$ exists with $\lambda(\overline{\bigcup B_n} \setminus \bigcup B_n) > 0$, then by (g) $\lambda \in \mathscr{M}(\mathscr{S})$ which is impossible since the set of purely additive elements of $\mathscr{E}(\mathscr{S})$ is a band of $\mathscr{E}(\mathscr{S})$.

Conversely, let the condition be satisfied, and let $\mu \in \mathscr{M}(\mathscr{S})$. Then $\lambda := \inf(|\mu|, |\nu|) \in \mathscr{M}(\mathscr{S})$ and therefore, by (g), $\lambda = 0$. Hence $\nu \in \mathscr{M}(\mathscr{S})^d$, i.e. ν is purely finitely additive. ▮

The condition in (g) is not very surprising: See e.g. [18; 18.7.2].

A representation for $\mathscr{E}(\mathscr{S})$ as the Riesz space $\mathscr{M}(Y)$ for some hyperstonian space Y was given by the author in [11; 4.5].

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