Discrete Mathematics 81 (1990) 211–215
North-Holland

COMMUNICATION

A NEW DERIVATION OF THE GENERATING FUNCTION FOR THE MAJOR INDEX

Sumanta GUHA and Sriram PADMANABHAN
EECS Department, The University of Michigan, Ann Arbor, MI 48109, USA

Communicated by I. Gessel
Received 7 November 1989

We present a new proof of the well-known combinatorial result $\left[{n \atop k}\right]_q = \sum_w q^{\mathrm{Maj}(w)}$ where w is a permutation of 0^k1^{n-k}, by showing a bijection between the set of partitions of an integer m that fit in a $k \times n-k$ rectangle and the set consisting of all permutations w of 0^k1^{n-k} having $\mathrm{Maj}(w) = m$.

1. Introduction

Consider the sequence $w = w_1w_2\cdots w_n$ where $w_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$. The *Descent Set* of w, $D(w)$ is $\{i \mid 1 \leqslant i < n, w_i > w_{i+1}\}$, and the *major index*, $\mathrm{Maj}(w)$ is the sum of all the elements (possibly zero) of $D(w)$. MacMahon [4, 5] showed that the major indices of the set of all permutations of w has the same generating function as the *inversion numbers*, $\mathrm{Inv}(w)$ of these permutations. A combinatorial proof of this correspondence between $\mathrm{Maj}(w)$ and $\mathrm{Inv}(w)$ was obtained by Foata [1]. Rawlings [7] uses a statistic called r-major index to describe a bijection that takes the major index to the inversion number.

This paper considers the case where the w's are permutations of 0^k1^{n-k}. A new combinatorial proof for the generating function of the major indices of these sequences is derived by showing a bijection between the set of partitions of a positive integer m that fit inside a $k \times n-k$ rectangle and the set of permutations w of 0^k1^{n-k} which have $\mathrm{Maj}(w) = m$. The bijections in [1, 7] can be shown, with some effort, to reduce to the bijection described in this paper in this case.

2. Notation and definitions

A *partition* of a positive integer m into k parts is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = m$. The *size* of the partition, denoted by $|\lambda|$, is m, and the *length*, $L(\lambda)$, is k.

The *diagram* of the partition is the set of lattice points $\{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant L(\lambda), 1 \leqslant j \leqslant \lambda_i\}$ and is denoted by D_λ. The diagram D_λ and the partition λ may be used interchangeably in this paper. The *rank* of a partition λ is the length of the largest square subdiagram of D_λ. Suppose the rank of λ is r. Then λ can be expressed as $(\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r)$ [Frobenius notation, see Macdonald [3] page 3], where α_i is the number of nodes in the ith row of D_λ to the right of (i, i), and β_i is the number of nodes in the ith column of D_λ below (i, i). A list of standard notation used to describe partitions and q-combinations is presented below.

- $p(m)$ = Number of partitions of m.
- $p_k(m)$ = Number of partitions, λ, of m, where $L(\lambda) \leqslant k$.
- $p_{n,k}(m)$ = Number of partitions of m that fit inside a $n \times k$ rectangle.
- $[k] = 1 + q + \cdots + q^{k-1}$, $k \geqslant 1$; $[0] = 1$.
- $[n]! = [1][2] \cdots [n]$.
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]!}{[k]!\,[n-k]!}$.

3. Theorem and proof

Theorem. *There is a bijection between the set of partitions of m (>0) that fit inside a $k \times n - k$ rectangle and the set consisting of permutations w of $0^k 1^{n-k}$ having* $\mathrm{Maj}(w) = m$.

This theorem and the well-known result $[{n \atop k}]_q = \sum_{m \geqslant 0} p_{k,n-k}(m) q^m$ (Stanley [8], page 29) directly imply that the generating function for the major index of permutations w of $0^k 1^{n-k}$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_w q^{\mathrm{Maj}(w)}.$$

Proof. Let $w = w_1 w_2 \cdots w_n$ be a permutation of $0^k 1^{n-k}$ such that $\mathrm{Maj}(w) = m$. Suppose $D(w) = \{i_1, \ldots, i_j, \ldots, i_r\}$ where $i_1 < i_2 < \cdots < i_r$. For each j, $1 \leqslant j \leqslant r$, let Zero($j$) denote the number of zeros, and One(j) denote the number of ones, respectively, found in the substring $w_1 w_2 \cdots w_{i_j - 1}$. Let partition λ_w be defined as (One(r), One($r-1$), ..., One(1) | Zero(r), Zero($r-1$), ..., Zero(1)). The height of λ_w is (Zero(r) + 1), which is at most k. Likewise, the width of λ_w is (One(r) + 1), which is at most $n-k$. Also, $|\lambda_w| = \sum_{j=1}^r (\mathrm{One}(j) + \mathrm{Zero}(j) + 1) = \sum_{j=1}^r i_j = m$. Thus, $w \to \lambda_w$ defines a mapping from the set of permutations of $0^k 1^{n-k}$ having $\mathrm{Maj}(w) = m$ to the set of partitions $p_{k,n-k}(m)$.

Example. Let $n = 10$, $k = 6$, and, $w = 00\mathbf{1}0\mathbf{0}\mathbf{1}0\mathbf{1}\mathbf{1}\mathbf{0}$. Then $D(w) = \{3, 6, 9\}$, Zero(1) = 2, One(1) = 0, Zero(2) = 4, One(2) = 1, Zero(3) = 5, One(3) = 3. *Fig. 1* shows the partition diagram for the example.

The inverse function can be obtained by using the following method. Let λ be a partition of m that fits inside a $k \times n - k$ rectangle. Suppose $\lambda =$

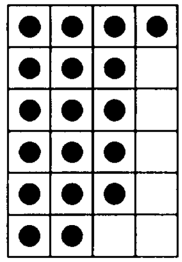

Fig. 1. D_λ for $w = 0010010110$.

$(\alpha_r, \alpha_{r-1}, \ldots, \alpha_1 \mid \beta_r, \beta_{r-1}, \ldots, \beta_1)$. The string $w_\lambda = w_1 w_2 \cdots w_n$ can be constructed as follows. Place β_1 zeros and α_1 ones before the first **10** descent pair. All these zeros must precede all the ones since they cannot give rise to any descents. Thus, the partial string w_λ up to and including the first descent pair is

$$\underbrace{00\cdots0}_{\beta_1}\overbrace{11\cdots1}^{\alpha_1}\mathbf{10}.$$

Place $\beta_2 - \beta_1 - 1$ zeros and $\alpha_2 - \alpha_1 - 1$ ones between the first and second descent pairs, such that the zeros precede the ones. The partial string w_λ up to the second descent pair is therefore

$$\underbrace{0\cdots0}_{\beta_1}\overbrace{1\cdots1}^{\alpha_1}\mathbf{10}\underbrace{0\cdots0}_{\beta_2-\beta_1-1}\overbrace{1\cdots1}^{\alpha_2-\alpha_1-1}\mathbf{10}.$$

Continuing similarly, the string w_λ can be compiled up to the last descent pair. Placing $k - \beta_r - 1$ zeros and $(n-k) - \alpha_r - 1$ ones in sequence after the last descent pair completes string w_λ. Note that, $\mathrm{Maj}(w_\lambda) = \sum_{i_j \in D(w)} i_j = \sum_{j=1}^{r} (\alpha_j + \beta_j + 1) = |\lambda| = m$. Thus, the bijection has been established. □

Consider permutations of $0^k 1^{n-k}$ which have exactly r descents. The following Corollary establishes the generating function for the major indices of these permutations. MacMahon [6] (pp. 169–170) describes a different bijection for the case $q = 1$. Goulden [2] provides a bijective proof of Stanley's shuffling theorem, which in a special case, provides an alternate proof, similar to MacMahon's, of this corollary.

Corollary.

$$q^{r^2}\begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q = \sum_w q^{\mathrm{Maj}(w)},$$

summed over permutations w of $0^k 1^{n-k}$ having r descents.

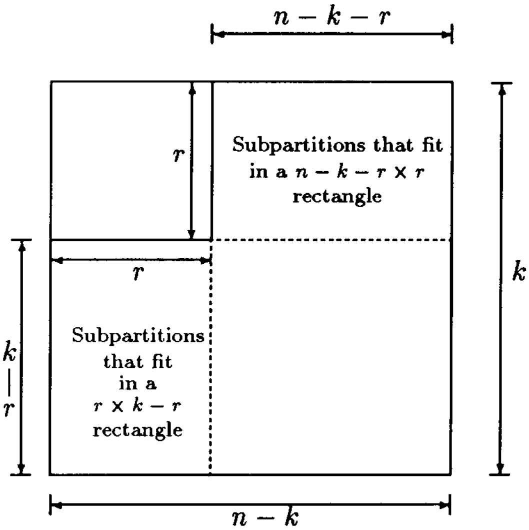

Fig. 2. Partitions that fit inside a $k \times n - k$ rectangle with rank r.

Proof. Use the bijection technique of the theorem. Each permutation, w, of 0^k1^{n-k} having r descents is mapped to a partition with rank r. The generating function for partitions of rank r that fit inside a $k \times n - k$ rectangle is

$$q^{r^2}\begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q,$$

as is evident from *Fig. 2.* □

Acknowledgement

We wish to thank Professor John Stembridge for suggesting this problem, and for the helpful discussions we had with him regarding this paper. We would also like to thank the referee for many valuable comments and suggestions.

References

[1] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968) 236–240.

[2] I.P. Goulden, A bijective proof of Stanley's shuffling theorem, Trans. Amer. Math. Soc. 288 (1985) 147–160.

[3] I.G. Macdonald, Symmetric Functions and Hall Polynomials (Oxford Univ. Press, Oxford; Clarendon Press, New York, 1979).

[4] P.A. MacMahon, The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects, Amer. J. Math. 35 (1913) 281–322.

[5] P.A. MacMahon, Two applications of general theorems in combinatory analysis, Proc. London Math. Soc. 15 (1916) 314–321.
[6] P.A. MacMahon, Combinatory Analysis, Vol. I (Cambridge Univ. Press, London, 1915–16, Reprinted Chelsea, New York, 1960).
[7] D. Rawlings, The (q, r)-Simon Newcomb problem, Linear and Multilinear Algebra 10 (1981) 253–260.
[8] R. Stanley, Enumerative Combinatorics, Vol. I (Wadsworth and Brooks, Monterey, Calif., 1986).