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Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$ ☆

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Abstract

We develop a theory of holomorphic functions in several noncommuting (free) variables and provide a framework for the study of tuples of bounded linear operators on Hilbert spaces. We introduce a free analytic functional calculus and study it in connection with Hausdorff derivations, noncommutative Cauchy and Poisson transforms, and von Neumann type inequalities. Several classical results from complex analysis have free analogues in our noncommutative multivariable setting.

Keywords: Multivariable operator theory; Free holomorphic functions; Analytic functional calculus; Hausdorff derivation; Cauchy transform; Poisson transform; Hardy space; Fock space; Creation operators

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0. Introduction

The Shilov–Arens–Calderon theorem [1,45] states that if $a_1, \ldots, a_n$ are elements of a commutative Banach algebra A with the joint spectrum included in a domain $\Omega \subset \mathbb{C}^n$, then the algebra homomorphism

$$\mathbb{C}[z_1, \ldots, z_n] \ni p \mapsto p(a_1, \ldots, a_n) \in A$$

extends to a continuous homomorphism from the algebra $Hol(\Omega)$, of holomorphic functions on Ω, to the algebra A. This result was greatly improved by the pioneering work of J.L. Taylor [47–49] who introduced a "better" notion of joint spectrum for n-tuples of commuting operators, which is now called Taylor spectrum, and developed an analytic functional calculus. Stated for the open unit ball of $\mathbb{C}^n$,

$$\mathbb{B}_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n\colon\ |\lambda|^2 + \cdots + |\lambda_n|^2 < 1\},$$

his result states that if $[T_1, \ldots, T_n] \in B(\mathcal{H})^n$ is an n-tuple of commuting bounded linear operators on a Hilbert space $\mathcal{H}$ with Taylor spectrum $\sigma(T_1, \ldots, T_n) \subset \mathbb{B}_n$, then there is a unique continuous unital algebra homomorphism

$$Hol(\mathbb{B}_n) \ni f \mapsto f(T_1, \ldots, T_n) \in B(\mathcal{H})$$

such that $z_i \mapsto T_i$, $i = 1, \ldots, n$. Due to a result of V. Müller [23], the condition that $\sigma(T_1, \ldots, T_n) \subset \mathbb{B}_n$ is equivalent to the fact that the joint spectral radius

$$r(T_1, \ldots, T_n) := \lim_{k\to\infty} \left\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right\|^{1/2k} < 1.$$

F.H. Vasilescu introduced and studied, in [50] and a joint paper with R.E. Curto [12], operator-valued Cauchy and Poisson transforms on the unit ball $\mathbb{B}_n$ associated with commuting operators with $r(T_1, \ldots, T_n) < 1$, in connection with commutative multivariable dilation theory.

In recent years, there has been exciting progress in noncommutative multivariable operator theory regarding noncommutative dilation theory ([8–10,17,19,25–27,37,39], etc.) and its applications concerning interpolation in several variables ([3,7,13,27,31,33,38], etc.) and unitary invariants for n-tuples of operators ([4–6,21,27,35–38], etc.).

Our program to develop a *free* analogue of Sz.-Nagy–Foiaş theory [46], for row contractions, fits perfectly the setting of the present paper, which includes that of free holomorphic functions on the open operatorial unit ball

$$[B(\mathcal{H})^n]_1 := \{[X_1, \ldots, X_n] \in B(\mathcal{H})^n\colon\ \|X_1 X_n^* + \cdots + X_n X_n^*\| < 1\}.$$

The present work is an attempt to develop a theory of holomorphic functions in several noncommuting (free) variables and thus provide a framework for the study of arbitrary n-tuples of operators, and to introduce and study a free analytic functional calculus in connection with Hausdorff derivations, noncommutative Cauchy and Poisson transforms, and von Neumann inequalities.

In Section 1, we introduce a notion of radius of convergence for formal power series in n noncommuting indeterminates $Z_1, \ldots, Z_n$ and prove noncommutative multivariable analogues of Abel theorem and Hadamard formula from complex analysis [11,43]. This enables us to define, in particular, the algebra $Hol(B(\mathcal{X})_1^n)$ of free holomorphic functions on the open operatorial unit n-ball, as the set of all power series $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha Z_\alpha$ with radius of convergence $\geqslant 1$. When $n=1$, $Hol(B(\mathcal{X})_1^1)$ coincides with the algebra of all analytic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. The algebra of free holomorphic functions $Hol(B(\mathcal{X})_1^n)$ has the following universal property.

Any strictly contractive representation $\pi : \mathbb{C}[Z_1, \ldots, Z_n] \to B(\mathcal{H})$, i.e.,

$$\big\|\big[\pi(Z_1), \ldots, \pi(Z_n)\big]\big\| < 1,$$

extends uniquely to a representation of $Hol(B(\mathcal{X})_1^n)$.

A free holomorphic function on the open operatorial unit ball $[B(\mathcal{H})^n]_1$ is the representation of an element $F \in Hol(B(\mathcal{X})_1^n)$ on the Hilbert space $\mathcal{H}$, that is, the mapping

$$\big[B(\mathcal{H})^n\big]_1 \ni (X_1, \ldots, X_n) \mapsto F(X_1, \ldots, X_n) \in B(\mathcal{H}).$$

As expected, we prove that any free holomorphic function is continuous on $[B(\mathcal{H})^n]_1$ in the operator norm topology. In the last part of this section, we show that the Hausdorff derivations $\frac{\partial}{\partial Z_i}$, $i = 1, \ldots, n$, on the algebra of noncommutative polynomials $\mathbb{C}[Z_1, \ldots, Z_n]$ [22,42] can be extended to the algebra of free holomorphic functions.

In Section 2, we obtain Cauchy type estimates for the coefficients of free holomorphic functions and a Liouville type theorem for free entire functions. Based on a noncommutative version of Gleason's problem [44], which is obtained here, and the noncommutative von Neumann inequality [29], we provide a free analogue of Schwartz lemma from complex analysis [11,43]. In particular, we prove that if f is a free holomorphic function on $[B(\mathcal{H})^n]_1$ such that $\|f\|_\infty \leqslant 1$ and $f(0) = 0$, then

$$\big\|f(X_1, \ldots, X_n)\big\| \leqslant \big\|[X_1, \ldots, X_n]\big\|, \qquad r\big(f(X_1, \ldots, X_n)\big) \leqslant r(X_1, \ldots, X_n), \quad \text{and}$$
$$\sum_{i=1}^n \left|\frac{\partial f}{\partial X_i}(0)\right|^2 \leqslant 1.$$

In Section 3, following the classical case [20,41], we introduce two Banach algebras of free holomorphic functions, $H^\infty(B(\mathcal{X})_1^n)$ and $A(B(\mathcal{X})_1^n)$, and prove that, together with a natural operator space structure, they are completely isometrically isomorphic to the noncommutative analytic Toeplitz algebra F_n^∞ and the noncommutative disc algebra $\mathcal{A}_n$, respectively, which were introduced in [29] in connection with a multivariable von Neumann inequality. We recall that the algebra F_n^∞ (respectively $\mathcal{A}_n$) is the weakly (respectively norm) closed algebra generated by the left creation operators $S_1, \ldots, S_n$ on the full Fock space with n generators, $F^2(H_n)$, and the identity. These algebras have been intensively studied in recent years by many authors [2,3,14–16,24,27–32,34–37]. The results of this section are used to obtain a maximum principle for free holomorphic functions.

In Section 4, we provide a free analytic functional calculus for n-tuples $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$. We show that there is a continuous unital algebra homomorphism

$$\Phi_T : Hol\big(B(\mathcal{X})_1^n\big) \to B(\mathcal{H}), \quad \Phi_T(f) = f(T_1, \ldots, T_n),$$

which is uniquely determined by the mapping $z_i \mapsto T_i$, $i = 1, \ldots, n$. (The continuity and uniqueness of Φ_T are proved in Section 5.) We introduce a noncommutative Cauchy transform $\mathcal{C}_T : B(F^2(H_n)) \to B(\mathcal{H})$ associated with any n-tuple of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$. The definition is based on the *reconstruction operator*

$$S_1 \otimes T_1^* + \cdots + S_n \otimes T_n^*,$$

which has played an important role in noncommutative multivariable operator theory [37–39]. We prove that

$$f(T_1, \ldots, T_n) = \mathcal{C}_T\big(f(S_1, \ldots, S_n)\big), \quad f \in H^\infty\big(B(\mathcal{X})_1^n\big),$$

where $f(S_1, \ldots, S_n)$ is the boundary function of f. Hence, we deduce that

$$\big\| f(T_1, \ldots, T_n)\big\| \leqslant M \|f\|_\infty,$$

where $M := \sum_{k=0}^\infty \| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \|^{1/2}$. Similar Cauchy representations are obtained for the k-order Hausdorff derivations of f. Finally, we show that the noncommutative Cauchy transform commutes with the action of the unitary group $\mathcal{U}(\mathbb{C}^n)$. More precisely, we prove that

$$\mathcal{C}_T\big(\beta_U(f)\big) = \mathcal{C}_{\beta_U(T)}(f) \quad \text{for any } U \in \mathcal{U}\big(\mathbb{C}^n\big),\ f \in \mathcal{A}_n,$$

where β_U denotes a natural isometric automorphism (generated by U) of the noncommutative disc algebra $\mathcal{A}_n$, or the open unit ball $[B(\mathcal{H})^n]_1$.

In Section 5, we obtain Weierstrass and Montel type theorems [11] for the algebra of free holomorphic functions on the open operatorial unit n-ball. This enables us to introduce a metric on $Hol(B(\mathcal{X})_1^n)$ with respect to which it becomes a complete metric space, and the Hausdorff derivations

$$\frac{\partial}{\partial Z_i} : Hol\big(B(\mathcal{X})_1^n\big) \to Hol\big(B(\mathcal{X})_1^n\big), \quad i = 1, \ldots, n,$$

are continuous. In the end of this section, we prove the continuity and uniqueness of the free functional calculus for n-tuples of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$. Connections with the F_n^∞-functional calculus for row contractions [30] and, in the commutative case, with Taylor's functional calculus [48] are also discussed.

Given an operator $A \in B(F^2(H_n))$, the noncommutative Poisson transform [34] generates a function

$$P[A] : \big[B(\mathcal{H})^n\big]_1 \to B(\mathcal{H}).$$

In Section 6, we provide classes of operators $A \in B(F^2(H_n))$ such that $P[A]$ is a free holomorphic (respectively pluriharmonic) function on $[B(\mathcal{H})^n]_1$. In this case, the operator A can be regarded as the boundary function of the Poisson extension $P[A]$. Using some results from [29, 30,34], we characterize the free holomorphic functions u on the open unit ball $[B(\mathcal{H})^n]_1$ such that $u = P[f]$ for some boundary function f in the noncommutative analytic Toeplitz algebra F_n^∞, or the noncommutative disc algebra $\mathcal{A}_n$. For example, we prove that there exists $f \in F_n^\infty$ such that $u = P[f]$ if and only if

$$\sup_{0 \leqslant r<1} \left\| u(rS_1, \dots, rS_n) \right\| < \infty.$$

We also obtain noncommutative multivariable versions of Herglotz theorem and Dirichlet extension problem [11,20] for free pluriharmonic functions.

In Section 7, we define the radial maximal Hardy space $H^p(B(\mathcal{X})_1^n)$, $p \geqslant 1$, as the set of all free holomorphic function F such that

$$\|F\|_p := \left(\int_0^1 \left\| F(rS_1, \dots, rS_n) \right\|^p dr \right)^{1/p} < \infty,$$

and prove that it is a Banach space. Moreover, we show that

$$\left\| f(T_1, \dots, T_n) \right\| \leqslant \frac{1}{(1 - \|[T_1, \dots, T_n]\|)^{1/p}} \|f\|_p$$

for any $[T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1$ and $f \in H^p(B(\mathcal{X})_1^n)$.

Finally, we introduce the symmetrized Hardy space $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$ as the set of all holomorphic function on $\mathbb{B}_n$ such that $\|f\|_{\mathrm{sym}} := \|f_{\mathrm{sym}}\|_\infty < \infty$, where $f_{\mathrm{sym}} \in Hol(B(\mathcal{X})_1^n)$ is the symmetrized functional calculus of $f \in Hol(\mathbb{B}_n)$. We prove that $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$ is a Banach space and

$$\left\| f(T_1, \dots, T_n) \right\| \leqslant M \|f_{\mathrm{sym}}\|_\infty,$$

for any commuting n-tuple of operators with $r(T_1, \dots, T_n) < 1$.

Several classical results from complex analysis are extended to our noncommutative multivariable setting. The present paper exhibits, in particular, a "very good" free analogue of the algebra of analytic functions on the open unit disc $\mathbb{D}$. This claim is also supported by the fact that numerous results in noncommutative multivariable operator theory [29,30,32,34,37] fit perfectly our setting and can be seen in a new light. We strongly believe that many other results in the theory of analytic functions have free analogues in our noncommutative multivariable setting.

In a forthcoming paper [40], we consider operator-valued Wiener and Bohr type inequalities for free holomorphic (respectively pluriharmonic) functions on the open operatorial unit n-ball. As consequences, we obtain operator-valued Bohr inequalities for the noncommutative Hardy algebra $H^\infty(B(\mathcal{X})_1^n)$ and the symmetrized Hardy space $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$.

1. Free holomorphic functions

We introduce a notion of radius of convergence for formal power series in n noncommuting indeterminates $Z_1, \dots, Z_n$ and prove noncommutative multivariable analogues of Abel theorem and Hadamard formula. This enables us to define algebras of free holomorphic functions on open operatorial n-balls. We show that the Hausdorff derivations $\frac{\partial}{\partial Z_i}$, $i = 1, \dots, n$, on the algebra of noncommutative polynomials $\mathbb{C}[Z_1,\dots,Z_n]$ (see [22,42]) can be extended to algebras of free holomorphic functions.

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1,\dots,g_n$ and the identity g_0. The length of $\alpha\in\mathbb{F}_n^+$ is defined by $|\alpha|=0$ if $\alpha=g_0$ and $|\alpha|:=k$ if $\alpha=g_{i_1}\cdots g_{i_k}$, where $i_1,\dots,i_k\in\{1,\dots,n\}$. We consider formal power series in n noncommuting indeterminates $Z_1,\dots,Z_n$ and coefficients in $B(\mathcal{K})$, the algebra of all bounded linear operators on the Hilbert space $\mathcal{K}$, of the form

$$\sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)}\otimes Z_\alpha,\quad A_{(\alpha)}\in B(\mathcal{K}),$$

where $Z_\alpha:=Z_{i_1}\cdots Z_{i_k}$ if $\alpha=g_{i_1}\cdots g_{i_k}$ and $Z_{g_0}:=I$. If $F=\sum_{\alpha\in\mathbb{F}_n^+}A_{(\alpha)}\otimes Z_\alpha$ and $G=\sum_{\alpha\in\mathbb{F}_n^+}B_{(\alpha)}\otimes Z_\alpha$ are such formal power series, we define their sum and product by setting

$$F+G:=\sum_{\alpha\in\mathbb{F}_n^+}(A_{(\alpha)}+B_{(\alpha)})\otimes Z_\alpha\quad\text{and}\quad FG:=\sum_{\alpha\in\mathbb{F}_n^+}C_{(\alpha)}\otimes Z_\alpha,$$

respectively, where $C_{(\alpha)}:=\sum_{\sigma,\beta\in\mathbb{F}_n^+:\ \alpha=\sigma\beta}A_{(\sigma)}B_{(\beta)}$.

By abuse of notation, throughout this paper, we will denote by $[T_1,\dots,T_n]$ either the n-tuple of operators $(T_1,\dots,T_n)\in B(\mathcal{H})^n$ or the row operator matrix $[T_1\cdots T_n]\in B(\mathcal{H}^{(n)},\mathcal{H})$ acting as an operator from $\mathcal{H}^{(n)}$, the direct sum of n copies of the Hilbert space $\mathcal{H}$, to $\mathcal{H}$. We also denote by $[T_\alpha\colon\ |\alpha|=k]$ the row operator matrix acting from $\mathcal{H}^{n^k}$ to $\mathcal{H}$, where the entries are arranged in the lexicographic order of the free semigroup $\mathbb{F}_n^+$.

In what follows we show that given a sequence of operators $A_{(\alpha)}\in B(\mathcal{K})$, $\alpha\in\mathbb{F}_n^+$, there is a unique $R\in[0,\infty]$ such that the series

$$\sum_{k=0}^\infty\sum_{|\alpha|=k}A_{(\alpha)}\otimes X_\alpha$$

converges in the operator norm of $B(\mathcal{K}\otimes\mathcal{H})$ ($\mathcal{K}\otimes\mathcal{H}$ is the Hilbert tensor product) for any Hilbert space $\mathcal{H}$ and any n-tuple $[X_1,\dots,X_n]\in B(\mathcal{H})^n$ with $\|[X_1,\dots,X_n]\|<R$, and it is divergent for some n-tuples $[Y_1,\dots,Y_n]$ of operators with $\|[Y_1,\dots,Y_n]\|>R$.

The result can be regarded as a noncommutative multivariable analogue of Abel theorem and Hadamard's formula from complex analysis.

Theorem 1.1. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces and let $A_{(\alpha)}\in B(\mathcal{K})$, $\alpha\in\mathbb{F}_n^+$, be a sequence of operators. Define $R\in[0,\infty]$ by setting*

$$\frac{1}{R}:=\limsup_{k\to\infty}\left\|\sum_{|\alpha|=k}A_{(\alpha)}^*A_{(\alpha)}\right\|^{\frac{1}{2k}}.$$

Then the following properties hold:

(i) *For any n-tuple of operators $[X_1,\dots,X_n]\in B(\mathcal{H})^n$, the series $\sum_{k=0}^\infty\|\sum_{|\alpha|=k}A_{(\alpha)}\otimes X_\alpha\|$ converges if $\|[X_1,\dots,X_n]\|<R$. Moreover, if $0\leqslant\rho<R$, then the convergence is uniform for $[X_1,\dots,X_n]$ with $\|[X_1,\dots,X_n]\|\leqslant\rho$.*

(ii) *If $R < R' < \infty$ and $\mathcal{H}$ is infinite-dimensional, then there is an n-tuple $[X_1, \ldots, X_n] \in B(\mathcal{H})^n$ of operators with*

$$\left\| X_1 X_1^* + \cdots + X_n X_n^* \right\|^{1/2} = R'$$

such that $\sum_{k=0}^{\infty}(\sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha)$ is divergent in the operator norm of $B(\mathcal{K} \otimes \mathcal{H})$.

Moreover, the number R satisfying properties (i) *and* (ii) *is unique.*

Proof. Assume that $R > 0$ and $[X_1, \ldots, X_n]$ is an n-tuple of operators on $\mathcal{H}$ such that $\|[X_1, \ldots, X_n]\| < R$. Let $\rho', \rho > 0$ be such that $\|[X_1, \ldots, X_n]\| < \rho' < \rho < R$. Since $\frac{1}{\rho} > \frac{1}{R}$, we can find $m_0 \in \mathbb{N} := \{1, 2, \ldots\}$ such that

$$\left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2k} < \frac{1}{\rho} \quad \text{for any } k \geqslant m_0.$$

Hence, we deduce that

$$\begin{aligned}
\left\| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \right\| &= \left\| \left[I \otimes X_\alpha \colon |\alpha| = k \right] \begin{bmatrix} A_{(\alpha)} \otimes I \\ \vdots \\ |\alpha| = k \end{bmatrix} \right\| \\
&= \left\| \sum_{|\alpha|=k} X_\alpha X_\alpha^* \right\|^{1/2} \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \\
&\leqslant \left\| \sum_{i=1}^{n} X_i X_i^* \right\|^{k/2} \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \\
&\leqslant \left(\frac{\rho'}{\rho} \right)^k
\end{aligned}$$

for any $k \geqslant m_0$. This proves the convergence of the series $\sum_{k=0}^{\infty} \| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \|$. Assume now that $0 \leqslant \rho < R$ and $\|[X_1, \ldots, X_n]\| \leqslant \rho$. Choose γ such that $0 \leqslant \rho < \gamma < R$ and notice that, due to similar calculations as above, there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \right\| \leqslant \left(\frac{\rho}{\gamma} \right)^k$$

for any $(X_1, \ldots, X_n)$ with $\|[X_1, \ldots, X_n]\| \leqslant \rho$, and $k \geqslant n_0$, which proves the uniform convergence of the above series. The case $R = \infty$, can be treated in a similar manner.

To prove part (ii), assume that $R < \infty$ and $\mathcal{H}$ is infinite-dimensional. Let $R', \rho > 0$ be such that $R < \rho < R'$ and define the operators $X_i := R' V_i$, $i = 1, \ldots, n$, where $V_1, \ldots, V_n$ are isometries with orthogonal ranges. Notice that $\|[X_1, \ldots, X_n]\| = R'$ and

$$\begin{aligned}\left\|\sum_{|\alpha|=k} A_{(\alpha)}\otimes X_\alpha\right\| &= R'^k\left\|\left(\sum_{|\alpha|=k} A_{(\alpha)}^*\otimes V_\alpha^*\right)\left(\sum_{|\alpha|=k} A_{(\alpha)}\otimes V_\alpha\right)\right\|^{1/2}\\ &= R'^k\left\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\otimes I\right\|^{1/2}\\ &= R'^k\left\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\right\|^{1/2}.\end{aligned}$$

On the other hand, since $\frac{1}{\rho} < \frac{1}{R}$, there are arbitrarily large $k \in \mathbb{N}$ such that

$$\left\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\right\|^{1/2} > \left(\frac{1}{\rho}\right)^k.$$

Consequently, we deduce that

$$\left\|\sum_{|\alpha|=k} A_{(\alpha)}\otimes X_\alpha\right\| > \left(\frac{R'}{\rho}\right)^k,$$

which proves that the series $\sum_{k=0}^\infty(\sum_{|\alpha|=k} A_{(\alpha)}\otimes X_\alpha)$ is divergent in the operator norm. The uniqueness of the number R satisfying properties (i) and (ii) is now obvious. □

As expected, the number R in the above theorem is called the radius of convergence of the power series $\sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)}\otimes Z_\alpha$.

Let us consider the full Fock space

$$F^2(H_n) = \mathbb{C}1 \oplus \bigoplus_{m\geqslant 1} H_n^{\otimes m},$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1,\dots,e_n\}$. Setting $e_\alpha := e_{i_1}\otimes\cdots\otimes e_{i_k}$ if $\alpha = g_{i_1}\cdots g_{i_k}$, and $e_{g_0} = 1$, it is clear that $\{e_\alpha\colon \alpha\in\mathbb{F}_n^+\}$ is an orthonormal basis of the full Fock space $F^2(H_n)$. For each $i = 1,2,\dots,$ we define the left creation operator $S_i \in B(F^2(H_n))$ by

$$S_i\xi = e_i\otimes\xi,\quad \xi\in F^2(H_n).$$

We can now obtain the following characterization of the radius of convergence, which will be useful later.

Corollary 1.2. *Let $\sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)}\otimes Z_\alpha$ be a formal power series with radius of convergence R.*

(i) *If $R > 0$ and $0 < r < R$, then there exists $C > 0$ such that*

$$\left\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\right\|^{1/2} \leqslant \frac{C}{r^k}\quad \text{for any } k = 0,1,\dots.$$

(ii) *The radius of convergence of the power series satisfies the relations*

$$R = \sup\left\{r \geqslant 0\text{: } \textit{the sequence } \left\{r^k \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2}\right\}_{k=0}^{\infty} \textit{ is bounded}\right\}$$

and

$$R = \sup\left\{r \geqslant 0\text{: } \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} A_{(\alpha)} \otimes S_\alpha \textit{ is convergent in the operator norm}\right\}.$$

Proof. Setting $X_i := rS_i$, $i = 1, \ldots, n$, where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space, we have $\|[X_1, \ldots, X_n]\| = r < R$. According to Theorem 1.1, the series $\sum_{k=0}^{\infty} \|r^k \sum_{|\alpha|=k} A_{(\alpha)} \otimes S_\alpha\|$ is convergent. Since $S_1, \ldots, S_n$ are isometries with orthogonal ranges, the above series is equal to $\sum_{k=0}^{\infty} r^k \| \sum_{|\alpha|=k} A_{(\alpha)}^* A_\alpha \|^{1/2}$. Consequently, there is a constant $C > 0$ such that

$$r^k \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_\alpha \right\|^{1/2} \leqslant C \quad \text{for any } k = 0, 1, \ldots.$$

Now, the second part of this corollary follows easily from part (i) and Theorem 1.1. This completes the proof. □

We establish terminology which will be used throughout the paper. Denote by $[B(\mathcal{H})^n]_\gamma$ the open ball of $B(\mathcal{H})^n$ of radius $\gamma > 0$, i.e.,

$$\left[B(\mathcal{H})^n\right]_\gamma := \left\{[X_1, \ldots, X_n]\text{: } \left\|X_1 X_1^* + \cdots + X_n X_n^*\right\|^{1/2} < \gamma\right\}.$$

We also use the notation $[B(\mathcal{H})^n]_1^-$ for the closed ball. A formal power series $F := \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ represents a free holomorphic function on the open operatorial n-ball of radius γ with coefficients in $B(\mathcal{K})$, if for any Hilbert space $\mathcal{H}$ and any representation

$$\pi : \mathbb{C}[Z_1, \ldots, Z_n] \to B(\mathcal{H}) \quad \text{such that} \quad \left[\pi(Z_1), \ldots, \pi(Z_n)\right] \in \left[B(\mathcal{H})^n\right]_\gamma$$

the series

$$F\big(\pi(Z_1), \ldots, \pi(Z_n)\big) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes \pi(Z_\alpha)$$

converges in the operator norm of $B(\mathcal{K} \otimes \mathcal{H})$. Due to Theorem 1.1, we must have $\gamma \leqslant R$, where R is the radius of convergence of F. The mapping

$$\left[B(\mathcal{H})^n\right]_\gamma \ni [X_1, \ldots, X_n] \mapsto F(X_1, \ldots, X_n) \in B(\mathcal{K} \otimes \mathcal{H})$$

is called the representation of F on the Hilbert space $\mathcal{H}$. Given a Hilbert space $\mathcal{H}$, we say that a function $G : [B(\mathcal{H})^n]_\gamma \to B(\mathcal{K} \otimes \mathcal{H})$ is a *free holomorphic function* on $[B(\mathcal{H})^n]_\gamma$ with coefficients in $B(\mathcal{K})$ if there exist operators $A_{(\alpha)} \in B(\mathcal{K})$, $\alpha \in \mathbb{F}_n^+$, such that the power series $\sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ has radius of convergence $\geqslant \gamma$ and

$$G(X_1, \dots, X_n) = \sum_{k=0}^{\infty}\biggl(\sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha\biggr),$$

where the series converges in the operator norm for any $[X_1, \dots, X_n] \in [B(\mathcal{H})^n]_\gamma$.

We remark that the coefficients of a free holomorphic function are uniquely determined by its representation on an infinite-dimensional Hilbert space. Indeed, let $0 < r < \gamma$ and assume $F(rS_1, \dots, rS_n) = 0$, where $S_1, \dots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Taking into account that $S_i^* S_j = \delta_{ij} I$, we have

$$\bigl\langle F(rS_1, \dots, rS_n)(x \otimes 1), (I_{\mathcal{K}} \otimes S_\alpha)(y \otimes 1)\bigr\rangle = \langle A_{(\alpha)} x, y\rangle = 0$$

for any $x, y \in \mathcal{K}$ and $\alpha \in \mathbb{F}_n^+$. Therefore $A_{(\alpha)} = 0$ for any $\alpha \in \mathbb{F}_n^+$.

We establish now the continuity of free holomorphic functions on the open ball $[B(\mathcal{H})^n]_\gamma$.

Theorem 1.3. *Let $f(X_1, \dots, X_n) = \sum_{k=0}^{\infty}(\sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha)$ be a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ with coefficients in $B(\mathcal{K})$. If $X := [X_1, \dots, X_n]$, $Y := [Y_1, \dots, Y_n]$ are in the closed ball $[B(\mathcal{H})^n]_r^-$, $0 < r < \gamma$, then*

$$\bigl\| f(X) - f(Y)\bigr\| \leqslant \|X - Y\| \sum_{k=1}^{\infty} k r^{k-1} \Biggl\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\Biggr\|^{1/2}.$$

In particular, f is continuous on $[B(\mathcal{H})^n]_\gamma$ and uniformly continuous on $[B(\mathcal{H})^n]_r^-$ in the operator norm topology.

Proof. Let $X^{[k]} := [X_\alpha\colon \alpha \in \mathbb{F}_n^+,\ |\alpha| = k]$, $k = 1, 2, \dots,$ be the row operator matrix with entries arranged in the lexicographic order of the free semigroup $\mathbb{F}_n^+$. First, we prove that if $\|X\| \neq \|Y\|$, then

$$\frac{\|X^{[k]} - Y^{[k]}\|}{\|X - Y\|} \leqslant \frac{\|X\|^k - \|Y\|^k}{\|X\| - \|Y\|}. \tag{1.1}$$

Notice that

$$\begin{aligned} X^{[k]} - Y^{[k]} &= \bigl[(X_1 - Y_1)X^{[k-1]}, \dots, (X_n - Y_n)X^{[k-1]}\bigr] \\ &\quad + \bigl[Y_1\bigl(X^{[k-1]} - Y^{[k-1]}\bigr), \dots, Y_n\bigl(X^{[k-1]} - Y^{[k-1]}\bigr)\bigr] \\ &= (X - Y)\operatorname{diag}_n\bigl(X^{[k-1]}\bigr) + Y \operatorname{diag}_n\bigl(X^{[k-1]} - Y^{[k-1]}\bigr), \end{aligned}$$

where $\operatorname{diag}_n(A)$ is the $n \times n$ block diagonal operator matrix with A on the diagonal and 0 otherwise. Hence, we deduce that

$$\bigl\|X^{[k]} - Y^{[k]}\bigr\| \leqslant \|X - Y\|\bigl\|X^{[k-1]}\bigr\| + \|Y\|\bigl\|X^{[k-1]} - Y^{[k-1]}\bigr\|$$

for any $k \geqslant 2$. Iterating this relation and taking into account that $\|X^{[k]}\| \leqslant \|X\|^k$ for $k = 1, 2, \ldots,$ we obtain

$$\begin{aligned}\big\|X^{[k]} - Y^{[k]}\big\| &\leqslant \|X - Y\|\big(\big\|X^{[k-1]}\big\| + \big\|X^{[k-2]}\big\|\big\|Y^{[1]}\big\| + \cdots + \big\|Y^{[k-1]}\big\|\big) \\ &\leqslant \|X - Y\|\big(\|X\|^{k-1} + \|X\|^{k-2}\|Y\| + \cdots + \|Y\|^{k-1}\big),\end{aligned}$$

which proves inequality (1.1). Assuming that $\|X\| \leqslant r$ and $\|Y\| \leqslant r$, we deduce that

$$\big\|X^{[k]} - Y^{[k]}\big\| \leqslant kr^{k-1}\|X - Y\|, \quad k = 1, 2, \ldots.$$

Hence, we obtain

$$\begin{aligned}\big\|f(X) - f(Y)\big\| &\leqslant \sum_{k=1}^{\infty}\bigg\|\sum_{|\alpha|=k} A_{(\alpha)} \otimes (X_\alpha - Y_\alpha)\bigg\| \leqslant \sum_{k=1}^{\infty}\bigg\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\bigg\|^{1/2}\big\|X^{[k]} - Y^{[k]}\big\| \\ &\leqslant \|X - Y\|\sum_{k=1}^{\infty} kr^{k-1}\bigg\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\bigg\|^{1/2}.\end{aligned}$$

Let ρ be a constant such that $r < \rho < \gamma$. Since $\gamma \leqslant R$ (R is the radius of convergence of f) and $\frac{1}{\rho} > \frac{1}{\gamma} \geqslant \frac{1}{R}$, we can find $m_0 \in \mathbb{N}$, such that

$$\bigg\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\bigg\|^{1/2k} < \frac{1}{\rho} \quad \text{for any } k \geqslant m_0.$$

Combining this with the above inequality, we deduce that

$$\big\|f(X) - f(Y)\big\| \leqslant \|X - Y\|\left(\sum_{k=1}^{m_0-1} kr^{k-1}\bigg\|\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)}\bigg\|^{1/2} + \sum_{k=m_0}^{\infty}\frac{k}{r}\left(\frac{r}{\rho}\right)^k\right).$$

Since $r < \rho$, the above series is convergent. Consequently, there exists a constant $M > 0$ such that

$$\big\|f(X) - f(Y)\big\| \leqslant M\|X - Y\| \quad \text{for any } X, Y \in \big[B(\mathcal{H})^n\big]_r^-.$$

This implies the uniform continuity of f on any closed ball $[B(\mathcal{H})^n]_r^-$, $0 < r < \gamma$, in the norm topology and, consequently, the continuity of f on $[B(\mathcal{H})^n]_\gamma$. $\square$

Theorem 1.4. *Let F and G be formal power series such that*

$$F(X_1, \ldots, X_n) = \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha\right) \quad \textit{and} \quad G(X_1, \ldots, X_n) = \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k} B_{(\alpha)} \otimes X_\alpha\right)$$

are free holomorphic functions on $[B(\mathcal{H})^n]_\gamma$, and let $a, b \in \mathbb{C}$. Then the power series $aF + bG$, and FG generate free holomorphic functions on $[B(\mathcal{H})^n]_\gamma$. Moreover,

$$aF(X_1,\dots,X_n)+bG(X_1,\dots,X_n)=\sum_{k=0}^{\infty}\Bigg(\sum_{|\alpha|=k}(aA_{(\alpha)}+bB_{(\alpha)})\otimes X_\alpha\Bigg) \quad \textit{and}$$

$$F(X_1,\dots,X_n)G(X_1,\dots,X_n)=\sum_{k=0}^{\infty}\Bigg(\sum_{|\alpha|=k}C_{(\alpha)}\otimes X_\alpha\Bigg)$$

for any $[X_1,\dots,X_n]\in[B(\mathcal{H})^n]_\gamma$, *where* $C_{(\alpha)}:=\sum_{\alpha=\sigma\beta}A_{(\sigma)}B_{(\beta)}$, $\alpha\in\mathbb{F}_n^+$.

Proof. According to the hypotheses, both power series F and G have radius of convergence $\geqslant\gamma$. Due to Theorem 1.1, we deduce that, given any $\epsilon>0$, there exists $k_0\in\mathbb{N}$ such that

$$\Bigg\|\sum_{|\alpha|=k}A_{(\alpha)}^*A_{(\alpha)}\Bigg\|^{1/2k}\leqslant\frac{1}{\gamma}+\epsilon \quad\text{and}\quad \Bigg\|\sum_{|\alpha|=k}B_{(\alpha)}^*B_{(\alpha)}\Bigg\|^{1/2k}\leqslant\frac{1}{\gamma}+\epsilon$$

for any $k\geqslant k_0$. Assume that $|a|+|b|\neq 0$. Since the left creation operators $S_1,\dots,S_n$ are isometries with orthogonal ranges, we have

$$\begin{aligned}
&\Bigg\|\sum_{|\alpha|=k}(aA_{(\alpha)}+bB_{(\alpha)})^*(aA_{(\alpha)}+bB_{(\alpha)})\Bigg\|^{1/2}\\
&\quad=\Bigg\|\sum_{|\alpha|=k}(aA_{(\alpha)}+bB_{(\alpha)})\otimes S_\alpha\Bigg\|\\
&\quad\leqslant\Bigg\|\sum_{|\alpha|=k}aA_{(\alpha)}\otimes S_\alpha\Bigg\|+\Bigg\|\sum_{|\alpha|=k}bB_{(\alpha)}\otimes S_\alpha\Bigg\|\\
&\quad=\Bigg\|\sum_{|\alpha|=k}|a|^2A_{(\alpha)}^*A_{(\alpha)}\Bigg\|^{1/2}+\Bigg\|\sum_{|\alpha|=k}|b|^2B_{(\alpha)}^*B_{(\alpha)}\Bigg\|^{1/2}\\
&\quad=\big(|a|+|b|\big)\left(\frac{1}{\gamma}+\epsilon\right)^k
\end{aligned}$$

for any $k\geqslant k_0$. Hence, we deduce that

$$\limsup_{k\to\infty}\Bigg\|\sum_{|\alpha|=k}(aA_{(\alpha)}+bB_{(\alpha)})^*(aA_{(\alpha)}+bB_{(\alpha)})\Bigg\|^{1/2k}\leqslant\frac{1}{\gamma}+\epsilon$$

for any $\epsilon>0$. Taking $\epsilon\to 0$, we deduce that the power series $aF+bG$ has the radius of convergence $\geqslant\gamma$. Now, we prove that the power series FG has radius of convergence $\geqslant\gamma$. If $0<r<\gamma$, then, due to Corollary 1.2, there is a constant $M>0$ such that

$$\Bigg\|\sum_{|\sigma|=k}C_{(\sigma)}^*C_{(\sigma)}\Bigg\|^{1/2}=\Bigg\|\sum_{|\sigma|=k}C_{(\sigma)}\otimes S_\sigma\Bigg\|$$

$$
\begin{aligned}
&= \left\| \sum_{p+q=k} \left(\sum_{|\alpha|=p} A_{(\alpha)} \otimes S_\alpha \right) \left(\sum_{|\beta|=q} B_{(\beta)} \otimes S_\beta \right) \right\| \\
&\leqslant \sum_{p+q=k} \left\| \sum_{|\alpha|=p} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \left\| \sum_{|\beta|=q} B_{(\beta)}^* B_{(\beta)} \right\|^{1/2} \\
&\leqslant \sum_{p+q=k} \frac{M}{r^p} \cdot \frac{M}{r^q} \\
&= (k+1) \frac{M^2}{r^k}
\end{aligned}
$$

for any $k = 0, 1, \ldots$. Hence, we obtain

$$
\limsup_{k \to \infty} \left\| \sum_{|\sigma|=k} C_{(\sigma)}^* C_{(\sigma)} \right\|^{1/2k} \leqslant \frac{1}{r}
$$

for any r such that $0 < r < \gamma$. Consequently, the radius of convergence of the power series FG is $\geqslant \gamma$. The last part of the theorem follows easily using Theorem 1.1. □

We are in position to give a characterization as well as models for free holomorphic functions on the open operatorial n-ball of radius γ.

Theorem 1.5. *A power series $F := \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ represents a free holomorphic function on the open operatorial n-ball of radius γ with coefficients in $B(\mathcal{K})$ if and only if the series*

$$
\sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} A_{(\alpha)} \otimes S_\alpha
$$

is convergent for any $r \in [0, \gamma)$, where $S_1, \ldots, S_n$ are the left creation operators on the Fock space $F^2(H_n)$. Moreover, in this case, the series

$$
\sum_{k=0}^{\infty} \left\| \sum_{|\alpha|=k} r^{|\alpha|} A_{(\alpha)} \otimes S_\alpha \right\| = \sum_{k=0}^{\infty} r^k \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \tag{1.2}
$$

are convergent for any $r \in [0, \gamma)$.

Proof. Assume that F represents a free holomorphic function on the open operatorial n-ball of radius γ. According to Theorem 1.1, $\gamma \leqslant R$, where R is the radius of convergence of F, and $\sum_{k=0}^{\infty} \| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \|$ converges for any n-tuple $[X_1, \ldots, X_n]$ with $\|[X_1, \ldots, X_n]\| = r < \gamma$. Since $\|[rS_1, \ldots, rS_n]\| = r < \gamma$, we deduce that the series (1.2) is convergent for any $r \in [0, \gamma)$.

Now, assume that the series (1.2) is convergent for any $r \in [0, \gamma)$. According to the noncommutative von Neumann inequality [29], we have

$$\sum_{k=0}^{\infty}\left\| \sum_{|\alpha|=k} r^{|\alpha|} A_{(\alpha)} \otimes T_\alpha \right\| \leqslant \sum_{k=0}^{\infty}\left\| \sum_{|\alpha|=k} r^{|\alpha|} A_{(\alpha)} \otimes S_\alpha \right\|$$

for any n-tuple $[T_1, \dots, T_n] \in B(\mathcal{H})^n$ with $T_1 T_1^* + \cdots + T_n T_n^* \leqslant I$ and any $r \in [0, \gamma)$. Hence, we deduce that the series

$$\sum_{k=0}^{\infty}\left\| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \right\|$$

converges for any n-tuple of operators $[X_1, \dots, X_n]$ with $\|[X_1, \dots, X_n]\| < \gamma$. Due to Theorem 1.1, the power series $F = \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ represents a free holomorphic function on the open operatorial n-ball of radius γ. This completes the proof. □

Corollary 1.6. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of complex numbers. Then the following statements are equivalent*:

(i) $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is an analytic function on the open unit disc* $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$.
(ii) $f_r(S) := \sum_{k=0}^{\infty} r^k a_k S^k$ *is convergent in the operator norm for each* $r \in [0, 1)$, *where S is the unilateral shift on the Hardy space H^2.*
(iii) $f(Z) := \sum_{k=0}^{\infty} a_k Z^k$ *is a free holomorphic function on the open operatorial unit* 1*-ball.*

Proof. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function on the open unit disc, then Hadamard's theorem implies $\limsup_{k\to\infty} |a_k|^{1/k} \leqslant 1$. Hence $\sum_{k=0}^{\infty} r^k |a_k| < \infty$ for any $r \in [0, 1)$ and, consequently, the series $\sum_{k=0}^{\infty} r^k a_k S^k$ is convergent in the operator norm. Conversely, if the latter series is norm convergent, then, due to von Neumann inequality [51], the series $\sum_{k=0}^{\infty} r^k a_k z$ converges for any $r \in [0, 1)$ and $z \in \mathbb{D}$. Hence, we deduce (i). The equivalence (ii) ⇔ (iii) is a particular case of Theorem 1.5. □

If $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$, then we set $\lambda_\alpha := \lambda_{i_1} \cdots \lambda_{i_k}$ and $\lambda_0 = 1$.

Corollary 1.7. *If $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$, $a_\alpha \in \mathbb{C}$, is a free holomorphic function on the open operatorial unit n-ball, then its representation on $\mathbb{C}$,*

$$f(\lambda_1, \dots, \lambda_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha \lambda_\alpha,$$

is a holomorphic function on $\mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$.

Proof. Due to Theorem 1.5, we have

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} |a_\alpha||\lambda_\alpha| \leqslant \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k} |a_\alpha|^2\right)^{1/2}\left(\sum_{|\alpha|=k} |\lambda_\alpha|^2\right)^{1/2}$$

$$\leqslant \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}|a_\alpha|^2\right)^{1/2}\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^{k/2}<\infty$$

for any $(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n$. Hence, the result follows. □

In the last part of this section, we show that the Hausdorff derivations on the algebra of noncommutative polynomials $\mathbb{C}[Z_1,\dots,Z_n]$ (see [22,42]) can be extended to the algebra of free holomorphic functions. For each $i=1,\dots,n$, we define the free partial derivation $\frac{\partial}{\partial Z_i}$ on $\mathbb{C}[Z_1,\dots,Z_n]$ as the unique linear operator on this algebra, satisfying the conditions

$$\frac{\partial I}{\partial Z_i}=0,\qquad \frac{\partial Z_i}{\partial Z_i}=I,\qquad \frac{\partial Z_j}{\partial Z_i}=0\quad \text{if } i\neq j,$$

and

$$\frac{\partial(fg)}{\partial Z_i}=\frac{\partial f}{\partial Z_i}g+f\frac{\partial g}{\partial Z_i}$$

for any $f,g\in\mathbb{C}[Z_1,\dots,Z_n]$ and $i,j=1,\dots,n$. The same definition extends to formal power series in the noncommuting indeterminates $Z_1,\dots,Z_n$.

Notice that if $\alpha=g_{i_1}\cdots g_{i_p}$, $|\alpha|=p$, and q of the $g_{i_1},\dots,g_{i_p}$ are equal to g_j, then $\frac{\partial Z_\alpha}{\partial Z_j}$ is the sum of the q words obtained by deleting each occurrence of Z_j in $Z_\alpha:=Z_{i_1}\cdots Z_{i_p}$. For example,

$$\frac{\partial(Z_1Z_2Z_1^2)}{\partial Z_1}=Z_2Z_1^2+Z_1Z_2Z_1+Z_1Z_2Z_1.$$

One can easily show that $\frac{\partial}{\partial Z_i}$ coincides with the Hausdorff derivative. If $\beta:=g_{i_1}\cdots g_{i_k}\in\mathbb{F}_n^+$, $i_1,\dots,i_k\in\{1,2,\dots,n\}$, we denote $Z_\beta:=Z_{i_1}\cdots Z_{i_k}$ and define the k-order free partial derivative of $G\in\mathbb{C}[Z_1,\dots,Z_n]$ with respect to $Z_{i_1},\dots,Z_{i_k}$ by

$$\frac{\partial^k G}{\partial Z_{i_1}\cdots\partial Z_{i_k}}:=\frac{\partial}{\partial Z_{i_1}}\left(\frac{\partial}{\partial Z_{i_2}}\cdots\left(\frac{\partial G}{\partial Z_{i_k}}\right)\cdots\right).$$

These definitions can easily be extended to formal power series. If $F:=\sum_{\alpha\in\mathbb{F}_n^+}A_{(\alpha)}\otimes Z_\alpha$ is a power series with operator-valued coefficients, then we define the k-order free partial derivative of F with respect to $Z_{i_1},\dots,Z_{i_k}$ to be the power series

$$\frac{\partial^k F}{\partial Z_{i_1}\cdots\partial Z_{i_k}}:=\sum_{\alpha\in\mathbb{F}_n^+}A_{(\alpha)}\otimes\frac{\partial^k Z_\alpha}{\partial Z_{i_1}\cdots\partial Z_{i_k}}.$$

Proposition 1.8. *If $i,j\in\{1,\dots,n\}$, then*

$$\frac{\partial^2 F}{\partial Z_i\partial Z_j}=\frac{\partial^2 F}{\partial Z_j\partial Z_i}$$

for any formal power series F.

Proof. Due to linearity, it is enough to prove the result for monomials. Let $\alpha := g_{i_1} \cdots g_{i_k}$ be a word in $\mathbb{F}_n^+$ and $Z_\alpha := Z_{i_1} \cdots Z_{i_k}$. Let $i, j \in \{1, \dots, n\}$ be such that $i \neq j$. Assume that Z_i occurs q times in Z_α, and Z_j occurs p times in Z_α. Then $\frac{\partial Z_\alpha}{\partial Z_i}$ is the sum of the q words obtained by deleting each occurrence of Z_i in Z_α. Notice that Z_j occurs p times in each of these q words. Therefore, $\frac{\partial^2 Z_\alpha}{\partial Z_j \partial Z_i}$ is the sum of the qp words obtained by deleting each occurrence of Z_i in Z_α and then deleting each occurrence of Z_j in the resulting words. Similarly, $\frac{\partial^2 Z_\alpha}{\partial Z_i \partial Z_j}$ is the sum of the qp words obtained by deleting each occurrence of Z_j in Z_α and then deleting each occurrence of Z_i in the resulting words. Hence, it is clear that

$$\frac{\partial^2 Z_\alpha}{\partial Z_i \partial Z_j} = \frac{\partial^2 Z_\alpha}{\partial Z_j \partial Z_i}.$$

This completes the proof. □

Theorem 1.9. *Let $F = \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ be a power series with radius of convergence R and let R' be the radius of convergence of the power series $\frac{\partial^k F}{\partial Z_{j_1} \cdots \partial Z_{j_k}}$, where $j_1, \dots, j_k \in \{1, \dots, n\}$. Then $R' \geqslant R$ and, in general, the inequality is strict.*

Proof. It is enough to prove the result for first order free partial derivatives. For any word $\omega := g_{i_1} \cdots g_{i_k}$, $|\omega| = k \geqslant 1$, and $0 \leqslant m \leqslant k$, we define the insertion mapping of g_j, $j = 1, \dots, n$, on the m position of ω by setting

$$\chi(g_j, m, \omega) := \begin{cases} g_j \omega & \text{if } m = 0, \\ g_{i_1} \cdots g_{i_m} g_j g_{i_{m+1}} \cdots g_{i_k} & \text{if } 1 \leqslant m \leqslant k-1, \\ \omega g_j & \text{if } m = |\omega| = k, \end{cases}$$

and $\chi(g_j, 0, g_0) := g_j$. Let

$$\frac{\partial F}{\partial Z_j} = \sum_{\beta \in \mathbb{F}_n^+} B_{(\beta)} \otimes Z_\beta.$$

Using the definition of the Hausdorff derivation and the insertion mapping, we deduce that

$$B_{(\beta)} = \sum_{m=0}^{k} A_{(\chi(g_j, m, \beta))}$$

for any $\beta \in \mathbb{F}_n^+$ with $|\beta| = k$. This is the case, since the monomial Z_β comes from free differentiation with respect to Z_j of the monomials $Z_{\chi(g_j, m, \beta)}$, $m = 0, 1, \dots, |\beta|$. Therefore, we have

$$\sum_{|\beta|=k} B_{(\beta)}^* B_{(\beta)} = \sum_{|\beta|=k} \left(\sum_{m=0}^{k} A_{(\chi(g_j, m, \beta))}^* \right) \left(\sum_{m=0}^{k} A_{(\chi(g_j, m, \beta))} \right)$$

$$\leqslant (k+1)\sum_{|\beta|=k}\sum_{m=0}^{k} A^*_{(\chi(g_j,m,\beta))}A_{(\chi(g_j,m,\beta))}$$
$$\leqslant (k+1)^2\sum_{|\alpha|=k+1} A^*_{(\alpha)}A_{(\alpha)}.$$

The last inequality holds since, for each $j=1,\dots,n$, each $\alpha\in\mathbb{F}_n^+$ with $|\alpha|=k+1$, and each $\beta\in\mathbb{F}_n^+$ with $|\beta|=k$, the cardinal of the set

$$\bigl\{(g_j,m,\beta):\ \chi(g_j,m,\beta)=\alpha,\ \text{where } m=0,1,\dots,k\bigr\}$$

is $\leqslant k+1$. Hence, we deduce that

$$\biggl(\sum_{|\beta|=k} B^*_{(\beta)}B_{(\beta)}\biggr)^{1/2k}\leqslant (k+1)^{1/k}\biggl(\sum_{|\alpha|=k+1} A^*_{(\alpha)}A_{(\alpha)}\biggr)^{1/2k}.$$

Consequently, due to Theorem 1.1, we have $\frac{1}{R'}\leqslant\frac{1}{R}$. Therefore, $R'\geqslant R$.

To prove the last part of the theorem, let $R_1,R_2>0$ be such that $R_1<R_2$. Let us consider two power series

$$F=\sum_{k=0}^{\infty}a_kZ_1^k\quad\text{and}\quad G=\sum_{k=0}^{\infty}b_kZ_2^k$$

with radius of convergence R_1 and R_2, respectively. We shall show that the power series

$$F+G=\sum_{k=0}^{\infty}\bigl(a_kZ_1^k+b_kZ_2^k\bigr)$$

has the radius of convergence equal to R_1. First, since

$$\sup_k\bigl(|a_k|^2+|b_k|^2\bigr)^{1/2k}\geqslant\sup|a_k|^{1/k}=\frac{1}{R_1},$$

we deduce that the radius of convergence of $F+G$ is $\leqslant R_1$. On the other hand, if $r<R_1$, Corollary 1.2 shows that both sequences $\{r^k|a_k|\}_{k=0}^{\infty}$ and $\{r^k|b_k|\}_{k=0}^{\infty}$ are bounded. This implies that the sequence $\{r^k(|a_k|^2+|b_k|^2)^{1/2}\}_{k=0}^{\infty}$ is bounded. Applying again Corollary 1.2, we can conclude that $F+G$ has radius of convergence R_1. Since

$$\frac{\partial(F+G)}{\partial Z_2}=\sum_{k=1}^{\infty}kb_kZ_2^{k-1},$$

the power series $\frac{\partial(F+G)}{\partial Z_2}$ has radius of convergence R_2, which is strictly larger than the radius of convergence of $F+G$. This completes the proof. □

2. Cauchy, Liouville, and Schwartz type results for free holomorphic functions

In this section, we obtain Cauchy type estimates for the coefficients of free holomorphic functions and a Liouville type theorem for free entire functions. Based on a noncommutative version of Gleason's problem [44] and the noncommutative von Neumann inequality [29], we provide a free analogue of Schwartz lemma.

First, we obtain Cauchy type estimates for the coefficients of free holomorphic functions on the open ball $[B(\mathcal{H})^n]_\gamma$ with coefficients in $B(\mathcal{K})$.

Theorem 2.1. *Let $F : [B(\mathcal{H})^n]_\gamma \to B(\mathcal{K})\bar{\otimes}B(\mathcal{H})$ be a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ with the representation*

$$F(X_1, \ldots, X_n) = \sum_{k=0}^{\infty}\biggl(\sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha\biggr),$$

and define

$$M(\rho) := \bigl\|F(\rho S_1, \ldots, \rho S_n)\bigr\| \quad \textit{for any } \rho \in (0, \gamma),$$

where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space. Then, for each $k = 0, 1, \ldots,$

$$\biggl\|\sum_{|\alpha|=k} A_\alpha^* A_\alpha\biggr\|^{1/2} \leqslant \frac{1}{\rho^k} M(\rho).$$

Proof. Let $\{Y_{(\alpha)}\}_{|\alpha|=k}$ be an arbitrary sequence of operators in $B(\mathcal{K})$. Using Theorem 1.5, we have

$$\begin{aligned}\biggl|\biggl\langle\biggl(\sum_{|\alpha|=k} Y_{(\alpha)}^* \otimes S_\alpha^*\biggr)F(\rho S_1, \ldots, \rho S_n)h \otimes 1, h \otimes 1\biggr\rangle\biggr| &\leqslant \biggl\|\sum_{|\alpha|=k} Y_{(\alpha)}^* \otimes S_\alpha^*\biggr\| M(\rho)\|h\|^2 \\ &= \biggl\|\sum_{|\alpha|=k} Y_{(\alpha)}^* Y_{(\alpha)}\biggr\|^{1/2} M(\rho)\|h\|^2\end{aligned}$$

for any $h \in \mathcal{K}$. On the other hand, since $S_1, \ldots, S_n$ are isometries with orthogonal ranges, we have

$$\begin{aligned}&\biggl|\biggl\langle\biggl(\sum_{|\alpha|=k} Y_{(\alpha)}^* \otimes S_\alpha^*\biggr)F(\rho S_1, \ldots, \rho S_n)h \otimes 1, h \otimes 1\biggr\rangle\biggr| \\ &\quad= \rho^k\biggl|\biggl\langle\biggl(\sum_{|\alpha|=k} Y_{(\alpha)}^* A_{(\alpha)} \otimes I\biggr)h \otimes 1, h \otimes 1\biggr\rangle\biggr| \\ &\quad= \rho^k\left|\left\langle\bigl[Y_{(\alpha)}^* \colon |\alpha| = k\bigr]\begin{bmatrix} A_{(\alpha)} \\ \vdots \\ |\alpha| = k\end{bmatrix}h, h\right\rangle\right|.\end{aligned}$$

Combining these relations and taking $Y_{(\alpha)} := A_{(\alpha)}$, $|\alpha| = k$, we deduce that

$$\rho^k \left\| \begin{bmatrix} A_{(\alpha)} \\ \vdots \\ |\alpha| = k \end{bmatrix} h \right\|^2 \leqslant \left\| \begin{bmatrix} A_{(\alpha)} \\ \vdots \\ |\alpha| = k \end{bmatrix} \right\| M(\rho) \|h\|^2$$

for any $h \in \mathcal{K}$. Therefore,

$$\left\| \sum_{|\alpha|=k} A_\alpha^* A_\alpha \right\|^{1/2} = \left\| \left[A_{(\alpha)}^* \colon |\alpha| = k \right] \right\| \leqslant \frac{1}{\rho^k} M(\rho),$$

which completes the proof. $\square$

A free holomorphic function with radius of convergence $R = \infty$ is called free entire function. We can prove now the following noncommutative multivariable generalization of Liouville's theorem.

Theorem 2.2. *Let F be an entire function and let*

$$F(X_1, \ldots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha$$

be its representation on an infinite-dimensional Hilbert space $\mathcal{H}$. Then F is a polynomial of degree $\leqslant m$, $m = 0, 1, \ldots,$ if and only if there are constants $M > 0$ and $C > 1$ such that

$$\left\| F(X_1, \ldots, X_n) \right\| \leqslant M \left\| [X_1, \ldots, X_n] \right\|^m$$

for any $[X_1, \ldots, X_n] \in B(\mathcal{H})^n$ such that $\|[X_1, \ldots, X_n]\| \geqslant C$.

Proof. If $F = \sum_{|\alpha| \leqslant m} A_{(\alpha)} \otimes X_\alpha$ is a polynomial, then

$$\|F\| \leqslant \sum_{k=0}^{m} \left\| \sum_{|\alpha|=m} A_{(\alpha)} \otimes X_\alpha \right\| \leqslant \sum_{k=0}^{m} \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{1/2} \left\| [X_1, \ldots, X_n] \right\|^k$$

if $\|[X_1, \ldots, X_n]\| \geqslant 1$. Therefore, there exists $M > 0$ and $R > 1$ such that

$$\left\| F(X_1, \ldots, X_n) \right\| \leqslant M \left\| [X_1, \ldots, X_n] \right\|^k \tag{2.1}$$

for any n-tuple of operators $[X_1, \ldots, X_n]$ with $\|[X_1, \ldots, X_n]\| \geqslant R$.

Conversely, if the inequality (2.1) holds, then

$$\left\| F(\rho S_1, \ldots, \rho S_n) \right\| \leqslant M \rho^m \quad \text{as } \rho \to \infty.$$

According to Theorem 2.1, we have

$$\Big\| \sum_{|\alpha|=k} A_\alpha^* A_\alpha \Big\|^{1/2} \leqslant \frac{1}{\rho^k} M(\rho),$$

where $M(\rho) := \|F(\rho S_1, \ldots, \rho S_n)\|$. Combining these inequalities, we deduce that

$$\Big\| \sum_{|\alpha|=k} A_\alpha^* A_\alpha \Big\|^{1/2} \leqslant M \frac{1}{\rho^{k-m}}.$$

Consequently, if $k > m$ and $\rho \to \infty$, we obtain $\sum_{|\alpha|=k} A_\alpha^* A_\alpha = 0$. This shows that $A_{(\alpha)} = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| > m$. $\square$

We say that a free holomorphic function F on the open operatorial n-ball of radius γ is bounded if

$$\|F\|_\infty := \sup \big\| F(X_1, \ldots, X_n) \big\| < \infty,$$

where the supremum is taken over all n-tuples of operators $[X_1, \ldots, X_n] \in (B(\mathcal{H})^n)_\gamma$ and any Hilbert space $\mathcal{H}$. In the particular case when $m = 0$, Theorem 2.2 implies the following free analogue of Liouville's theorem from complex analysis (see [11,43]).

Corollary 2.3. *If F is a bounded free entire function, then it is constant.*

We recall that the joint spectral radius of an n-tuple of operators $[T_1, \ldots, T_n] \in B(\mathcal{H})^n$,

$$r(T_1, \ldots, T_n) := \lim_{k\to\infty} \Big\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \Big\|^{1/2k},$$

is also equal to the spectral radius of the reconstruction operator $S_1 \otimes T_1^* + \cdots + S_n \otimes T_n^*$ (see [37]). Consequently, $r(T_1, \ldots, T_n) < 1$ if and only if

$$\sigma\big(S_1 \otimes T_1^* + \cdots + S_n \otimes T_n^*\big) \subset \mathbb{D}.$$

Moreover, the joint right spectrum $\sigma_r(T_1, \ldots, T_n)$ is included in the closed ball of $\mathbb{C}^n$ of radius equal to $r(T_1, \ldots, T_n)$. We recall that $\sigma_r(T_1, \ldots, T_n)$ is the set of all n-tuples $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that the right ideal of $B(\mathcal{H})$ generated by $\lambda_1 I - T_1, \ldots, \lambda_n I - T_n$ does not contain the identity.

Now, we prove an analogue of Schwartz lemma, in our multivariable operatorial setting.

Theorem 2.4. *Let $F(X_1, \ldots, X_n) = \sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)} \otimes X_\alpha$, $A_{(\alpha)} \in B(\mathcal{K})$, be a free holomorphic function on $[B(\mathcal{H})^n]_1$ with the properties*:

(i) $\|F\|_\infty \leqslant 1$ *and*
(ii) $A_{(\beta)} = 0$ *for any $\beta \in \mathbb{F}_n^+$ with $|\beta| \leqslant m - 1$, where $m = 1, 2, \ldots.$*

Then

$$\left\|F(X_1,\ldots,X_n)\right\| \leqslant \left\|[X_1,\ldots,X_n]\right\|^m \quad \textit{and} \quad r\big(F(X_1,\ldots,X_n)\big) \leqslant r(X_1,\ldots,X_n)^m$$

for any $[X_1,\ldots,X_n] \in [B(\mathcal{H})^n]_1$. *Moreover,*

$$\left\|\sum_{|\alpha|=k} A_\alpha A_\alpha^*\right\|^{1/2} \leqslant 1 \quad \textit{for any } k \geqslant m.$$

Proof. For each $\beta \in \mathbb{F}_n^+$ with $|\beta| \leqslant m$, define the formal power series

$$\Phi_{(\beta)}(Z_1,\ldots,Z_n) := \sum_{\alpha\in\mathbb{F}_n^+} A_{(\beta\alpha)} \otimes Z_\alpha.$$

Since

$$\left\|\sum_{|\alpha|=k} A_{(\beta\alpha)}^* A_{(\beta\alpha)}^*\right\| \leqslant \left\|\sum_{|\gamma|=m+k} A_{(\gamma)}^* A_{(\gamma)}\right\|,$$

we deduce that

$$\limsup_{k\to\infty}\left\|\sum_{|\alpha|=k} A_{(\beta\alpha)}^* A_{(\beta\alpha)}^*\right\|^{1/2k} \leqslant \limsup_{k\to\infty}\left\|\sum_{|\gamma|=m+k} A_{(\gamma)}^* A_{(\gamma)}\right\|^{\frac{1}{2(m+k)}}.$$

Consequently, due to Theorem 1.1, the radius of convergence of $\Phi_{(\beta)}$ is greater than the radius of convergence of F. Therefore, $\Phi_{(\beta)}$ represents a free holomorphic function on the open operatorial unit n-ball. Since $A_{(\beta)} = 0$ for any $\beta \in \mathbb{F}_n^+$ with $|\beta| \leqslant m-1$, and due to Theorem 1.4, we have the following Gleason type decomposition:

$$F(Z_1,\ldots,Z_n) = \sum_{|\beta|=m}\left[(I_{\mathcal{K}} \otimes Z_\beta)\sum_{\alpha\in\mathbb{F}_n^+} A_{(\beta\alpha)} \otimes Z_\alpha\right] = \sum_{|\beta|=m}(I_{\mathcal{K}} \otimes Z_\beta)\Phi_{(\beta)}(Z_1,\ldots,Z_n).$$

Therefore,

$$F(rS_1,\ldots,rS_n) = \sum_{|\beta|=m}\big(I_{\mathcal{K}} \otimes r^{|\beta|}S_\beta\big)\Phi_{(\beta)}(rS_1,\ldots,rS_n) \tag{2.2}$$

for any $r \in [0,1)$. Since $S_1,\ldots,S_n$ are isometries with orthogonal ranges, S_β, $|\beta| = m$, are also isometries with orthogonal ranges and we have

$$F(rS_1,\ldots,rS_n)^* F(rS_1,\ldots,rS_n) = r^{2m}\sum_{|\beta|=m}\Phi_{(\beta)}(rS_1,\ldots,rS_n)^*\Phi_{(\beta)}(rS_1,\ldots,rS_n).$$

Now, due to the noncommutative von Neumann inequality [29] and Theorem 1.5, we deduce that

$$\left\| \begin{bmatrix} \Phi_{(\beta)}(rX_1,\dots,rX_n) \\ \vdots \\ |\beta|=m \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} \Phi_{(\beta)}(rS_1,\dots,rS_n) \\ \vdots \\ |\beta|=m \end{bmatrix} \right\| \tag{2.3}$$

for any $[X_1,\dots,X_n]\in[B(\mathcal{H})^n]_1$. Consequently, using relations (2.2) and (2.3), we obtain

$$\begin{aligned}
\left\|F(rX_1,\dots,rX_n)\right\| &= \left\| \sum_{|\beta|=m} \bigl(I_{\mathcal{K}}\otimes r^{|\beta|}X_\beta\bigr)\Phi_{(\beta)}(rX_1,\dots,rX_n)\right\| \\
&\leqslant \left\|\bigl[r^mX_\beta\colon |\beta|=m\bigr]\right\| \left\| \begin{bmatrix} \Phi_{(\beta)}(rX_1,\dots,rX_n) \\ \vdots \\ |\beta|=m \end{bmatrix} \right\| \\
&\leqslant r^m \left\| \sum_{|\beta|=m} X_\beta X_\beta^* \right\|^{1/2} \left\| \begin{bmatrix} \Phi_{(\beta)}(rS_1,\dots,rS_n) \\ \vdots \\ |\beta|=m \end{bmatrix} \right\| \\
&= r^m \left\| \sum_{|\beta|=m} X_\beta X_\beta^* \right\|^{1/2} \left\| \sum_{|\beta|=m} \Phi_{(\beta)}(rS_1,\dots,rS_n)^*\Phi_{(\beta)}(rS_1,\dots,rS_n)\right\|^{1/2} \\
&= r^m \left\| \sum_{|\beta|=m} X_\beta X_\beta^* \right\|^{1/2} \left\|F(rS_1,\dots,rS_n)^*F(rS_1,\dots,rS_n)\right\| \\
&\leqslant r^m \left\| \sum_{|\beta|=m} X_\beta X_\beta^* \right\|^{1/2} \|F\|_\infty \\
&\leqslant r^m \left\| \sum_{|\beta|=m} X_\beta X_\beta^* \right\|^{1/2} \leqslant r^m \left\|[X_1,\dots,X_n]\right\|^m.
\end{aligned}$$

Taking $r\to 1$ and using the continuity of the free holomorphic function F on $[B(\mathcal{H})^n]_1$ (see Theorem 1.3), we infer that

$$\left\|F(X_1,\dots,X_n)\right\| \leqslant \left\| \sum_{|\beta|=m} X_\beta X_\beta^* \right\|^{1/2} \leqslant \left\|[X_1,\dots,X_n]\right\|^m$$

for any $[X_1,\dots,X_n]\in[B(\mathcal{H})^n]_1$.

Due to Theorem 1.4, the power series $F^k=\sum_{\alpha\in\mathbb{F}_n^+} B_{(\alpha)}\otimes Z_\alpha$ represents a free holomorphic function on the open operatorial unit n-ball, with $B_{(\alpha)}=0$ for any $\alpha\in\mathbb{F}_n^+$ with $|\alpha|\leqslant mk$. Applying the above inequality to F^k, we obtain

$$\left\|F(X_1,\dots,X_n)^k\right\| \leqslant \left\| \sum_{|\beta|=mk} X_\beta X_\beta^* \right\|^{1/2} \leqslant \left\| \sum_{|\beta|=k} X_\beta X_\beta^* \right\|^{m/2}.$$

Hence, and using the definition of the joint spectral radius, we deduce that $r(F(X_1,\dots,X_n)) \leqslant r(X_1,\dots,X_n)^m$.

To prove the last part of the theorem, notice that, according to Theorem 2.1, we have

$$\left\| \sum_{|\alpha|=k} A_{(\alpha)} A_{(\alpha)}^* \right\|^{1/2} \leqslant \frac{1}{\rho^k} M(\rho)$$

for any $\rho \in (0,1)$, where $M(\rho) = \|F(\rho S_1,\dots,\rho S_n)\|$. Since $M(\rho) \leqslant \|F\|_\infty \leqslant 1$, we take $\rho \to 1$ and deduce that $\|\sum_{|\alpha|=k} A_{(\alpha)} A_{(\alpha)}^*\|^{1/2} \leqslant 1$ for any $k \geqslant m$. The proof is complete. □

In the scalar case we get a little bit more.

Corollary 2.5. *Let $f(X_1,\dots,X_n) = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$, $a_\alpha \in \mathbb{C}$, be a free holomorphic function on $[B(\mathcal{H})^n]_1$ with scalar coefficients and the properties*:

(i) $\|f\|_\infty \leqslant 1$; *and*
(ii) $f(0) = 0$.

Then

(iii) $\|f(X_1,\dots,X_n)\| \leqslant \|[X_1,\dots,X_n]\|$ *and* $r(f(X_1,\dots,X_n)) \leqslant r(X_1,\dots,X_n)$ *for any n-tuple* $[X_1,\dots,X_n] \in [B(\mathcal{H})^n]_1$;
(iv) $\sum_{i=1}^n |\frac{\partial f}{\partial X_i}(0)|^2 \leqslant 1$.

Moreover, if $\sum_{i=1}^n |\frac{\partial f}{\partial X_i}(0)|^2 = 1$, *then* $\|f\|_\infty = 1$.

Proof. The first part of this corollary is a particular case of Theorem 2.4, when $m = 1$ and $\mathcal{K} = \mathbb{C}$. To prove the second part, assume that $\sum_{i=1}^n |\frac{\partial f}{\partial X_i}(0)|^2 = 1$. Consequently, we have $\sum_{i=1}^n |a_i|^2 = 1$. Hence, and due to Theorem 2.1, we have

$$1 \leqslant \sum_{i=1}^n |a_i|^2 \leqslant \frac{1}{\rho}\|f\|_\infty$$

for any $0 < \rho < 1$. Therefore, $\|f\|_\infty = 1$. This completes the proof. □

3. Algebras of free holomorphic functions

In this section, we introduce two Banach algebras of free holomorphic functions, $H^\infty(B(\mathcal{X})_1^n)$ and $A(B(\mathcal{X})_1^n)$, and prove that they are isometrically isomorphic to the noncommutative analytic Toeplitz algebra F_n^∞ and the noncommutative disc algebra $\mathcal{A}_n$, respectively. The results of this section are used to obtain a maximum principle for free holomorphic functions.

We denote by $Hol(B(\mathcal{X})^n_\gamma)$ the set of all free holomorphic functions with scalar coefficients on the open operatorial n-ball of radius γ. Due to Theorems 1.4 and 1.1, $Hol(B(\mathcal{X})^n_\gamma)$ is an algebra and an element $F = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ is in $Hol(B(\mathcal{X})^n_\gamma)$ if and only if

$$\limsup_{k \to \infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2 \right)^{1/2k} \leqslant 1.$$

Let $H^\infty(B(\mathcal{X})^n_1)$ denote the set of all elements F in $Hol(B(\mathcal{X})^n_1)$ such that

$$\|F\|_\infty := \sup \big\| F(X_1, \ldots, X_n) \big\| < \infty,$$

where the supremum is taken over all n-tuples $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$ and any Hilbert space $\mathcal{H}$. We denote by $A(B(\mathcal{X})^n_1)$ be the set of all elements F in $Hol(B(\mathcal{X})^n_1)$ such that, for any Hilbert space $\mathcal{H}$, the mapping

$$\big[B(\mathcal{H})^n\big]_1 \ni (X_1, \ldots, X_n) \mapsto F(X_1, \ldots, X_n) \in B(\mathcal{H})$$

has a continuous extension to the closed unit ball $[B(\mathcal{H})^n]_1^-$. In this section, we will show that $H^\infty(B(\mathcal{X})^n_1)$ and $A(B(\mathcal{X})^n_1)$ are Banach algebras under pointwise multiplication and the norm $\|\cdot\|_\infty$, which can be identified with the noncommutative analytic Toeplitz algebra F_n^∞ and the noncommutative disc algebra $\mathcal{A}_n$, respectively.

Let us recall (see [29,30,32,34]) a few facts about the Banach algebras $\mathcal{A}_n$ and F_n^∞. Any element f in the full Fock space $F^2(H_n)$ has the form:

$$f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha, \quad \text{with } a_\alpha \in \mathbb{C}, \text{ such that } \|f\|_2 := \left(\sum_{\alpha \in \mathbb{F}_n^+} |a_\alpha|^2 \right)^{1/2} < \infty.$$

If $g = \sum_{\alpha \in \mathbb{F}_n^+} b_\alpha e_\alpha \in F^2(H_n)$, we define the product $f \otimes g$ to be the formal power series

$$f \otimes g := \sum_{\gamma \in \mathbb{F}_n^+} c_\gamma e_\gamma, \quad \text{where } c_\gamma := \sum_{\substack{\alpha, \beta \in \mathbb{F}_n^+ \\ \alpha\beta = \gamma}} a_\alpha b_\beta, \ \gamma \in \mathbb{F}_n^+.$$

We also make the natural identification of $e_\alpha \otimes 1$ and $1 \otimes e_\alpha$ with e_α. Let $\mathcal{P}$ denote the set of all polynomials $p \in F^2(H_n)$, i.e., elements of the form $p = \sum_{|\alpha| \leqslant m} a_\alpha e_\alpha$, where $m = 0, 1, \ldots.$

In [29], we introduced the noncommutative Hardy algebra F_n^∞ as the set of all $f \in F^2(H_n)$ such that

$$\|f\|_\infty := \sup \big\{ \|f \otimes p\|_2 \colon\ p \in \mathcal{P},\ \|p\|_2 \leqslant 1 \big\} < \infty. \tag{3.1}$$

If $f \in F^2(H_n)$, then $f \in F_n^\infty$ if and only if $f \otimes g \in F^2(H_n)$ for any $g \in F^2(H_n)$. Moreover, if $f \in F_n^\infty$, then the left multiplication mapping $L_f : F^2(H_n) \to F^2(H_n)$ defined by

$$L_f g := f \otimes g, \quad g \in F^2(H_n),$$

is a bounded linear operator with $\|L_f\| = \|f\|_\infty$. The noncommutative Hardy algebra F_n^∞ is isometrically isomorphic to the left multiplier algebra of the full Fock space $F^2(H_n)$, which is also called the noncommutative Toeplitz algebra. Under this identification, F_n^∞ is the weakly closed algebra generated by the left creation operators $S_1, \ldots, S_n$ and the identity. The noncommutative disc algebra $\mathcal{A}_n$ was introduced in [29] as is the norm closed algebra generated by the left creation operators and the identity.

Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ be an element in $F^2(H_n)$ and define

$$f_r := \sum_{\alpha \in \mathbb{F}_n^+} r^{|\alpha|} a_\alpha e_\alpha \quad \text{for } 0 < r < 1.$$

In [30,34], we proved that if $f \in F_n^\infty$ then $\|f_r\|_\infty \leqslant \|f\|_\infty$ for $0 \leqslant r < 1$, and

$$L_f = \text{SOT-}\lim_{r \to 1} f_r(S_1, \ldots, S_n), \tag{3.2}$$

where $f_r(S_1, \ldots, S_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha S_\alpha$. Moreover, if $f \in \mathcal{A}_n$ then the above limit exists in the operator norm topology.

We identify $M_m(B(\mathcal{H}))$, the set of $m \times m$ matrices with entries from $B(\mathcal{H})$, with $B(\mathcal{H}^{(m)})$, where $\mathcal{H}^{(m)}$ is the direct sum of m copies of $\mathcal{H}$. Thus we have a natural C^*-norm on $M_m(B(\mathcal{H}))$. If X is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_m(X)$ as a subspace of $M_m(B(\mathcal{H}))$ with the induced norm. Let X, Y be operator spaces and $u : X \to Y$ be a linear map. Define the map $u_m : M_m(X) \to M_m(Y)$ by

$$u_m([x_{ij}]) := [u(x_{ij})].$$

We say that u is completely bounded (*cb* in short) if

$$\|u\|_{cb} := \sup_{m \geqslant 1} \|u_m\| < \infty.$$

If $\|u\|_{cb} \leqslant 1$ (respectively u_m is an isometry for any $m \geqslant 1$) then u is completely contractive (respectively isometric), and if u_m is positive for all m, then u is called completely positive.

For each $m = 1, 2, \ldots,$ we define the norms $\|\cdot\|_m : M_m(H^\infty(B(\mathcal{X})_1^n)) \to [0, \infty)$ by setting

$$\big\|[F_{ij}]_m\big\|_m := \sup \big\|\big[F_{ij}(X_1, \ldots, X_n)\big]_m\big\|,$$

where the supremum is taken over all n-tuples $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_m$, $m = 1, 2, \ldots,$ determine an operator space structure on $H^\infty(B(\mathcal{X})_1^n)$, in the sense of Ruan (see [18]).

Theorem 3.1. *Let $F := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ be a free holomorphic function on the open operatorial unit n-ball. Then the following statements are equivalent*:

(i) *F is in $H^\infty(B(\mathcal{X})_1^n)$*;
(ii) *$f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ is in F_n^∞*;
(iii) $\sup_{0 \leqslant r < 1} \|F(rS_1, \ldots, rS_n)\| < \infty$;

(iv) *The map* $\varphi : [0,1) \to B(F^2(H_n))$ *defined by*

$$\varphi(r) := F(rS_1, \dots, rS_n) \quad \textit{for any } r \in [0,1)$$

has a continuous extension to $[0,1]$ *with respect to the strong operator topology of* $B(F^2(H_n))$.

In this case, we have

$$\|L_f\| = \|f\|_\infty = \sup_{0 \leqslant r < 1} \big\| F(rS_1, \dots, rS_n) \big\| = \lim_{r \to 1} \big\| F(rS_1, \dots, rS_n) \big\| = \|F\|_\infty. \tag{3.3}$$

Moreover, the map

$$\Phi : H^\infty\big(B(\mathcal{X})_1^n\big) \to F_n^\infty \quad \textit{defined by} \quad \Phi(F) := f$$

is a completely isometric isomorphism of operator algebras.

Proof. Assume (ii) holds. Since $f \in F_n^\infty$, we have

$$\big\| F(rS_1, \dots, rS_n) \big\| = \big\| f(rS_1, \dots, rS_n) \big\| = \|L_{f_r}\| = \|f_r\| \leqslant \|f\|_\infty \tag{3.4}$$

for any $r \in [0,1)$. Therefore, (ii) $\Rightarrow$ (iii). To prove that (iii) $\Rightarrow$ (ii), assume that (iii) holds. Consequently, we have

$$\sum_{\alpha \in \mathbb{F}_n^+} r^{2|\alpha|} |a_\alpha|^2 = \bigg\| \sum_{\alpha \in \mathbb{F}_n^+} r^{|\alpha|} a_\alpha S_\alpha(1) \bigg\| \leqslant \sup_{0 \leqslant r < 1} \big\| F(rS_1, \dots, rS_n) \big\| < \infty$$

for any $0 \leqslant r < 1$. Hence, $\sum_{\alpha \in \mathbb{F}_n^+} |a_\alpha|^2 < \infty$, which shows that $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ is in $F^2(H_n)$. Now assume that $f \notin F_n^\infty$. Due to the definition of F_n^∞, given an arbitrary positive number M, there exists a polynomial $q \in \mathcal{P}$ with $\|q\|_2 = 1$ such that

$$\|f \otimes q\|_2 > M.$$

Since $\|f_r - f\|_2 \to 0$ as $r \to 1$, we have

$$\|f \otimes q - f_r \otimes q\|_2 = \big\| (f - f_r) \otimes q \big\|_2 \to 0 \quad \text{as } r \to 1.$$

Therefore, there is $r_0 \in (0,1)$ such that $\|f_{r_0} \otimes q\|_2 > M$. Hence,

$$\big\| f_{r_0}(S_1, \dots, S_n) \big\| = \|L_{f_{r_0}}\| = \|f_{r_0}\|_\infty > M.$$

Since $M > 0$ is arbitrary, we deduce that

$$\sup_{0 \leqslant r < 1} \big\| f(rS_1, \dots, rS_n) \big\| = \infty,$$

which is a contradiction. Consequently, (ii) $\Leftrightarrow$ (iii). Now, let us prove that (ii) $\Rightarrow$ (iv). Assume (ii) and define the map $\tilde{\varphi}: [0,1] \to B(F^2(H_n))$ by setting

$$\tilde{\varphi}(r) := \begin{cases} F(rS_1, \ldots, rS_n) & \text{if } 0 \leqslant r < 1, \\ L_f & \text{if } r = 1. \end{cases}$$

Since $f(rS_1, \ldots, rS_n) = F(rS_1, \ldots, rS_n)$, $0 \leqslant r < 1$, the SOT-continuity of $\tilde{\varphi}$ at $r = 1$ is due to relation (3.2), while the continuity of $\tilde{\varphi}$ on $[0, 1)$ is a consequence of Theorem 1.3. Therefore, item (iv) holds.

Assume now that (iv) holds. For each $x \in F^2(H_n)$, the map $[0, 1) \ni r \mapsto \|\varphi(r)x\| \in \mathbb{R}^+$ is bounded, i.e., $\sup_{0 \leqslant r<1} \|\varphi(r)x\| < \infty$. Due to the principle of uniform boundedness, we deduce condition (iii).

The implication (i) $\Rightarrow$ (iii) is obvious, and the implication (iii) $\Rightarrow$ (i) is due to Theorem 1.1 and the noncommutative von Neumann inequality. Indeed, if $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$, $\mathcal{H}$ is an arbitrary Hilbert space, and $\|[X_1, \ldots, X_n]\| = r < 1$, then

$$\left\| \sum_{k=0}^{m} \sum_{|\alpha|=k} a_\alpha X_\alpha \right\| \leqslant \left\| \sum_{k=0}^{m} \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha S_\alpha \right\|, \quad m = 1, 2, \ldots.$$

Hence, and taking into account Theorem 1.1, we deduce that

$$\left\| F(X_1, \ldots, X_n) \right\| \leqslant \left\| F(rS_1, \ldots, rS_n) \right\| \quad \text{for any } r \in [0, 1).$$

Consequently,

$$\sup_{[X_1,\ldots,X_n] \in [B(\mathcal{H})^n]_1} \left\| F(X_1, \ldots, X_n) \right\| \leqslant \sup_{0 \leqslant r<1} \left\| F(rS_1, \ldots, rS_n) \right\| < \infty, \tag{3.5}$$

whence (i) holds.

We prove now the last part of the theorem. If $f \in F_n^\infty$ and $\epsilon > 0$, then there exists a polynomial $q \in \mathcal{P}$ with $\|q\|_2 = 1$ such that

$$\|f \otimes q\|_2 > \|f\|_\infty - \epsilon.$$

Due to relation (3.2), there exists $r_0 \in (0, 1)$ such that $\|f_{r_0}(S_1, \ldots, S_n)q\| > \|f\|_\infty - \epsilon$. Using now relation (3.4), we deduce that

$$\sup_{0 \leqslant r<1} \left\| f(rS_1, \ldots, rS_n) \right\| = \|f\|_\infty.$$

Now, let $r_1, r_2 \in [0, 1)$ with $r_1 < r_2$ and let $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$. Since $g := \sum_{\alpha \in \mathbb{F}_n^+} r_2^{|\alpha|} a_\alpha e_\alpha$ is in the noncommutative disc algebra $\mathcal{A}_n$, we have $\|g_r\|_\infty \leqslant \|g\|_\infty$ for any $0 \leqslant r < 1$. In particular, when $r := r_1/r_2$, we deduce that

$$\left\| f_{r_1}(S_1, \ldots, S_n) \right\| \leqslant \left\| f_{r_2}(S_1, \ldots, S_n) \right\|.$$

Consequently, the function $[0, 1] \ni r \to \|f(rS_1, \ldots, rS_n)\| \in \mathbb{R}^+$ is increasing. Hence, and using relation (3.5), we deduce (3.3). Using the same techniques, one can prove a matrix form of relation (3.3). In particular, we have $\|[F_{ij}]_m\|_m = \|[L_{f_{ij}}]_m\|$ for any $[F_{ij}]_m \in M_m(H^\infty(B(\mathcal{X})_1^n))$ and $m = 1, 2, \ldots$. Hence, we deduce that Φ is a complete isometry of $H^\infty(B(\mathcal{X})_1^n)$ onto F_n^∞. The proof is complete. □

Theorem 3.2. *Let $F := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ be a free holomorphic function on the open operatorial unit n-ball. Then the following statements are equivalent*:

(i) *F is in $A(B(\mathcal{X})_1^n)$*;
(ii) *$f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ is in $\mathcal{A}_n$*;
(iii) *the map $\varphi : [0, 1) \to B(F^2(H_n))$ defined by*

$$\varphi(r) := F(rS_1, \ldots, rS_n)$$

has a continuous extension to $[0, 1]$, *with respect to the operator norm topology of* $B(F^2(H_n))$.

Moreover, the map

$$\Psi : A\big(B(\mathcal{X})_1^n\big) \to \mathcal{A}_n \quad \textit{defined by} \quad \Psi(F) := f$$

is a completely isometric isomorphism of operator algebras.

Proof. The implication (i) ⇒ (iii) is due to the definition of $A(B(\mathcal{X})_1^n)$. Assume that item (ii) holds, i.e., $f \in \mathcal{A}_n$. The norm continuity of φ on [0,1) is due to Theorem 1.3, while the continuity of φ at $r = 1$ is due to the fact that $\lim_{r\to 1} f_r(S_1, \ldots, S_n) = L_f$ in the operator norm for any $f \in \mathcal{A}_n$ (see the remarks preceding this theorem). Therefore, the implication (ii) ⇒ (iii) is true. Conversely, assume item (iii) holds. Then $\lim_{r\to\infty} F(rS_1, \ldots, rS_n)$ exists in the operator norm. Since $F(rS_1, \ldots, rS_n) \in \mathcal{A}_n$ and $\mathcal{A}_n$ is a Banach algebra, there exists $g \in \mathcal{A}_n$ such that $L_g = \lim_{r\to\infty} F(rS_1, \ldots, rS_n)$ in the operator norm. On the other hand, due to Theorem 3.1, we deduce that $f := \sum_{\alpha \in \mathbb{F}_n} a_\alpha e_\alpha \in F_n^\infty$. Since $f(rS_1, \ldots, rS_n) = F(rS_1, \ldots, rS_n)$, $0 \leqslant r < 1$, and $L_f = \text{SOT-}\lim_{r\to\infty} f(rS_1, \ldots, rS_n)$, we conclude that $L_f = L_g$, i.e., $f = g$. Therefore, condition (ii) holds.

It remains to prove that (ii) ⇒ (i). According to [30] (see also [34]), if $f \in \mathcal{A}_n$ then, for any n-tuple $[Y_1, \ldots, Y_n] \in [B(\mathcal{H})^n]_1^-$,

$$\tilde{F}(Y_1, \ldots, Y_n) := \lim_{r\to 1} f(rY_1, \ldots, rY_n),$$

exists in the operator norm, and

$$\big\|\tilde{F}(Y_1, \ldots, Y_n)\big\| \leqslant \|f\|_\infty \quad \text{for any } [Y_1, \ldots, Y_n] \in \big[B(\mathcal{H})^n\big]_1^-.$$

Notice also that $\tilde{F}$ is an extension of the free holomorphic function F on $[B(\mathcal{H})^n]_1$. Indeed, if $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$, then

$$\tilde{F}(X_1, \ldots, X_n) = \lim_{r\to 1} f(rX_1, \ldots, rX_n) = \lim_{r\to 1} F(rX_1, \ldots, rX_n) = F(X_1, \ldots, X_n).$$

The last equality is due to Theorem 1.3.

Let us prove that $\tilde{F}:[B(\mathcal{H})^n]_1^- \to B(\mathcal{H})$ is continuous. Since $f \in \mathcal{A}_n$, for any $\epsilon > 0$ there exists $r_0 \in [0,1)$ such that $\|L_f - f(r_0S_1,\dots,r_0S_n)\| < \epsilon$. Applying the above mentioned result from [34] to $f - f_{r_0} \in \mathcal{A}_n$, we deduce that

$$\bigl\|\tilde{F}(T_1,\dots,T_n) - f_{r_0}(T_1,\dots,T_n)\bigr\| \leqslant \|L_f - L_{f_{r_0}}\| < \frac{\epsilon}{3} \tag{3.6}$$

for any $[T_1,\dots,T_n] \in [B(\mathcal{H})^n]_1^-$. Due to Theorem 1.3, F is a continuous function on $[B(\mathcal{H})^n]_1$. Therefore, there exists $\delta > 0$ such that

$$\bigl\|F_{r_0}(T_1,\dots,T_n) - F_{r_0}(Y_1,\dots,Y_n)\bigr\| < \frac{\epsilon}{3}$$

for any n-tuples $[T_1,\dots,T_n]$ and $[Y_1,\dots,Y_n]$ in $[B(\mathcal{H})^n]_1^-$ such that $\|[T_1 - Y_1,\dots, T_n - Y_n]\| < \delta$. Hence, and using (3.6), we have

$$\begin{aligned}\bigl\|\tilde{F}(T_1,\dots,T_n) - \tilde{F}(Y_1,\dots,Y_n)\bigr\| \leqslant{}& \bigl\|\tilde{F}(T_1,\dots,T_n) - f_{r_0}(T_1,\dots,T_n)\bigr\| \\ &+ \bigl\|f_{r_0}(T_1,\dots,T_n) - f_{r_0}(Y_1,\dots,Y_n)\bigr\| \\ &+ \bigl\|f_{r_0}(Y_1,\dots,Y_n) - \tilde{F}(Y_1,\dots,Y_n)\bigr\| < \epsilon,\end{aligned}$$

whenever $\|[T_1 - Y_1,\dots,T_n - Y_n]\| < \delta$. This proves the continuity of $\tilde{F}$ on $[B(\mathcal{H})^n]_1^-$. Therefore, $F \in A(B(\mathcal{X})_1^n)$.

To prove the last part of the theorem, notice that if $f_{ij} \in \mathcal{A}_n \subset F_n^\infty$, then by Theorem 3.1 (see relation (3.3) and its matrix form), we have $\|[L_{f_{ij}}]_m\| = \|[F_{ij}]_m\|_m$. Since $\mathcal{A}_n \subset B(F^2(H_n))$ is an operator algebra, we deduce that Ψ is a completely isometric isomorphism of operator algebras. This completes the proof. □

Here is our version of the maximum principle for free holomorphic functions.

Theorem 3.3. *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Assume that $f:[B(\mathcal{H})^n]_1^- \to B(\mathcal{H})$ is a continuous function in the operator norm, and it is free holomorphic on $[B(\mathcal{H})^n]_1$. Then*

$$\begin{aligned}&\max\bigl\{\bigl\|f(X_1,\dots,X_n)\bigr\|:\ \bigl\|[X_1,\dots,X_n]\bigr\| \leqslant 1\bigr\} \\ &\quad= \max\bigl\{\bigl\|f(X_1,\dots,X_n)\bigr\|:\ \bigl\|[X_1,\dots,X_n]\bigr\| = 1\bigr\}.\end{aligned}$$

Proof. Due to the continuity of f, for any $[X_1,\dots,X_n] \in [B(\mathcal{H})^n]_1^-$,

$$\bigl\|f(X_1,\dots,X_n)\bigr\| = \lim_{r\to 1}\bigl\|f(rX_1,\dots,rX_n)\bigr\|.$$

On the other hand, the noncommutative von Neumann inequality implies

$$\bigl\|f(rX_1,\dots,rX_n)\bigr\| \leqslant \bigl\|f(rS_1,\dots,rS_n)\bigr\| \quad \text{for } 0 \leqslant r < 1.$$

By Theorem 3.2, $f \in \mathcal{A}_n$ and, consequently,

$$\lim_{r\to 1} \left\| f(rS_1, \dots, rS_n) \right\| = \|L_f\| = \|f\|_\infty.$$

Combining these relations, we deduce that

$$\left\| f(X_1, \dots, X_n) \right\| \leqslant \|f\|_\infty \quad \text{for any } [X_1, \dots, X_n] \in \left[B(\mathcal{H})^n\right]_1. \tag{3.7}$$

Since $\mathcal{H}$ is infinite-dimensional, there exists a subspace $\mathcal{K} \subset \mathcal{H}$ and a unitary operator $U : F^2(H_n) \to \mathcal{K}$. Define the operators

$$V_i := \begin{pmatrix} US_iU^* & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, \dots, n,$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, where $S_1, \dots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Notice that $\|[V_1, \dots, V_n]\| = 1$ and

$$f(V_1, \dots, V_n) = \lim_{r\to 1} \begin{pmatrix} Uf_r(S_1, \dots, S_n)U^* & 0 \\ 0 & 0 \end{pmatrix}$$

in the operator norm. Consequently,

$$\left\| f(V_1, \dots, V_n) \right\| = \lim_{r\to 1} \left\| f_r(S_1, \dots, S_n) \right\| = \|f\|_\infty.$$

Hence, and using inequality (3.7), we deduce that

$$\begin{aligned} &\max\left\{ \left\| f(X_1, \dots, X_n) \right\| : \left\| [X_1, \dots, X_n] \right\| \leqslant 1 \right\} \\ &\quad = \max\left\{ \left\| f(X_1, \dots, X_n) \right\| : \left\| [X_1, \dots, X_n] \right\| = 1 \right\} = \|f\|_\infty. \end{aligned}$$

This completes the proof. □

Corollary 3.4. *Let f be a free holomorphic function on $[B(\mathcal{H})^n]_1$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space, and let $r \in [0, 1)$. Then*

$$\begin{aligned} \max\left\{ \left\| f(X_1, \dots, X_n) \right\| : \left\| [X_1, \dots, X_n] \right\| \leqslant r \right\} &= \max\left\{ \left\| f(X_1, \dots, X_n) \right\| : \left\| [X_1, \dots, X_n] \right\| = r \right\} \\ &= \left\| f(rS_1, \dots, rS_n) \right\|. \end{aligned}$$

In a forthcoming paper [40], we obtain operator-valued multivariable Bohr type inequalities for free holomorphic functions on the open operatorial unit n-ball. As consequences, we obtain operator-valued Bohr inequalities for the noncommutative disc algebra $\mathcal{A}_n$ and the noncommutative analytic Toeplitz algebra F_n^∞.

4. Free analytic functional calculus and noncommutative Cauchy transforms

In this section, we introduce a free analytic functional calculus for n-tuples $T := [T_1, \dots, T_n] \in B(\mathcal{H})^n$ of operators with joint spectral radius $r(T_1, \dots, T_n) < 1$. We introduce a noncommutative Cauchy transform $\mathcal{C}_T : B(F^2(H_n)) \to B(\mathcal{H})$ associated with any such n-tuple of operators and prove that

$$f(T_1, \dots, T_n) = \mathcal{C}_T\big(f(S_1, \dots, S_n)\big), \quad f \in H^\infty\big(B(\mathcal{X})_1^n\big),$$

where $f(S_1, \dots, S_n)$ is the boundary function of f. Similar Cauchy representations are obtained for the k-order Hausdorff derivations of f. Finally, we show that the noncommutative Cauchy transform commutes with the action of the unitary group $\mathcal{U}(\mathbb{C}^n)$.

Theorem 4.1. *Let $F := \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ be a free holomorphic function on the open operatorial n-ball of radius γ. Then, for any Hilbert space $\mathcal{H}$ and any n-tuple of operators $[X_1, \dots, X_n] \in B(\mathcal{H})^n$ with $r(X_1, \dots, X_n) < \gamma$, the series*

$$F(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha$$

is convergent in the operator norm of $B(\mathcal{K} \otimes \mathcal{H})$. Moreover, if $0 < r < 1$, then

$$\lim_{r \to 1} F_r(X_1, \dots, X_n) = F(X_1, \dots, X_n) \tag{4.1}$$

and

$$\lim_{r \to 1} \left(\frac{\partial^k F_r}{\partial Z_{i_1} \cdots Z_{i_k}} \right)(X_1, \dots, X_n) = \left(\frac{\partial^k F}{\partial Z_{i_1} \cdots \partial Z_{i_k}} \right)(X_1, \dots, X_n) \tag{4.2}$$

for $i_1, \dots, i_k \in \{1, \dots, n\}$, where the limits are in the operator norm.

Proof. Assume that $[X_1, \dots, X_n]$ is an n-tuple of operators on $\mathcal{H}$ such that $r(X_1, \dots, X_n) < R$, where R is the radius of convergence of F. Let $\rho', \rho > 0$ be such that $r(X_1, \dots, X_n) < \rho' < \rho < R$. Due to the definition of $r(X_1, \dots, X_n)$, there exists $k_0 \in \mathbb{N}$ such that

$$\left\| \sum_{|\alpha|=k} X_\alpha X_\alpha^* \right\|^{1/2k} < \rho' \quad \text{for any } k \geqslant k_0. \tag{4.3}$$

Since $1/\rho > 1/R$, we can find m_0 such that

$$\left\| \sum_{|\alpha|=k} A_{(\alpha)} A_{(\alpha)}^* \right\|^{1/2k} < \frac{1}{\rho} \quad \text{for any } k \geqslant m_0. \tag{4.4}$$

If $k \geqslant \max\{k_0, m_0\}$, then relations (4.3) and (4.4) imply

$$\left\| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \right\| = \left\| \left[I \otimes X_\alpha \colon |\alpha| = k \right] \begin{bmatrix} A_{(\alpha)} \otimes I \\ \vdots \\ |\alpha| = k \end{bmatrix} \right\|$$
$$= \left\| \sum_{|\alpha|=k} X_\alpha X_\alpha^* \right\|^{1/2} \left\| \sum_{|\alpha|=k} A_{(\alpha)} A_{(\alpha)}^* \right\|^{1/2}$$
$$\leqslant \left(\frac{\rho'}{\rho} \right)^k.$$

This proves the convergence of the series $\sum_{k=0}^\infty (\sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha)$ in the operator norm. Now, using the above inequalities, we obtain

$$\left\| \sum_{k=0}^\infty \sum_{|\alpha|=k} \left(r^{|\alpha|} - 1 \right) A_\alpha \otimes X_\alpha \right\| \leqslant \sum_{k=1}^\infty \left(r^k - 1 \right) \left\| \sum_{|\alpha|=k} A_\alpha \otimes X_\alpha \right\| \leqslant \sum_{k=1}^\infty \left(r^k - 1 \right) \left(\frac{\rho'}{\rho} \right)^k$$
$$\leqslant (r-1) \sum_{k=1}^\infty k \left(\frac{\rho'}{\rho} \right)^k.$$

Since $\rho' < \rho$, the latter series is convergent and therefore relation (4.1) holds. Due to Theorem 1.9, $\frac{\partial F}{\partial Z_i}$ is a free holomorphic function on the open operatorial n-ball of radius γ, and

$$\frac{\partial^k F_r}{\partial Z_{i_1} \cdots \partial Z_{i_k}} (X_1, \ldots, X_n) = r^k \frac{\partial^k F}{\partial Z_{i_1} \cdots \partial Z_{i_k}} (r X_1, \ldots, r X_n), \quad 0 < r < 1.$$

Applying relation (4.1) to $\frac{\partial^k F}{\partial Z_{i_1} \cdots \partial Z_{i_k}}$, we deduce (4.2). The proof is complete. $\square$

Let $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ be an n-tuple of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$. We introduce the *Cauchy kernel* associated with T to be the operator $C_T(S_1, \ldots, S_n) \in B(F^2(H_n) \otimes \mathcal{H})$ defined by

$$C_T(S_1, \ldots, S_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} S_\alpha \otimes T_{\tilde{\alpha}}^*, \tag{4.5}$$

where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$, and $\tilde{\alpha}$ is the reverse of α, i.e., $\tilde{\alpha} = g_{i_k} \cdots g_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k}$. Applying Theorem 1.1, when $A_{(\alpha)} := T_\alpha^*$, $\alpha \in \mathbb{F}_n^+$ and $X_i := S_i$, $i = 1, \ldots, n$, we deduce that

$$\frac{1}{R} = \lim_{k \to \infty} \left\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right\|^{1/2k} = r(T_1, \ldots, T_n) < 1$$

and $\|[S_1, \ldots, S_n]\| = 1 < R$. Consequently, the series in (4.5) is convergent in the operator norm and $C_T(S_1, \ldots, S_n) \in \mathcal{A}_n \bar{\otimes} B(\mathcal{H}) \subset B(F^2(H_n) \otimes \mathcal{H})$. Now, one can easily see that

$$C_T(S_1, \ldots, S_n) = \left(I - S_1 \otimes T_1^* - \cdots - S_n \otimes T_n^* \right)^{-1}. \tag{4.6}$$

We call the operator

$$S_1 \otimes T_1^* + \cdots + S_n \otimes T_n^*$$

the *reconstruction operator* associated with the n-tuple $[T_1, \ldots, T_n]$. We should mention that this operator plays an important role in noncommutative multivariable operator theory (see [37, 39]). We remark that if 1 is not in the spectrum of the reconstruction operator, then the Cauchy kernel defined by (4.6) makes sense. In this case, $C_T(S_1, \ldots, S_n)$ is in $F_n^\infty \bar{\otimes} B(\mathcal{H})$, the WOT-closed operator algebra generated by the spatial tensor product, and not necessarily in $\mathcal{A}_n \bar{\otimes} B(\mathcal{H})$. Moreover, we can think of the series $\sum_{k=0}^\infty \sum_{|\alpha|=k} S_\alpha \otimes T_{\tilde{\alpha}}^*$ as the Fourier representation of the Cauchy kernel.

In what follows we also use the notation $C_T := C_T(S_1, \ldots, S_n)$.

Proposition 4.2. *Let $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ be an n-tuple of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$. Then*:

(i) $\|C_T\| \leqslant \sum_{k=0}^\infty \|\sum_{|\alpha|=k} T_\alpha T_\alpha^*\|^{1/2}$. *In particular, if $T := [T_1, \ldots, T_n] \in [B(\mathcal{H})^n]_1$, then* $\|C_T\| \leqslant \frac{1}{1-\|T\|}$.

(ii) $C_T - C_X = C_T[\sum_{i=1}^n S_i \otimes (T_i^* - X_i^*)]C_X$ *and*

$$\|C_T - C_X\| \leqslant \|C_T\| \|C_X\| \big\|[T_1 - X_1, \ldots, T_n - X_n]\big\|$$

for any n-tuple $X := [X_1, \ldots, X_n] \in B(\mathcal{H})^n$ with joint spectral radius $r(X_1, \ldots, X_n) < 1$.

Proof. Since $S_1, \ldots, S_n$ are isometries with orthogonal ranges, we have

$$\|C_T\| \leqslant \sum_{k=0}^\infty \left\| \sum_{|\alpha|=k} S_\alpha \otimes T_{\tilde{\alpha}}^* \right\| = \sum_{k=0}^\infty \left\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right\|^{1/2}.$$

If $\|[T_1, \ldots, T_n]\| < 1$, then

$$\sum_{k=0}^\infty \left\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right\|^{1/2} \leqslant \sum_{k=0}^\infty \left\| \sum_{i=1}^n T_i T_i^* \right\|^{k/2} = \frac{1}{1 - \|T\|}.$$

To prove (ii), notice that

$$\begin{aligned} &C_T - C_X \\ &= \left(I - \sum_{i=1}^n S_i \otimes T_i^*\right)^{-1} \left[I - \sum_{i=1}^n S_i \otimes X_i^* - \left(I - \sum_{i=1}^n S_i \otimes T_i^*\right)\right] \left(I - \sum_{i=1}^n S_i \otimes X_i^*\right)^{-1} \\ &= C_T \left[\sum_{i=1}^n S_i \otimes \left(T_i^* - X_i^*\right)\right] C_X, \end{aligned}$$

and

$$\begin{aligned}\|C_T - C_X\| &\leqslant \|C_T\|\|C_X\| \left\| \sum_{i=1}^n S_i \otimes \left(T_i^* - X_i^*\right) \right\| \\ &= \|C_T\|\|C_X\| \left\| \sum_{i=1}^n (T_i - X_i)(T_i - X_i)^* \right\|^{1/2},\end{aligned}$$

which completes the proof. □

The *Cauchy transform* at $T := [T_1, \dots, T_n] \in B(\mathcal{H})^n$ is the mapping

$$\mathcal{C}_T : B\big(F^2(H_n)\big) \to B(\mathcal{H})$$

defined by

$$\big\langle \mathcal{C}_T(A)x, y\big\rangle := \big\langle (A \otimes I_{\mathcal{H}})(1 \otimes x), C_T(R_1, \dots, R_n)(1 \otimes y)\big\rangle$$

for any $x, y \in \mathcal{H}$, where $R_1, \dots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$. The operator $\mathcal{C}_T(A)$ is called the Cauchy transform of A at T. Given $A \in B(F^2(H_n))$, the Cauchy transform generates a function (the Cauchy transform of A)

$$\mathcal{C}[A] : \big[B(\mathcal{H})^n\big]_1 \to B(\mathcal{H})$$

by setting

$$\mathcal{C}[A](X_1, \dots, X_n) := \mathcal{C}_X(A) \quad \text{for any } X := [X_1, \dots, X_n] \in \big[B(\mathcal{H})^n\big]_1.$$

Indeed, it is enough to see that $r(X_1, \dots, X_n) \leqslant \|[X_1, \dots, X_n]\| < 1$, and therefore $\mathcal{C}_X(A)$ is well defined. This gives rise to an important question: when is $\mathcal{C}[A]$ a free holomorphic function on $[B(\mathcal{H})^n]_1$.

Due to Theorem 4.1, if $f = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ is a free holomorphic function on the open operatorial unit n-ball and $[T_1, \dots, T_n] \in B(\mathcal{H})^n$ is any n-tuple of operators with $r(T_1, \dots, T_n) < 1$ then, we can define a bounded linear operator

$$f(T_1, \dots, T_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha T_\alpha,$$

where the series converges in norm. This provides the *free analytic functional calculus*.

If $F = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ is in the Hardy algebra $H^\infty(B(\mathcal{X})_1^n)$, we denote by $F(S_1, \dots, S_n)$ the boundary function of F, i.e., $F(S_1, \dots, S_n) := L_f \in B(F^2(H_n))$, where $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$.

Theorem 4.3. *Let $T := [T_1, \dots, T_n] \in B(\mathcal{H})^n$ be an n-tuple of operators with joint spectral radius $r(T_1, \dots, T_n) < 1$. Then, for any $f \in H^\infty(B(\mathcal{X})_1^n)$,*

$$f(T_1, \dots, T_n) = \mathcal{C}_T\big(f(S_1, \dots, S_n)\big),$$

where $f(T_1,\ldots,T_n)$ is defined by the free analytic functional calculus, and $f(S_1,\ldots,S_n)$ is the boundary function of f. Moreover,

$$\|f(T_1,\ldots,T_n)\| \leqslant \left(\sum_{k=0}^{\infty}\left\|\sum_{|\alpha|=k} T_\alpha T_\alpha^*\right\|^{1/2}\right)\|f\|_\infty.$$

Proof. First, we prove the above equality for monomials. Notice that

$$\begin{aligned}\langle \mathcal{C}_T(S_\alpha)x, y\rangle &= \langle (S_\alpha \otimes I_{\mathcal{H}})(1\otimes x), C_T(R_1,\ldots,R_n)(1\otimes y)\rangle \\ &= \left\langle e_\alpha \otimes x, \left(\sum_{\beta\in\mathbb{F}_n^+} R_\beta \otimes T_{\tilde\beta}^*\right)(1\otimes y)\right\rangle \\ &= \left\langle e_\alpha \otimes x, \sum_{\beta\in\mathbb{F}_n^+} e_{\tilde\beta}\otimes T_{\tilde\beta}^* y\right\rangle \\ &= \langle T_\alpha x, y\rangle\end{aligned}$$

for any $x, y \in \mathcal{H}$. Now, assume that $f := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ is in $H^\infty(B(\mathcal{X})_1^n)$ and $0 < r < 1$. Then, due to Theorem 4.1, we have

$$\lim_{m\to\infty}\sum_{k=0}^m r^k \sum_{|\alpha|=k} a_\alpha S_\alpha = f_r(S_1,\ldots,S_n) \in \mathcal{A}_n$$

in the operator norm of $B(F^2(H_n))$, and

$$\lim_{m\to\infty}\sum_{k=0}^m r^k \sum_{|\alpha|=k} a_\alpha T_\alpha = f_r(T_1,\ldots,T_n)$$

in the operator norm of $B(\mathcal{H})$. Now, due to the continuity of the noncommutative Cauchy transform in the operator norm, we deduce that

$$f_r(T_1,\ldots,T_n) = \mathcal{C}_T\big(f_r(S_1,\ldots,S_n)\big). \tag{4.7}$$

Since $f(S_1,\ldots,S_n) \in F_n^\infty$, we know that $\lim_{r\to 1} f_r(S_1,\ldots,S_n) = f(S_1,\ldots,S_n)$ in the strong operator topology. Since $\|f_r(S_1,\ldots,S_n)\| \leqslant \|f\|_\infty$, we deduce that

$$\text{SOT-}\lim_{r\to 1} f_r(S_1,\ldots,S_n)\otimes I_{\mathcal{H}} = f(S_1,\ldots,S_n)\otimes I_{\mathcal{H}}.$$

On the other hand, by Theorem 4.1, $\lim_{r\to 1} f_r(T_1,\ldots,T_n) = f(T_1,\ldots,T_n)$ in the operator norm. Passing to the limit, as $r \to 1$, in the equality

$$\langle f_r(T_1,\ldots,T_n)x, y\rangle = \big\langle \big(f_r(S_1,\ldots,S_n)\otimes I_{\mathcal{H}}\big)(1\otimes x), C_T(R_1,\ldots,R_n)(1\otimes y)\big\rangle, \quad x, y \in \mathcal{H},$$

we obtain $f(T_1,\dots,T_n)=\mathcal{C}_T(f(S_1,\dots,S_n))$, which proves the first part of the theorem. Now, we can deduce the second part of the theorem using Proposition 4.2. This completes the proof. $\square$

Using the Cauchy representation provided by Theorem 4.3, one can deduce the following result.

Corollary 4.4. *Let $[T_1,\dots,T_n]\in B(\mathcal{H})^n$ be an n-tuple of operators with $r(T_1,\dots,T_n)<1$.*

(i) *If $\{f_k\}_{k=1}^\infty$ and f are free holomorphic functions in $Hol(B(\mathcal{X})_1^n)$ such that $\|f_k-f\|_\infty\to 0$ as $k\to\infty$, then $f_k(T_1,\dots,T_n)\to f(T_1,\dots,T_n)$ in the operator norm of $B(\mathcal{H})$.*
(ii) *If $\{f_k\}_{k=1}^\infty$ and f are in the algebra $H^\infty(B(\mathcal{X})_1^n)$ such that $f_k(S_1,\dots,S_n)\to f(S_1,\dots,S_n)$ in the w^*-topology (or strong operator topology) and $\|f_k\|_\infty\leqslant M$ for any $k=1,2,\dots$, then $f_k(T_1,\dots,T_n)\to f(T_1,\dots,T_n)$ in the weak operator topology.*

We can extend Theorem 4.3 and obtain Cauchy representations for the k-order Hausdorff derivations of bounded free holomorphic functions.

Theorem 4.5. *Let $T:=[T_1,\dots,T_n]\in B(\mathcal{H})^n$ be an n-tuple of operators with the joint spectral radius $r(T_1,\dots,T_n)<1$ and let $f\in H^\infty(B(\mathcal{X})_1^n)$. Then*

$$\begin{aligned}&\left\langle\left(\frac{\partial^k f}{\partial Z_{i_1}\cdots\partial Z_{i_k}}\right)(T_1,\dots,T_n)x,y\right\rangle\\&\quad=\left\langle\left[\frac{\partial^k(C_T(R_1,\dots,R_n)^*)}{\partial T_{i_1}\cdots\partial T_{i_k}}\right]\bigl(f(S_1,\dots,S_n)\otimes I_{\mathcal{H}}\bigr)(1\otimes x),1\otimes y\right\rangle\end{aligned}\tag{4.8}$$

for any $i_1,\dots,i_k\in\{1,\dots,n\}$ and $x,y\in\mathcal{H}$, where $f(S_1,\dots,S_n)$ is the boundary function of f. Moreover,

$$\left\|\left(\frac{\partial f}{\partial Z_i}\right)(T_1,\dots,T_n)\right\|\leqslant\|f\|_\infty\sum_{k=1}^\infty k^{3/2}\left\|\sum_{|\beta|=k-1}T_\beta T_\beta\right\|^{1/2},\quad i=1,\dots,n.\tag{4.9}$$

Proof. First, notice that

$$C_X(R_1,\dots,R_n)^*=\sum_{k=0}^\infty\sum_{|\alpha|=k}R_{\tilde\alpha}^*\otimes X_\alpha,$$

where the series is convergent in norm for each n-tuple $[X_1,\dots,X_n]$ with $r(X_1,\dots,X_n)<1$. Therefore,

$$G:=\sum_{k=0}^\infty\sum_{|\alpha|=k}R_{\tilde\alpha}^*\otimes Z_\alpha$$

is a free holomorphic function on the open operatorial unit n-ball. Due to Theorem 1.9, $\frac{\partial^k G}{\partial Z_{i_1}\cdots\partial Z_{i_k}}$ is also a free holomorphic function. By Theorem 4.1, $\frac{\partial^k G}{\partial Z_{i_1}\cdots\partial Z_{i_k}}(X_1,\dots,X_n)$ is a bounded operator for any n-tuple $[X_1,\dots,X_n]$ with spectral radius $r(X_1,\dots,X_n)<1$.

Now, notice that, for each $\alpha\in\mathbb{F}_n^+$, $i=1,\dots,n$, and $x,y\in\mathcal{H}$, we have

$$\begin{aligned}
&\left\langle\left[\frac{\partial(C_T(R_1,\dots,R_n)^*)}{\partial T_i}\right](S_\alpha\otimes I_{\mathcal{H}})(1\otimes x),1\otimes y\right\rangle\\
&\quad=\left\langle\left(\sum_{k=0}^\infty\sum_{|\beta|=k}R_\beta^*\otimes\frac{\partial T_{\tilde\beta}}{\partial T_i}\right)(S_\alpha\otimes I_{\mathcal{H}})(1\otimes x),1\otimes y\right\rangle\\
&\quad=\left\langle e_\alpha\otimes x,\sum_{k=0}^\infty\sum_{|\beta|=k}e_{\tilde\beta}\otimes\left(\frac{\partial T_{\tilde\beta}}{\partial T_i}\right)^*y\right\rangle\\
&\quad=\left\langle\frac{\partial T_\alpha}{\partial T_i}x,y\right\rangle=\left\langle\left(\frac{\partial Z_\alpha}{\partial Z_i}\right)(T_1,\dots,T_n)x,y\right\rangle.
\end{aligned}$$

Hence, we deduce relation (4.8) for polynomials. Let $f=\sum_{k=1}^\infty\sum_{|\alpha|=k}a_\alpha Z_\alpha$ be in $H^\infty(B(\mathcal{X})_1^n)$. Due to Theorem 4.1, we have

$$\left(\frac{\partial f_r}{\partial Z_i}\right)(T_1,\dots,T_n)=\lim_{m\to\infty}\sum_{k=0}^m\sum_{|\alpha|=k}r^{|\alpha|}a_\alpha\left(\frac{\partial Z_\alpha}{\partial Z_i}\right)(T_1,\dots,T_n),$$

where the convergence is in the operator norm of $B(\mathcal{H})$, and

$$f_r(S_1,\dots,S_n)=\lim_{m\to\infty}\sum_{k=0}^m\sum_{|\alpha|=k}r^{|\alpha|}a_\alpha S_\alpha\in\mathcal{A}_n,$$

where the convergence is in the operator norm of $B(F^2(H_n))$. Since (4.8) holds for polynomials, the last two relations imply

$$\begin{aligned}
&\left\langle\left(\frac{\partial f_r}{\partial Z_i}\right)(T_1,\dots,T_n)x,y\right\rangle\\
&\quad=\left\langle\left[\frac{\partial(C_T(R_1,\dots,R_n)^*)}{\partial T_i}\right]\big(f_r(S_1,\dots,S_n)\otimes I_{\mathcal{H}}\big)(1\otimes x),1\otimes y\right\rangle
\end{aligned}$$

for any $x,y\in\mathcal{H}$ and $0<r<1$. Using again Theorem 4.1, we have

$$\lim_{r\to1}\left(\frac{\partial f_r}{\partial Z_i}\right)(T_1,\dots,T_n)=\left(\frac{\partial f}{\partial Z_i}\right)(T_1,\dots,T_n)$$

in the operator norm. Since $f(S_1,\dots,S_n)\in F_n^\infty$ (see Theorem 3.1), as in the proof of Theorem 4.3, we deduce that

$$\text{SOT-}\lim_{r\to\infty}f_r(S_1,\dots,S_n)\otimes I_{\mathcal{H}}=f(S_1,\dots,S_n)\otimes I_{\mathcal{H}}.$$

Passing to the limit, as $r \to \infty$, in the above equality, we deduce relation (4.8) in the particular case when $k = 1$. Repeating this argument, one can prove the general case when $\frac{\partial}{\partial T_i}$ is replaced by $\frac{\partial^k}{\partial T_{i_1} \cdots \partial T_{i_k}}$.

Now, we prove the second part of the theorem. Notice that

$$\left\| \frac{\partial G}{\partial Z_i}(X_1, \ldots, X_n) \right\| \leqslant \sum_{k=0}^{\infty} \left\| \sum_{|\alpha|=k} R_{\tilde{\alpha}} \otimes \left(\frac{\partial X_\alpha}{\partial X_i} \right)^* \right\| \leqslant \sum_{k=0}^{\infty} \left\| \sum_{|\alpha|=k} \left(\frac{\partial X_\alpha}{\partial X_i} \right) \left(\frac{\partial X_\alpha}{\partial X_i} \right)^* \right\|^{1/2}.$$

For each $\alpha \in \mathbb{F}_n^+$, $|\alpha| = k$, we can prove that

$$\left(\frac{\partial X_\alpha}{\partial X_i} \right) \left(\frac{\partial X_\alpha}{\partial X_i} \right)^* \leqslant k^2 {\sum_{\gamma}}_d^{\alpha} X_\gamma X_\gamma^*, \tag{4.10}$$

where the sum is taken over all distinct words γ obtained by deleting each occurrence of g_i in α. Indeed, notice first that $\frac{\partial X_\alpha}{\partial X_i} = \sum_\beta^\alpha X_\beta$, where the sum is taken over all words β obtained by deleting each occurrence of g_i in α. Since the above some contains at most k terms, one can show that

$$\left(\frac{\partial X_\alpha}{\partial X_i} \right) \left(\frac{\partial X_\alpha}{\partial X_i} \right)^* \leqslant k \sum_{\beta}^{\alpha} X_\beta X_\beta^*.$$

Indeed, it enough to use the following result which is an easy consequence of the classical Cauchy inequality: if $A_1, \ldots, A_k \in B(\mathcal{H})$, then

$$\left(\sum_{i=1}^{k} A_i \right) \left(\sum_{i=1}^{k} A_i^* \right) \leqslant k \sum_{i=1}^{k} A_i A_i^*.$$

Now, the X_β's in the above sum are not necessarily distinct but each of them can occur at most k times. Consequently,

$$\sum_{\beta}^{\alpha} X_\beta X_\beta^* \leqslant k {\sum_{\gamma}}_d^{\alpha} X_\gamma X_\gamma^*.$$

Combining these inequalities, we deduce (4.10). (We remark that the inequality (4.10) is sharp and the equality occurs, for example, when $\alpha = g_i^k$.) Therefore, we have

$$\sum_{k=0}^{\infty} \left\| \sum_{|\alpha|=k} \left(\frac{\partial X_\alpha}{\partial X_i} \right) \left(\frac{\partial X_\alpha}{\partial X_i} \right)^* \right\|^{1/2} \leqslant \sum_{k=0}^{\infty} \left\| \sum_{|\alpha|=k} k^2 {\sum_{\gamma}}_d^{\alpha} X_\gamma X_\gamma^* \right\|^{1/2}.$$

We remark that if $\beta \in \mathbb{F}_n^+$, $|\beta| = k-1$, then X_β can come from free differentiation with respect to X_i of the monomials $X_{\chi(g_i, m, \beta)}$, $m = 0, 1, \ldots, k-1$, where $\chi(g_i, m, \beta)$ is the insertion mapping of g_i on the m position of β (see the proof of Theorem 1.9). Consequently, we have

$$\sum_{|\alpha|=k} {\sum_{\gamma}}_d^{\alpha} X_\gamma X_\gamma^* \leqslant k \sum_{|\beta|=k-1} X_\beta X_\beta^*.$$

Using the above inequalities, we obtain

$$\left\| \frac{\partial G}{\partial Z_i}(X_1, \ldots, X_n) \right\| \leqslant \sum_{k=1}^{\infty} k^{3/2} \left\| \sum_{|\beta|=k-1} X_\beta X_\beta \right\|^{1/2}.$$

Hence, and due to relation (4.8), we deduce inequality (4.9). The proof is complete. □

We remark that inequalities of type (4.8) can be obtained for k-order Hausdorff derivations. On the other hand, a similar result to Corollary 4.4 can be obtain for k-order Hausdorff derivations, if one uses Theorem 4.5.

In the last part of this section, we show that the noncommutative Cauchy transform commutes with certain classes of automorphisms. Let $\mathcal{U}(H_n)$ be the group of all unitaries on H_n and let $U \in \mathcal{U}(H_n)$. If $U := [\lambda_{ij}]_{i,j=1}^n$ and $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$, we define

$$\beta_U(T_j) := \sum_{i=1}^{n} \lambda_{ij} T_i, \quad j = 1, \ldots, n,$$

and the map $\beta_U : B(\mathcal{H})^n \to B(\mathcal{H})^n$ by setting $\beta_U(T) := [\beta_U(T_1), \ldots, \beta_U(T_n)]$.

Theorem 4.6. *If $U \in \mathcal{U}(H_n)$, $U := [\lambda_{ij}]_{i,j=1}^n$, then the map β_U is an isometric automorphism of the open unit ball $[B(\mathcal{H})^n]_1$ and also of the ball*

$$\big\{[T_1, \ldots, T_n] \in B(\mathcal{H})^n\colon\ r(T_1, \ldots, T_n) < 1\big\}.$$

Moreover, there is a unique completely isometric automorphism of the noncommutative disc algebra $\mathcal{A}_n$, denoted also by β_U, such that

$$\beta_U(S_j) := \sum_{i=1}^{n} \lambda_{ij} S_i, \quad j = 1, \ldots, n,$$

where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space.

Proof. For each $j = 1, \ldots, n$, we define the operators

$$\mathbf{U}_j := \begin{bmatrix} \lambda_{1j} I_{\mathcal{H}} \\ \vdots \\ \lambda_{nj} I_{\mathcal{H}} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^{(n)},$$

where $\mathcal{H}^{(n)}$ is the direct sum of n copies of $\mathcal{H}$. Notice that

$$\mathbf{U}_i^* \mathbf{U}_j = \delta_{ij} I_{\mathcal{H}^{(n)}} \quad \text{and} \quad \sum_{i=1}^{n} \mathbf{U}_i \mathbf{U}_i^* = I_{\mathcal{H}^{(n)}}. \tag{4.11}$$

We have $\beta_U(T) = [B_1, \ldots, B_n]$, where $B_i := T\mathbf{U}_i$, $i = 1, \ldots, n$ and $T := [T_1, \ldots, T_n]$. Now, it is clear that $\sum_{i=1}^n B_i B_i^* = \sum_{i=1}^n T_i T_i$. If $A \in B(\mathcal{H})$ then

$$\mathbf{U}_i A = \operatorname{diag}_n(A)\mathbf{U}_i, \quad i = 1, \ldots, n,$$

where $\operatorname{diag}_n(A)$ is the $n \times n$ block diagonal operator matrix having A on the diagonal and 0 otherwise. Using this relation and (4.11), we deduce that

$$\begin{aligned}
\sum_{|\alpha|=2} B_\alpha B_\alpha^* &= \sum_{i=1}^n B_i \left(\sum_{|\alpha|=1} B_\alpha B_\alpha^* \right) B_i \\
&= T \left[\sum_{i=1}^n \mathbf{U}_i \left(T T^* \right) \mathbf{U}_i^* \right] T^* \\
&= T \operatorname{diag}_n\left(T T^* \right) \left(\sum_{i=1}^n \mathbf{U}_i \mathbf{U}_i^* \right) T^* \\
&= T \operatorname{diag}_n\left(T T^* \right) T = \sum_{|\alpha|=2} T_\alpha T_\alpha^*.
\end{aligned}$$

By induction over k, one can similarly prove that

$$\sum_{|\alpha|=k} B_\alpha B_\alpha^* = \sum_{|\alpha|=k} T_\alpha T_\alpha^*, \quad k = 1, 2, \ldots. \tag{4.12}$$

Consequently, we have

$$\left\| \beta_U(T) \right\| = \|T\| \quad \text{and} \quad r\left(\beta_U(T) \right) = r(T).$$

Hence, and since $\beta_U(T) = T\mathbf{U}$, where $\mathbf{U} := [\mathbf{U}_1, \ldots, \mathbf{U}_n]$ is a unitary operator, we deduce that the map $\beta_U : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ is an isometric automorphism of the open unit ball of $B(\mathcal{H})^n$ and

$$\beta_U^{-1}(Y) = Y\mathbf{U}^*, \quad Y \in \left[B(\mathcal{H})^n \right]_1.$$

Moreover, β_U is an isometric automorphism of the operatorial ball

$$\left\{ [T_1, \ldots, T_n] \in B(\mathcal{H})^n \colon r(T_1, \ldots, T_n) < 1 \right\}.$$

Now, let us prove the second part of the theorem. Using the same notation for the unitary operator $\mathbf{U}$, when $\mathcal{H} := F^2(H_n)$, we deduce that $[\beta_U(S_1), \ldots, \beta_U(S_n)] = S\mathbf{U}$, where $S := [S_1, \ldots, S_n]$. Setting $V_i := \beta_U(S_i)$, $i = 1, \ldots, n$, one can easily see that $V_1, \ldots, V_n$ are isometries with orthogonal ranges. For any polynomial $p(S_1, \ldots, S_n)$ in the noncommutative disc algebra $\mathcal{A}_n$, we have $\beta_U(p(S_1, \ldots, S_n)) = p(V_1, \ldots, V_n)$. According to [32], we have

$$\left\| \left[p_{ij}(S_1, \ldots, S_n) \right]_m \right\| = \left\| \left[p_{ij}(V_1, \ldots, V_n) \right]_m \right\|.$$

Since $\mathcal{A}_n$ is the norm closure of all polynomials in $S_1, \dots, S_n$ and the identity, β_U can be uniquely extended to a completely isometric homomorphism from $\mathcal{A}_n$ to $\mathcal{A}_n$. Define the n-tuple $[X_1, \dots, X_n] := [S_1, \dots, S_n]\mathbf{U}^*$ and notice that each entry X_i is a homogeneous polynomial of degree one in $S_1, \dots, S_n$. Since

$$\big[\beta_U(X_1), \dots, \beta_U(X_n)\big] = [X_1, \dots, X_n]\mathbf{U} = [S_1, \dots, S_n],$$

we deduce that $\beta_U(X_i) = S_i$, $i = 1, \dots, n$, and consequently, $\beta_U(X_\alpha) = S_\alpha$, $\alpha \in \mathbb{F}_n^+$. Hence, the range of $\beta_U : \mathcal{A}_n \to \mathcal{A}_n$ contains all polynomials in $\mathcal{A}_n$. Using again the norm density of polynomials in $\mathcal{A}_n$, we conclude that β_U is a completely isometric automorphism of $\mathcal{A}_n$. □

In what follows we show that the noncommutative Cauchy transform commutes with the action of the unitary group $\mathcal{U}(H_n)$.

Theorem 4.7. *Let $T := [T_1, \dots, T_n] \in B(\mathcal{H})^n$ be an n-tuple of operators with joint spectral radius $r(T_1, \dots, T_n) < 1$ and $U \in \mathcal{U}(H_n)$. Then*

$$\mathcal{C}_T\big(\beta_U(f)\big) = \mathcal{C}_{\beta_U(T)}(f), \quad f \in \mathcal{A}_n,$$

where β_U is the canonical automorphism generated by U.

Proof. Remember that $\mathcal{A}_n$ is the norm closure of the polynomials in $S_1, \dots, S_n$ and the identity. Due to the continuity of the noncommutative Cauchy transform in the operator norm, it is enough to prove the above relation for $f := S_\alpha$, $\alpha \in \mathbb{F}_n^+$. By Theorem 4.3, we have

$$\big\langle \mathcal{C}_T\big(\beta_U(S_\alpha)\big)x, y\big\rangle = \big\langle C_T(R_1, \dots, R_n)^*\big(\beta_U(S_\alpha) \otimes I_{\mathcal{H}}\big)(1 \otimes x), 1 \otimes y\big\rangle = \langle B_\alpha x, y\rangle$$

for any $x, y \in \mathcal{H}$, where $[B_1, \dots, B_n] := \beta_U(T)$. On the other hand, due to Theorem 4.6, we have $r(\beta_U(T)) < 1$. Applying again Theorem 4.3, we obtain

$$\big\langle \mathcal{C}_{\beta_U(T)}(S_\alpha)x, y\big\rangle = \big\langle C_{\beta_U(T)}(R_1, \dots, R_n)^*(S_\alpha \otimes I_{\mathcal{H}})(1 \otimes x), 1 \otimes y\big\rangle = \langle B_\alpha x, y\rangle.$$

Hence, $\mathcal{C}_T(\beta_U(S_\alpha)) = \mathcal{C}_{\beta_U(T)}(S_\alpha)$, and the result follows. □

The continuity and the uniqueness of the free analytic functional calculus for n-tuples of operators with joint spectral radius $r(T_1, \dots, T_n) < 1$ will be proved in the next section.

5. Weierstrass and Montel theorems for free holomorphic functions

In this section, we obtain Weierstrass and Montel type theorems for the algebra of free holomorphic functions with scalar coefficients on the open operatorial unit n-ball. This enables us to introduce a metric on $Hol(B(\mathcal{X})_1^n)$ with respect to which it becomes a complete metric space, and the Hausdorff derivations are continuous. In the end of this section, we prove the continuity and uniqueness of the free functional calculus. Connections with the F_n^∞-functional calculus for row contractions [30] and, in the commutative case, with Taylor's functional calculus [48] are also discussed.

We say that a sequence $\{F_m\}_{m=1}^\infty \subset Hol(B(\mathcal{X})_1^n)$ of free holomorphic functions converges uniformly on the closed operatorial n-ball of radius $r \in [0,1)$ if it converges uniformly on the closed ball

$$\big[B(\mathcal{H})^n\big]_r^- := \big\{[X_1,\dots,X_n] \in B(\mathcal{H})^n\colon\ \|X_1X_1^* + \cdots + X_nX_n^*\| \leqslant r\big\},$$

where $\mathcal{H}$ is an infinite-dimensional Hilbert space. According to the maximum principle of Corollary 3.4, this is equivalent to the fact that the sequence $\{F_m(rS_1,\dots,rS_n)\}_{m=1}^\infty$ is convergent in the operator norm topology of $B(F^2(H_n))$.

The first result of this section is a multivariable operatorial version of Weierstrass theorem [11].

Theorem 5.1. *Let $\{F_m\}_{m=1}^\infty \subset Hol(B(\mathcal{X})_1^n)$ be a sequence of free holomorphic functions which is uniformly convergent on any closed operatorial n-ball of radius $r \in [0,1)$. Then there is a free holomorphic function $F \in Hol(B(\mathcal{X})_1^n)$ such that F_m converges to F on any closed operatorial n-ball of radius $r \in [0,1)$.*

Moreover, given $i_1,\dots,i_k \in \{1,\dots,n\}$, the sequence $\{\frac{\partial^k F_m}{\partial Z_{i_1}\cdots\partial Z_{i_k}}\}_{m=1}^\infty$ is uniformly convergent to $\frac{\partial^k F}{\partial Z_{i_1}\cdots\partial Z_{i_k}}$ on any closed operatorial n-ball of radius $r \in [0,1)$, where $\frac{\partial^k}{\partial Z_{i_1}\cdots\partial Z_{i_k}}$ is the k-order Hausdorff derivation.

Proof. Let $F_m := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(m)} Z_\alpha$ and fix $r \in (0,1)$. Then, due to Theorem 1.5,

$$F_m(rS_1,\dots,rS_n) = \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha^{(m)} S_\alpha$$

is in the noncommutative disc algebra $\mathcal{A}_n$. Since $\{F_m\}_{m=1}^\infty$ is uniformly convergent on the closed operatorial n-ball of radius r, the sequence $\{F_m(rS_1,\dots,rS_n)\}_{m=1}^\infty$ is convergent in the operator norm of $B(F^2(H_n))$. On the other hand, since the noncommutative disc algebra $\mathcal{A}_n$ is closed in the operator norm, there exists $g \in \mathcal{A}_n$ such that

$$F_m(rS_1,\dots,rS_n) \to L_g \quad \text{as } m \to \infty. \tag{5.1}$$

Assume $g = \sum_{\alpha\in\mathbb{F}_n^+} b_\alpha(r) e_\alpha$, and notice also that

$$b_\alpha(r) = \big\langle S_\alpha^* L_g(1), 1\big\rangle, \quad \alpha \in \mathbb{F}_n^+.$$

If $\lambda_{(\beta)} \in \mathbb{C}$ for $\beta \in \mathbb{F}_n^+$ with $|\beta| = k$, we have

$$\left|\left\langle \sum_{|\beta|=k} \lambda_{(\beta)} S_\beta^*\big(F_m(rS_1,\dots,rS_n) - L_g\big)1, 1\right\rangle\right| \leqslant \big\|F_m(rS_1,\dots,rS_n) - L_g\big\| \left\|\sum_{|\beta|=k} \lambda_{(\beta)} S_\beta^*\right\|.$$

Since $S_1,\dots,S_n$ are isometries with orthogonal ranges, we deduce that

$$\left|\sum_{|\beta|=k} \big(r^k a_\beta^{(m)} - b_\beta(r)\big)\lambda_{(\beta)}\right| \leqslant \big\|F_m(rS_1,\dots,rS_n) - L_g\big\| \left(\sum_{|\beta|=k} |\lambda_{(\beta)}|^2\right)^{1/2}$$

for any $\lambda_{(\beta)} \in \mathbb{C}$ with $|\beta| = k$. Consequently, we have

$$\left(\sum_{|\beta|=k} \left| r^k a_\beta^{(m)} - b_\beta(r) \right|^2 \right)^{1/2} \leqslant \left\| F_m(rS_1, \dots, rS_n) - L_g \right\|$$

for any $k = 0, 1, \dots$. Since $\| F_m(rS_1, \dots, rS_n) - L_g \| \to 0$, as $m \to \infty$, we deduce that $r^k a_\beta^{(m)} \to b_\beta(r)$, as $m \to \infty$, for any $|\beta| = k$ and $k = 0, 1, \dots$. Hence, $a_\beta := \lim_{m\to\infty} a_\beta^{(m)}$ exists and $b_\beta(r) = r^k a_\beta$ for any $\beta \in \mathbb{F}_n^+$ with $|\beta| = k$ and $k = 0, 1, \dots$. Consider the formal power series $F := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$. We show now that F is a free holomorphic function on the open operatorial unit n-ball. Due to the above calculations, we have

$$r^k \left| \left(\sum_{|\beta|=k} |a_\beta^{(m)}|^2 \right)^{1/2} - \left(\sum_{|\beta|=k} |a_\beta|^2 \right)^{1/2} \right| \leqslant \left\| F_m(rS_1, \dots, rS_n) - L_g \right\|.$$

Therefore,

$$\sum_{|\beta|=k} \left| a_\beta^{(m)} \right|^2 \to \sum_{|\beta|=k} |a_\beta|^2 \quad \text{as } m \to \infty, \tag{5.2}$$

uniformly with respect to $k = 0, 1, \dots$. Let us show that the radius of convergence of F is $\geqslant 1$. To this end, assume that $\gamma > 1$ and

$$\limsup_{k\to\infty} \left(\sum_{|\beta|=k} |a_\beta|^2 \right)^{1/2k} > \gamma.$$

Then there is $k \in \mathbb{N}$ as large as we want such that

$$\left(\sum_{|\beta|=k} |a_\beta|^2 \right)^{1/2} > \gamma^k. \tag{5.3}$$

Choose λ such that $1 < \lambda < \gamma$ and let $\epsilon > 0$ be such that $\epsilon < \gamma - \lambda$. Notice that $\epsilon < \gamma^k - \lambda^k$ for any $k = 1, 2, \dots$. Now, due to relation (5.2), there exists $N_\epsilon \in \mathbb{N}$ such that

$$\left| \left(\sum_{|\beta|=k} |a_\beta^{(m)}|^2 \right)^{1/2} - \left(\sum_{|\beta|=k} |a_\beta|^2 \right)^{1/2} \right| < \epsilon$$

for any $m > N_\epsilon$ and any $k = 0, 1, \dots$. Hence, and using inequality (5.3), we deduce that

$$\left(\sum_{|\beta|=k} |a_\beta^{(m)}|^2 \right)^{1/2} \geqslant \gamma^k - \epsilon > \lambda^k$$

for any $m > N_\epsilon$ and some k as large as we want. Consequently, we have

$$\limsup_{k\to\infty}\left(\sum_{|\beta|=k}\left|a_\beta^{(m)}\right|^2\right)^{1/2k} \geqslant \lambda > 1$$

for $m \geqslant N_\epsilon$. Due to Theorem 1.1, this shows that the radius of convergence of F_m is < 1, which contradicts the fact that F_m is a free holomorphic function with radius of convergence $\geqslant 1$. Therefore,

$$\limsup_{k\to\infty}\left(\sum_{|\beta|=k}|a_\beta|^2\right)^{1/2k} \leqslant 1$$

and, consequently, Theorem 1.1 shows that F is a free holomorphic function on the open operatorial unit ball. The same theorem implies that $F(rS_1,\dots,rS_n) = \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|}a_\alpha S_\alpha$ is convergent in norm. Since L_g and $F(rS_1,\dots,rS_n)$ have the same Fourier coefficients, we must have $L_g = F(rS_1,\dots,rS_n)$. Due to relation (5.1), we have

$$\left\|F_m(rS_1,\dots,rS_n) - F(rS_1,\dots,rS_n)\right\| \to 0 \quad \text{as } m\to\infty.$$

If $[X_1,\dots,X_n] \in [B(\mathcal{H})^n]_1$ and $\|[X_1,\dots,X_n]\| = r < 1$, the noncommutative von Neumann inequality implies

$$\left\|F_m(X_1,\dots,X_n) - F(X_1,\dots,X_n)\right\| \leqslant \left\|F_m(rS_1,\dots,rS_n) - F(rS_1,\dots,rS_n)\right\|.$$

Taking $m\to\infty$, we deduce that F_m converges to F on any closed operatorial n-ball of radius $r\in[0,1)$.

Now, we show that for each $\gamma\in(0,1)$

$$\left(\frac{\partial F_m}{\partial Z_i}\right)(\gamma S_1,\dots,\gamma S_n) \to \left(\frac{\partial F}{\partial Z_i}\right)(\gamma S_1,\dots,\gamma S_n) \tag{5.4}$$

in the operator norm, as $m\to\infty$. Let $r, r' \in (0,1)$ such that $\gamma = rr'$. Since $(F_m)_r$ and $F_r \in \mathcal{A}_n$ are in the noncommutative disc algebra $\mathcal{A}_n$, we can apply Theorem 4.5 (see inequality (4.9)) and obtain

$$\left\|\left(\frac{\partial((F_m)_r - F_r)}{\partial Z_i}\right)(r'S_1,\dots,r'S_n)\right\| \leqslant M\left\|(F_m)_r - F_r\right\|_\infty,$$

where M is an appropriate constant which does not depend on m. Since $\|(F_m)_r - F_r\|_\infty \to 0$ as $m\to\infty$ and

$$\left(\frac{\partial((F_m)_r - F_r)}{\partial Z_i}\right)(r'S_1,\dots,r'S_n) = r\left(\frac{\partial(F_m - F)}{\partial Z_i}\right)(\gamma S_1,\dots,\gamma S_n),$$

we deduce relation (5.4). Using the result for $\frac{\partial}{\partial Z_i}$, one can obtain the general case for k-order Hausdorff partial derivations. The proof is complete. □

We say that a set $\mathcal{F} \subset Hol(B(\mathcal{X})_1^n)$ is normal if each sequence in $\mathcal{F}$ has a subsequence which converges to a function in $Hol(B(\mathcal{X})_1^n)$ uniformly on any closed operatorial ball of radius $r \in [0, 1)$. The set $\mathcal{F}$ is called locally bounded if, for any $r \in [0, 1)$, there exists $M > 0$ such that $\|f(X_1, \ldots, X_n)\| \leqslant M$ for any $f \in \mathcal{F}$ and $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_r$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space.

We can prove now the following noncommutative version of Montel theorem (see [11]).

Theorem 5.2. *Let $\mathcal{F} \subset Hol(B(\mathcal{X})_1^n)$ be a family of free holomorphic functions. Then the following statements are equivalent*:

(i) $\sup_{f\in\mathcal{F}} \|f(rS_1, \ldots, rS_n)\| < \infty$ *for each* $r \in [0, 1)$.
(ii) $\mathcal{F}$ *is a normal set.*
(iii) $\mathcal{F}$ *is locally bounded.*

Proof. Assume that condition (i) holds. For each $f \in \mathcal{F}$, let $\{a_\alpha(f)\}_{\alpha\in\mathbb{F}_n^+}$ be the sequence of coefficients. Due to (i), for each $r \in [0, 1)$, there exists $M_r > 0$ such that

$$\big\|f(rS_1, \ldots, rS_n)\big\| \leqslant M_r \quad \text{for any } f \in \mathcal{F}. \tag{5.5}$$

By the Cauchy type estimate of Theorem 2.1, if $r \in (0, 1)$, then

$$\left(\sum_{|\alpha|=k} \big|a_\alpha(f)\big|^2\right)^{1/2} \leqslant \frac{1}{r^k} M_r \quad \text{for any } f \in \mathcal{F},\ k = 0, 1, \ldots. \tag{5.6}$$

Let $\{F_m\}_{m=1}^\infty$ be a sequence of elements in $\mathcal{F}$. Then, relation (5.5) implies

$$\big|a_0(F_m)\big| \leqslant M_0 \quad \text{for any } m = 1, 2, \ldots.$$

Due to the classical Bolzano–Weierstrass theorem for bounded sequences of complex numbers, there is a subsequence $\{F_{m_k^{(0)}}\}_{k=1}^\infty$ of $\{F_m\}_{m=1}^\infty$ such that the scalar sequence $\{a_0(F_{m_k^{(0)}})\}_{k=1}^\infty$ is convergent in $\mathbb{C}$, as $k \to \infty$. Inductively, using relation (5.6), we find, for each $\alpha \in \mathbb{F}_n^+$, $|\alpha| \geqslant 1$, a subsequence $\{F_{m_k^{(\alpha)}}\}_{k=1}^\infty$ of $\{F_{m_k^{(\beta)}}\}_{k=1}^\infty$, where α is the successor of β in the lexicographic order of $\mathbb{F}_n^+$, such that the sequence $\{a_\alpha(F_{m_k^{(\alpha)}})\}_{k=1}^\infty$ is convergent in $\mathbb{C}$, as $k \to \infty$. Using the diagonal process, we find a subsequence $\{F_{p_k}\}_{k=1}^\infty$ of $\{F_m\}_{m=1}^\infty$ such that $\{a_\alpha(F_{p_k})\}_{k=1}^\infty$ converges in $\mathbb{C}$ as $k \to \infty$, for any $\alpha \in \mathbb{F}_n^+$.

Now let us prove that, if $\gamma > 1$, then $\{F_{p_k}(\frac{r}{\gamma}S_1, \ldots, \frac{r}{\gamma}S_n)\}_{k=1}^\infty$ converges in the norm topology of $B(F^2(H_n))$. Indeed, if $N \in \mathbb{N}$, then relation (5.6) implies

$$\begin{aligned}
&\left\|F_{p_k}\left(\frac{r}{\gamma}S_1, \ldots, \frac{r}{\gamma}S_n\right) - F_{p_s}\left(\frac{r}{\gamma}S_1, \ldots, \frac{r}{\gamma}S_n\right)\right\| \\
&\quad\leqslant \sum_{j=1}^N \frac{r^j}{\gamma^j}\left(\sum_{|\alpha|=j} \big|a_\alpha(F_{p_k}) - a_\alpha(F_{p_s})\big|^2\right)^{1/2} + \sum_{j=N+1} \frac{r^j}{\gamma^j}\left(\sum_{|\alpha|=j} \big|a_\alpha(F_{p_k}) - a_\alpha(F_{p_s})\big|^2\right)^{1/2} \\
&\quad\leqslant \sum_{j=1}^N \frac{r^j}{\gamma^j}\left(\sum_{|\alpha|=j} \big|a_\alpha(F_{p_k}) - a_\alpha(F_{p_s})\big|^2\right)^{1/2} + \sum_{j=N+1}^\infty \frac{r^j}{\gamma^j}\frac{2M_r}{r^j}
\end{aligned}$$

$$\leqslant \sum_{j=1}^{N} \frac{r^j}{\gamma^j} \left(\sum_{|\alpha|=j} \left| a_\alpha(F_{p_k}) - a_\alpha(F_{p_s}) \right|^2 \right)^{1/2} + \frac{2M_r}{\gamma^N(\gamma-1)}.$$

Given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $\frac{2M_r}{\gamma^N} < \frac{\epsilon}{2}$. On the other hand, since $\{a_\alpha(F_{p_k})\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{j=1}^{N} \frac{r^j}{\gamma^j} \left(\sum_{|\alpha|=j} \left| a_\alpha(F_{p_k}) - a_\alpha(F_{p_s}) \right|^2 \right)^{1/2} < \frac{\epsilon}{2} \quad \text{for any } k, s \geqslant k_0.$$

Summing up the above results, we deduce that

$$\left\| F_{p_k}\left(\frac{r}{\gamma} S_1, \ldots, \frac{r}{\gamma} S_n \right) - F_{p_s}\left(\frac{r}{\gamma} S_1, \ldots, \frac{r}{\gamma} S_n \right) \right\| < \epsilon \quad \text{for any } k, s \geqslant k_0.$$

This proves that the sequence $\{F_{p_k}(\frac{r}{\gamma} S_1, \ldots, \frac{r}{\gamma} S_n)\}_{k=1}^\infty$ converges in the norm topology of $B(F^2(H_n))$, for any $r \in [0,1)$ and $\gamma > 1$. Since the set $A := \{\frac{r}{\gamma}\colon\ 0 \leqslant r < 1,\ \gamma > 1\}$ is equal to $[0,1)$, one can choose an increasing sequence $\{t_q\}_{q=1}^\infty$ such that $t_q \in A$ and $t_q \to 1$ as $q \to \infty$.

Now, if $\{F_m\}_{m=1}^\infty \subset \mathcal{F}$, then, using the above result, there is a subsequence $\{F_{n_k^{(1)}}\}_{k=1}^\infty$ of $\{F_m\}_{m=1}^\infty$ such that $\{F_{n_k^{(1)}}(t_1 S_1, \ldots, t_1 S_n)\}$ is convergent in the norm topology of $B(F^2(H_n))$, as $k \to \infty$. Inductively, for each $q = 2, 3, \ldots,$ we find a subsequence $\{F_{n_k^{(q)}}\}_{k=1}^\infty$ of $\{F_{n_k^{(q-1)}}\}_{k=1}^\infty$ such that $\{F_{n_k^{(q)}}(t_q S_1, \ldots, t_q S_n)\}$ is convergent in the norm topology of $B(F^2(H_n))$, as $k \to \infty$. Using again the diagonal process, we find a subsequence $\{F_{m_k}\}_{k=1}^\infty$ of $\{F_m\}_{m=1}^\infty$ such that, for each $r \in [0,1)$, the subsequence $\{F_{m_k}(rS_1, \ldots, rS_n)\}$ is convergent in the norm topology of $B(F^2(H_n))$, as $k \to \infty$. Applying Theorem 5.1, we deduce that $\mathcal{F}$ is a normal set. Therefore, the implication (i) $\Rightarrow$ (ii) is true.

To prove the converse, assume that there is $r_0 \in (0,1)$ such that

$$\sup_{f \in \mathcal{F}} \left\| f(r_0 S_1, \ldots, r_0 S_n) \right\| = \infty.$$

Let $\{f_m\}_{m=1}^\infty \subset \mathcal{F}$ be such that

$$\left\| f_m(r_0 S_1, \ldots, r_0 S_n) \right\| \to \infty \quad \text{as } m \to \infty. \tag{5.7}$$

Since (ii) holds, there exists a subsequence $\{f_{m_k}\}_{k=1}^\infty$ such that $\{f_{m_k}(rS_1, \ldots, rS_n)\}_{k=1}^\infty$ is convergent for any $r \in [0,1)$. This contradicts relation (5.7). The equivalence (i) $\Leftrightarrow$ (ii) follows from Corollary 3.4. The proof is complete. □

Now, we can obtain the following Vitali type result in our setting.

Theorem 5.3. *Let $\{F_m\}_{m=1}^\infty$ be a sequence of free holomorphic functions on $[B(\mathcal{H})^n]_1$ with scalar coefficients such that, for each $r \in [0,1)$,*

$$\sup_m \left\| F_m(rS_1, \ldots, rS_n) \right\| < \infty.$$

If there exists $0 < \gamma < 1$ *such that* $F_m(\gamma S_1, \ldots, \gamma S_n)$ *converges in norm as* $m \to \infty$*, then* F_m *converges uniformly on* $[B(\mathcal{H})^n]_r^-$ *for any* $r \in [0, 1)$.

Proof. Suppose that $\{F_m\}_{m=1}^\infty$ does not converge uniformly on $[B(\mathcal{H})^n]_{r_0}^-$ for some $r_0 \in (0, 1)$. Then there exist $\delta > 0$, subsequences $\{F_{m_k}\}_{k=1}^\infty$ and $\{F_{n_k}\}_{k=1}^\infty$ of $\{F_m\}_{m=1}^\infty$, and n-tuples of operators $[X_1^{(k)}, \ldots, X_n^{(k)}] \in [B(\mathcal{H})^n]_{r_0}^-$ such that

$$\left\| F_{n_k}\big(X_1^{(k)}, \ldots, X_n^{(k)}\big) - F_{m_k}\big(X_1^{(k)}, \ldots, X_n^{(k)}\big) \right\| \geqslant \delta \tag{5.8}$$

for any $k = 1, 2, \ldots$. By Theorem 5.2, we find a subsequence $\{k_p\}_{p=1}^\infty$ of $\{k\}_{k=1}^\infty$ such that $\{F_{m_{k_p}}\}_{k=1}^\infty$ and $\{F_{n_{k_p}}\}_{k=1}^\infty$ are uniformly convergent to f and g, respectively, on any closed operatorial n-ball of radius $r \in [0, 1)$. Using Theorem 5.1, we deduce that f, g are free holomorphic functions on $[B(\mathcal{H})^n]_1$. Now, the inequality (5.8) and the noncommutative von Neumann inequality imply

$$\left\| F_{n_{k_p}}(r_0 S_1, \ldots, r_0 S_n) - F_{m_{k_p}}(r_0 S_1, \ldots, r_0 S_n) \right\| \geqslant \delta > 0$$

for any $k = 1, 2, \ldots$. Consequently, we have

$$\left\| f(r_0 S_1, \ldots, r_0 S_n) - g(r_0 S_1, \ldots, r_0 S_n) \right\| \geqslant \delta > 0. \tag{5.9}$$

On the other hand, since $\{F_m(\gamma S_1, \ldots, \gamma S_n)\}_{m=1}^\infty$ converges in norm as $m \to \infty$, we must have

$$f(\gamma S_1, \ldots, \gamma S_n) = g(\gamma S_1, \ldots, \gamma S_n).$$

Since $0 < \gamma < 1$ and f, g are free holomorphic functions on $[B(\mathcal{H})^n]_1$, we deduce that $f = g$, which contradicts inequality (5.9). The proof is complete. □

Let $\mathcal{H}$ be a Hilbert space and let $C(B(\mathcal{H})_1^n, B(\mathcal{H}))$ be the vector space of all continuous functions from the open operatorial unit ball $[B(\mathcal{H})^n]_1$ to $B(\mathcal{H})$. If $f, g \in C(B(\mathcal{H})_1^n, B(\mathcal{H}))$ and $0 < r < 1$, we define

$$\rho_r(f, g) := \sup_{[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_r^-} \left\| f(X_1, \ldots, X_n) - g(X_1, \ldots, X_n) \right\|.$$

Let $0 < r_m < 1$ be such that $\{r_m\}_{m=1}^\infty$ is an increasing sequence convergent to 1. For any $f, g \in C(B(\mathcal{H})_1^n, B(\mathcal{H}))$, we define

$$\rho(f, g) := \sum_{m=1}^\infty \left(\frac{1}{2}\right)^m \frac{\rho_{r_m}(f, g)}{1 + \rho_{r_m}(f, g)}.$$

Based on standard arguments, one can prove that ρ is a metric on $C(B(\mathcal{H})_1^n, B(\mathcal{H}))$. Following the corresponding result (see [11]) for the set of all continuous functions from a set $G \subset \mathbb{C}$ to a metric space Ω, one can easily obtain the following operator version. We leave the proof to the reader.

Lemma 5.4. *If $\epsilon > 0$, then there exist $\delta > 0$ and $m \in \mathbb{N}$ such that for any $f, g \in C(B(\mathcal{H})_1^n, B(\mathcal{H}))$*

$$\sup_{[X_1,\ldots,X_n]\in[B(\mathcal{H})^n]^-_{r_m}} \left\| f(X_1,\ldots,X_n) - g(X_1,\ldots,X_n)\right\| < \delta \quad \Rightarrow \quad \rho(f,g) < \epsilon.$$

Conversely, if $\delta > 0$ and $m \in \mathbb{N}$ are fixed, then there is $\epsilon > 0$ such that for any $f, g \in C(B(\mathcal{H})_1^n, B(\mathcal{H}))$

$$\rho(f,g) < \epsilon \quad \Rightarrow \quad \sup_{[X_1,\ldots,X_n]\in[B(\mathcal{H})^n]^-_{r_m}} \left\| f(X_1,\ldots,X_n) - g(X_1,\ldots,X_n)\right\| < \delta.$$

An immediate consequence of Lemma 5.4 is the following: if $\{f_m\}_{k=1}^\infty$ and f are in $C(B(\mathcal{H})_1^n, B(\mathcal{H}))$, then f_k is convergent to f in the metric ρ if and only if $f_m \to f$ uniformly on any closed ball $[B(\mathcal{H})^n]^-_{r_m}$, $m = 1, 2, \ldots$. This result is needed to prove the following.

Theorem 5.5. $(C(B(\mathcal{H})_1^n, B(\mathcal{H})), \rho)$ *is a complete metric space.*

Proof. Suppose that $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $(C(B(\mathcal{H})_1^n, B(\mathcal{H})), \rho)$. Due to Lemma 5.4, the sequence $\{f_k|_{[B(\mathcal{H})^n]^-_r}\}_{k=1}^\infty$ is Cauchy in $C([B(\mathcal{H})^n]^-_r, B(\mathcal{H}))$. Consequently, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$\sup_{[X_1,\ldots,X_n]\in[B(\mathcal{H})^n]^-_{r}} \left\| f_m(X_1,\ldots,X_n) - f_k(X_1,\ldots,X_n)\right\| < \epsilon \quad \text{for any } k, m \geqslant N. \tag{5.10}$$

In particular, $\{f_k(X_1,\ldots,X_n)\}_{k=1}^\infty$ is a Cauchy sequence in the operator norm of $B(\mathcal{H})$. Therefore, there is an operator $f(X_1,\ldots,X_n) \in B(\mathcal{H})$ such that

$$f(X_1,\ldots,X_n) = \lim_{k\to\infty} f_k(X_1,\ldots,X_n) \tag{5.11}$$

in the operator norm. This gives rise to a function $f : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$. We need to show that $\rho(f_k, f) \to 0$, as $k \to \infty$, and that f is continuous. If $[X_1,\ldots,X_n] \in [B(\mathcal{H})^n]^-_r$, then, due to relations (5.10) and (5.11), there exists $m \geqslant N$ such that

$$\left\| f(X_1,\ldots,X_n) - f_m(X_1,\ldots,X_n)\right\| < \epsilon \quad \text{and} \quad \left\| f(X_1,\ldots,X_n) - f_k(X_1,\ldots,X_n)\right\| < \epsilon$$

for any $k \geqslant N$. Since N does not depend on $[X_1,\ldots,X_n] \in [B(\mathcal{H})^n]^-_r$, we deduce that $\{f_k\}_{k=1}^\infty$ converges to f uniformly on any closed ball $[B(\mathcal{H})^n]^-_r$. Due to Lemma 5.4, this shows that $\rho(f_k, f) \to 0$, as $k \to \infty$. The continuity of f can be proved using standard arguments in the theory of metric spaces. We leave it to the reader. □

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and denote by $Hol(B(\mathcal{H})_1^n)$ the algebra of free holomorphic functions on $[B(\mathcal{H})^n]_1$.

Theorem 5.6. $(Hol(B(\mathcal{H})_1^n), \rho)$ *is a complete metric space and the Hausdorff derivations*

$$\frac{\partial}{\partial Z_i} : \bigl(Hol\bigl(B(\mathcal{H})_1^n\bigr), \rho\bigr) \to \bigl(Hol\bigl(B(\mathcal{H})_1^n\bigr), \rho\bigr), \quad i = 1, \ldots, n,$$

are continuous.

Proof. First, note that Theorem 1.3 implies that $Hol(B(\mathcal{H})_1^n) \subset C(B(\mathcal{H})_1^n, B(\mathcal{H}))$. Due to Theorem 5.5, it is enough to show that $(\mathcal{H}ol(B(\mathcal{H})_1^n), \rho)$ is closed in $(C(B(\mathcal{H})_1^n, B(\mathcal{H})), \rho)$. Let $\{f_m\}_{m=1}^\infty \subset Hol(B(\mathcal{H})_1^n)$ and $f \in C(B(\mathcal{H})_1^n, B(\mathcal{H}))$ be such that $\rho(f_m, f) \to 0$, as $m \to \infty$. Due to Lemma 5.4, $f_m \to f$ uniformly on any closed ball $[B(\mathcal{H})^n]_{r_m}^-$, $m = 1, 2, \ldots$. Applying now Theorem 5.1, we deduce that $f \in Hol(B(\mathcal{H})_1^n)$ and that

$$\frac{\partial f_m}{\partial Z_i} \to \frac{\partial f}{\partial Z_i}$$

uniformly on any closed ball $[B(\mathcal{H})^n]_{r_m}^-$ and, therefore, in the metric ρ. This completes the proof of the theorem. □

Now, Theorem 5.2 implies the following compactness criterion for subsets of $Hol(B(\mathcal{H})_1^n)$.

Corollary 5.7. *A subset $\mathcal{F}$ of $(Hol(B(\mathcal{H})_1^n), \rho)$ is compact if and only if it is closed and locally bounded.*

We return now to the setting of Section 4, where we showed that if $f = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ is a free holomorphic function on the open operatorial unit n-ball and $[T_1, \ldots, T_n] \in B(\mathcal{H})^n$ is any n-tuple of operators with $r(T_1, \ldots, T_n) < 1$, then we can define the bounded linear operator

$$f(T_1, \ldots, T_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha T_\alpha,$$

where the series converges in norm. This provides a *free analytic functional calculus*, which now turns out to be continuous and unique.

Theorem 5.8. *If $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ is any n-tuple of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$ then the mapping $\Phi_T : Hol(B(\mathcal{X})_1^n) \to B(\mathcal{H})$ defined by*

$$\Phi_T(f) := f(T_1, \ldots, T_n)$$

is a continuous unital algebra homomorphism. Moreover, the free analytic functional calculus is uniquely determined by the mapping

$$Z_i \mapsto T_i, \quad i = 1, \ldots, n.$$

Proof. Due to Theorems 4.1 and 1.4, we deduce that Φ_T is a well-defined unital algebra homomorphism. To prove the continuity of Φ_T, let f_m and f be in $Hol(B(\mathcal{X})_1^n)$ such that $f_m \to f$ in the metric ρ of $Hol(B(\mathcal{X})_1^n)$, as $m \to \infty$. Due to Lemma 5.4 and Corollary 3.4, this is equivalent to the fact that, for each $r \in [0, 1)$,

$$f_m(rS_1, \ldots, rS_n) \to f(rS_1, \ldots, rS_n) \quad \text{as } m \to \infty, \tag{5.12}$$

where the convergence is in the operator norm of $B(F^2(H_n))$. We shall prove that

$$\left\| f_m(T_1, \ldots, T_n) - f(T_1, \ldots, T_n) \right\| \to 0 \quad \text{as } m \to \infty. \tag{5.13}$$

Let $f := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ and $f_m := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(m)} Z_\alpha$. Due to Theorem 4.1, the series defining $f_m(T_1,\dots,T_n)$ and $f(T_1,\dots,T_n)$ are norm convergent. Notice that

$$\begin{aligned}\|f_m(T_1,\dots,T_n)-f(T_1,\dots,T_n)\| &= \left\|\sum_{k=0}^\infty \sum_{|\alpha|=k}\bigl(a_\alpha^{(m)}-a_\alpha\bigr)T_\alpha\right\| \leqslant \sum_{k=0}^\infty \left\|\sum_{|\alpha|=k}\bigl(a_\alpha^{(m)}-a_\alpha\bigr)T_\alpha\right\| \\ &\leqslant \sum_{k=0}^\infty \left\|\sum_{|\alpha|=k} T_\alpha T_\alpha^*\right\|^{1/2}\left(\sum_{|\alpha|=k}\bigl|a_\alpha^{m)}-a_\alpha\bigr|^2\right)^{1/2}.\end{aligned}$$

If $r(T_1,\dots,T_n)<\rho<r<1$, then there exists $k_0\in\mathbb{N}$ such that

$$\left\|\sum_{|\alpha|=k} T_\alpha T_\alpha^*\right\|^{1/2} \leqslant \rho^k \quad \text{for any } k\geqslant k_0.$$

According to Theorem 2.1, we have

$$\left(\sum_{|\alpha|=k}\bigl|a_\alpha^{m)}-a_\alpha\bigr|^2\right)^{1/2} \leqslant \frac{1}{r^k}\|f_m(rS_1,\dots,rS_n)-f(rS_1,\dots,rS_n)\|.$$

Combining this with the above inequalities, we obtain

$$\|f_m(T_1,\dots,T_n)-f(T_1,\dots,T_n)\| \leqslant M(T,\rho,r)\|f_m(rS_1,\dots,rS_n)-f(rS_1,\dots,rS_n)\|,$$

where

$$M(T,\rho,r) := \sum_{k=0}^{k_0}\left\|\sum_{|\alpha|=k} T_\alpha T_\alpha^*\right\|^{1/2}\frac{1}{r^k} + \sum_{k=k_0+1}^\infty \left(\frac{\rho}{r}\right)^k.$$

Now, using relation (5.12), we deduce (5.13), which proves the continuity of Φ_T.

To prove the uniqueness of the free analytic functional calculus, let $\Phi : Hol(B(\mathcal{X})_1^n) \to B(\mathcal{H})$ be a continuous unital algebra homomorphism such that $\Phi(Z_i)=T_i$, $i=1,\dots,n$. Hence, we deduce that

$$\Phi_T\bigl(p(Z_1,\dots,Z_n)\bigr) = \Phi\bigl(p(Z_1,\dots,Z_n)\bigr) \tag{5.14}$$

for any polynomial $p(Z_1,\dots,Z_n)$ in $Hol(B(\mathcal{X})_1^n)$. Let $f=\sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ be an element in $Hol(B(\mathcal{X})_1^n)$ and let $p_m := \sum_{k=0}^m \sum_{|\alpha|=k} a_\alpha Z_\alpha$, $m=1,2,\dots$. Since

$$f(rS_1,\dots,rS_n) = \sum_{k=0}^\infty \sum_{|\alpha|=k} r^k a_\alpha S_\alpha$$

and the series $\sum_{k=0}^\infty r^k \|\sum_{|\alpha|=k} a_\alpha S_\alpha\|$ converges due to Theorem 1.5, we deduce that

$$p_m(rS_1,\dots,rS_n) \to f(rS_1,\dots,rS_n)$$

in the operator norm, as $m \to \infty$. Therefore, $p_m \to f$ in the metric ρ of $Hol(B(\mathcal{X})_1^n)$. Hence, using (5.14) and the continuity of Φ and Φ_T, we deduce that $\Phi = \Phi_T$. This completes the proof. $\square$

Using Theorems 3.1, 4.1, and the results from [30] concerning the F_n^∞-functional calculus for row contractions, one can make the following observation.

Remark 5.9. For strict row contractions, i.e., $\|[T_1, \ldots, T_n]\| < 1$, and $F \in H^\infty(B(\mathcal{X})_1^n)$, the free analytic functional calculus $F(T_1, \ldots, T_n)$ coincides with the F_n^∞-functional calculus for row contractions.

Let $\{F_m\}_{m=1}^\infty$ and F be in $Hol(B(\mathcal{X})_1^n)$ and let $\{f_m\}_{m=1}^\infty$ and f be the corresponding representations on $\mathbb{C}$, respectively (see Corollary 1.7). Due to the noncommuting von Neumann inequality, we have

$$\sup_{|\lambda_1|^2+\cdots+|\lambda_n|^2 \leqslant r^2} \left| f_m(\lambda_1, \ldots, \lambda_n) - f(\lambda_1, \ldots, \lambda_n)\right| \leqslant \left\| F_m(rS_1, \ldots, rS_n) - F(rS_1, \ldots, rS_n)\right\|$$

for any $r \in [0, 1)$. Hence, we deduce that if $F_m \to F$ in the metric ρ of $Hol(B(\mathcal{X})_1^n)$, then $f_m \to f$ uniformly on compact subsets of $\mathbb{B}_n$. Since there is a sequence of polynomials $\{p_m\}_{m=1}^\infty$ such that $p_m \to F$ in the metric ρ, one can use the continuity of Taylor's functional calculus and the continuity of the free analytic functional calculus as well as the fact that they coincide on polynomials, to deduce the following result.

Remark 5.10. If f is the representation of a free holomorphic function $F \in Hol(B(\mathcal{X})_1^n)$ on $\mathbb{C}$ and $[T_1, \ldots, T_n] \in B(\mathcal{H})^n$ is an n-tuple of commuting operators with Taylor spectrum $\sigma(T_1, \ldots, T_n) \subset \mathbb{B}_n$, then the free analytic calculus $F(T_1, \ldots, T_n)$ coincides with Taylor's functional calculus $f(T_1, \ldots, T_n)$.

6. Free pluriharmonic functions and noncommutative Poisson transforms

Given an operator $A \in B(F^2(H_n))$, the noncommutative Poisson transform [34] generates a function

$$P[A] : \left[B(\mathcal{H})^n\right]_1 \to B(\mathcal{H}).$$

In this section, we provide classes of operators $A \in B(F^2(H_n))$ such that $P[A]$ is a free holomorphic (respectively pluriharmonic) function on $[B(\mathcal{H})^n]_1$. We characterize the free holomorphic functions u on $[B(\mathcal{H})^n]_1$ such that $u = P[f]$ for some boundary function f in the noncommutative analytic Toeplitz algebra F_n^∞, or the noncommutative disc algebra $\mathcal{A}_n$. We also obtain noncommutative multivariable versions of Herglotz theorem and Dirichlet extension problem (see [11,20]), for free pluriharmonic functions.

We define the operator $K_T(S_1, \ldots, S_n) \in B(F^2(H_n) \otimes \mathcal{H})$ associated with a row contraction $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ by setting

$$K_T(S_1, \ldots, S_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} S_\alpha \otimes \Delta_T T_\alpha^*,$$

where $\Delta_T := (I_{\mathcal{H}} - \sum_{i=1}^n T_i T_i^*)^{1/2}$. Due to Theorem 1.1, when $A_{(\alpha)} := \Delta_T T_\alpha^*$ and $X_i := S_i$, $i = 1, \dots, n$, the above series is convergent in the operator norm if

$$\limsup_{k\to\infty} \left\| \sum_{|\alpha|=k} T_\alpha T_\alpha^* - \sum_{|\alpha|=k+1} T_\alpha T_\alpha^* \right\|^{1/2k} < 1. \tag{6.1}$$

In particular, if $\|[T_1, \dots, T_n]\| < 1$, then relation (6.1) holds and the operator $K_T(S_1, \dots, S_n)$ is in $\mathcal{A}_n \bar{\otimes} B(\mathcal{H})$. Notice also that

$$\left(S_\alpha^* \otimes I_{\mathcal{H}}\right) K_T(S_1, \dots, S_n) = K_T(S_1, \dots, S_n)\left(I_{F^2(H_n)} \otimes T_\alpha^*\right), \quad \alpha \in \mathbb{F}_n^+.$$

Introduced in [34], the noncommutative Poisson transform at $T := [T_1, \dots, T_n]$ is the map $P_T : B(F^2(H_n)) \to B(\mathcal{H})$ defined by

$$\begin{aligned} \left\langle P_T(A)x, y\right\rangle &:= \left\langle K_T(S_1, \dots, S_n)^*(A \otimes I_{\mathcal{H}}) K_T(S_1, \dots, S_n)(1 \otimes x), 1 \otimes y\right\rangle \\ &:= \left\langle K_T^*(A \otimes I_{\mathcal{H}}) K_T x, y\right\rangle \end{aligned}$$

for any $x, y \in B(\mathcal{H})$, where $K_T := K_T(S_1, \dots, S_n)|_{1\otimes\mathcal{H}} : \mathcal{H} \to F^2(H_n) \otimes \mathcal{H}$. We recall that the Poisson kernel K_T is an isometry if $\|T\| < 1$, and

$$p(T_1, \dots, T_n) = K_T^*\left(p(S_1, \dots, S_n) \otimes I_{\mathcal{H}}\right) K_T \tag{6.2}$$

for any polynomial p. We refer to [34–37] for more on noncommutative Poisson transforms on C^*-algebras generated by isometries.

Given an operator $A \in B(F^2(H_n))$, the noncommutative Poisson transform generates a function

$$P[A] : \left[B(\mathcal{H})^n\right]_1 \to B(\mathcal{H})$$

by setting

$$P[A](X_1, \dots, X_n) := P_X(A) \quad \text{for } X := [X_1, \dots, X_n] \in \left[B(\mathcal{H})^n\right]_1.$$

In what follows, we provide classes of operators $A \in B(F^2(H_n))$ such that the mapping $P[A]$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$. In this case, the operator A can be seen as the boundary function of the Poisson transform $P[A]$.

As in the previous sections, we identify $f \in F_n^\infty$ with the multiplication operator $L_f \in B(F^2(H_n))$.

Theorem 6.1. *Let $\mathcal{H}$ be a Hilbert space and u be a free holomorphic function on $[B(\mathcal{H})^n]_1$.*

(i) *There exists $f \in F_n^\infty$ with $u = P[f]$ if and only if $\sup_{0\leqslant r<1} \|u(rS_1, \dots, rS_n)\| < \infty$. In this case, $u(rS_1, \dots, rS_n) \to f$, as $r \to 1$, in the w^*-topology (or strong operator topology).*
(ii) *There exists $f \in \mathcal{A}_n$ with $u = P[f]$ if and only if $\{u(rS_1, \dots, rS_n)\}_{0\leqslant r<1}$ is convergent in norm as, $r \to 1$. In this case, $u(rS_1, \dots, rS_n) \to f$ in the operator norm, as $r \to 1$.*

Proof. To prove (i), assume that $f \in F_n^\infty$ and $u = P[f]$, where f is identified with the multiplication operator $L_f \in B(F^2(H_n))$. Then

$$u(X_1, \ldots, X_n) = K_X^*(L_f \otimes I_{\mathcal{H}})K_X, \quad [X_1, \ldots, X_n] \in \left[B(\mathcal{H})^n\right]_1,$$

and $\|u(X_1, \ldots, X_n)\| \leqslant \|L_f\| = \|f\|_\infty$ for any $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$. In particular,

$$\sup_{0 \leqslant r < 1} \left\|u(rS_1, \ldots, rS_n)\right\| \leqslant \|f\|_\infty < \infty. \tag{6.3}$$

Conversely, assume that $u(X_1, \ldots, X_n) := \sum_{k=0} \sum_{|\alpha|=k} a_\alpha X_\alpha$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$ such that (6.3) holds. By Theorem 3.1, $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ is in F_n^∞. Due to Theorem 1.1, we have that $u_r(X_1, \ldots, X_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha X_\alpha$ is convergent in norm for any $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$ and $r \in [0, 1]$. Similarly, we have that $f_r(S_1, \ldots, S_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha S_\alpha$ is convergent in norm for any $r \in [0, 1)$. Using relation (6.2), we deduce that

$$\sum_{k=0}^m \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha X_\alpha = K_X^* \left(\sum_{k=0}^m \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha S_\alpha \otimes I_{\mathcal{H}} \right) K_X.$$

Taking $m \to \infty$ and using the above convergences, we get

$$u_r(X_1, \ldots, X_n) = K_X^*\big(f_r(S_1, \ldots, S_n) \otimes I_{\mathcal{H}}\big)K_X, \quad r \in [0, 1). \tag{6.4}$$

By Theorem 1.3, we have

$$\lim_{r \to 1} u_r(X_1, \ldots, X_n) = u(X_1, \ldots, X_n)$$

in the operator norm. On the other hand, due to relation (3.2), we have

$$\text{SOT-}\lim_{r \to 1} f_r(S_1, \ldots, S_n) = L_f. \tag{6.5}$$

Since $\|f_r(S_1, \ldots, S_n)\| \leqslant \|f\|_\infty$ and the map $A \mapsto A \otimes I_{\mathcal{H}}$ is SOT-continuous on bounded subsets of $B(F^2(H_n))$, we take $r \to 1$ in relation (6.4) and deduce that $u(X_1, \ldots, X_n) = P_X(f)$ for any $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$. Since $u_r(S_1, \ldots, S_n) = f(rS_1, \ldots, rS_n)$ and the strong operator topology coincides with the w^*-topology on F_n^∞ (see [15]), one can use (6.5) to complete the proof of part (i).

To prove (ii), assume that $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ is in $\mathcal{A}_n$ and $u = P[f]$, i.e.,

$$u(X_1, \ldots, X_n) = K_X^*(L_f \otimes I_{\mathcal{H}})K_X$$

for any $X = [X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$. Due to Theorem 3.2, we have $\lim_{r \to 1} f_r(S_1, \ldots, S_n) = L_f$ in the operator norm. Hence, using relation (6.2) and Theorem 1.3, we deduce that

$$\begin{aligned} K_X^*(L_f \otimes I_{\mathcal{H}})K_X &= \lim_{r \to 1} K_X\big(f_r(S_1, \ldots, S_n) \otimes I_{\mathcal{H}}\big)K_X \\ &= \lim_{r \to 1} f(rX_1, \ldots, rX_n) = f(X_1, \ldots, X_n). \end{aligned}$$

This proves that $u(X_1,\dots,X_n)=f(X_1,\dots,X_n)$ for any $[X_1,\dots,X_n]\in[B(\mathcal{H})^n]_1$. In particular, we deduce that

$$u(rS_1,\dots,rS_n)=f_r(S_1,\dots,S_n)\to L_f \quad \text{as } r\to 1,$$

in the operator norm.

Conversely, assume that $u:=\sum_{k=0}\sum_{|\alpha|=k}a_\alpha Z_\alpha$ is a free holomorphic function on the open operatorial unit n-ball, such that $\{u(rS_1,\dots,rS_n)\}_{0\leqslant r<1}$ is convergent in norm, as $r\to 1$. By Theorem 1.5, we have that $u(rS_1,\dots,rS_n)\in\mathcal{A}_n$. Since $\mathcal{A}_n$ is a Banach algebra, there exists $f\in\mathcal{A}_n$ such that $u(rS_1,\dots,rS_n)\to f$ in norm, as $r\to 1$. Due to Theorem 3.2, we must have $f=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha e_\alpha$. As in the proof of part (i), we have

$$u(X_1,\dots,X_n)=\lim_{r\to 1}f_r(X_1,\dots,X_n)=\lim_{r\to 1}K_X^*\big(f_r(S_1,\dots,S_n)\otimes I_{\mathcal{H}}\big)K_X$$

for any $[X_1,\dots,X_n]\in[B(\mathcal{H})^n]_1$. Now, since $\lim_{r\to 1}f_r(S_1,\dots,S_n)=L_f$ in norm, we deduce that $u=P[f]$. This completes the proof. □

We now turn our attention to a noncommutative generalization of the harmonic functions on the open unit disc $\mathbb{D}$. We say that G is a self-adjoint free pluriharmonic function on $[B(\mathcal{H})^n]_1$ if there exists a free holomorphic function F on $[B(\mathcal{H})^n]_1$ such that

$$G(X_1,\dots,X_n)=\operatorname{Re}F(X_1,\dots,X_n):=\frac{1}{2}\big(F(X_1,\dots,X_n)+F(X_1,\dots,X_n)^*\big).$$

We remark that if $\mathcal{H}$ be an infinite-dimensional Hilbert space, then G determines F up to an imaginary complex number. Indeed, if we assume that $\operatorname{Re}F=0$ and take the representation on the full Fock space $F^2(H_n)$, we obtain $F(rS_1,\dots,rS_n)=-F(rS_1,\dots,rS_n)^*$, $0<r<1$. If $F(rS_1,\dots,rS_n)$ has the representation $\sum_{k=0}^\infty\sum_{|\alpha|=k}r^{|\alpha|}a_\alpha S_\alpha$, $a_\alpha\in\mathbb{C}$, the above relation implies

$$\sum_{k=0}^\infty\sum_{|\alpha|=k}r^{|\alpha|}a_\alpha e_\alpha=F(rS_1,\dots,rS_n)1=-F(rS_1,\dots,rS_n)^*1=-\bar{a}_0.$$

Hence, $a_\alpha=0$ if $|\alpha|\geqslant 1$ and $a_0+\bar{a}_0=0$. Therefore, $F=a_0$, where a_0 is an imaginary complex number. This proves our assertion. Due to Theorem 1.1,

$$G(X_1,\dots,X_n):=\sum_{k=1}^\infty\sum_{|\alpha|=k}\bar{a}_\alpha X_\alpha^*+a_0I+\sum_{k=1}^\infty\sum_{|\alpha|=k}a_\alpha X_\alpha$$

represents a self-adjoint free pluriharmonic function on $[B(\mathcal{H})^n]_1$ if and only if

$$\limsup_{k\to\infty}\bigg(\sum_{|\alpha|=k}|a_\alpha|^2\bigg)^{1/2k}\leqslant 1.$$

If H_1 and H_2 are self-adjoint free pluriharmonic functions on $[B(\mathcal{H})^n]_1$, we say that $H:=H_1+iH_2$ is a free pluriharmonic function on $[B(\mathcal{H})^n]_1$. Notice that any free holomorphic function on $[B(\mathcal{H})^n]_1$ is a free pluriharmonic function. This is due to the fact that $f=\frac{f+f^*}{2}+i\frac{f-f^*}{2i}$.

Proposition 6.2. *Let g be a free pluriharmonic function on the open operatorial n-ball of radius $1+\epsilon$, $\epsilon > 0$. Then*

$$g(X_1, \dots, X_n) = P_X\big(g(S_1, \dots, S_n)\big), \quad X := [X_1, \dots, X_n] \in \big[B(\mathcal{H})^n\big]_1,$$

where P_X is the noncommutative Poisson transform at X. Moreover, if $\mathcal{H}$ is an infinite-dimensional Hilbert space, then $g(S_1, \dots, S_n) \geqslant 0$ if and only if $g(X_1, \dots, X_n) \geqslant 0$ for any $[X_1, \dots, X_n] \in [B(\mathcal{H})^n]_1$.

Proof. Without loss of generality, we can assume that g is a self-adjoint free pluriharmonic function and $g(X_1, \dots, X_n) = f(X_1, \dots, X_n) + f(X_1, \dots, X_n)^*$ for any $[X_1, \dots, X_n] \in [B(\mathcal{H})^n]_{1+\epsilon}$, where the function $f(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha$ is free holomorphic on $[B(\mathcal{H})^n]_{1+\epsilon}$. According to Theorem 1.5, the series $\sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} a_\alpha S_\alpha$ converges in the operator norm for any $r \in [0, 1+\epsilon)$. Due to relation (6.2) and taking limits in the operator norm, we have

$$f(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha = P_X\big[f(S_1, \dots, S_n)\big] \quad \text{and}$$

$$f(X_1, \dots, X_n)^* = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \bar{a}_\alpha X_\alpha^* = P_X\big[f(S_1, \dots, S_n)^*\big].$$

Consequently,

$$g(X_1, \dots, X_n) = P_X\big[g(S_1, \dots, S_n)\big], \quad [X_1, \dots, X_n] \in \big[B(\mathcal{H})^n\big]_1.$$

We prove now the last part of the proposition. One implication is obvious due to the above relation. Conversely, assume that $g(X_1, \dots, X_n) \geqslant 0$ for any $[X_1, \dots, X_n] \in [B(\mathcal{H})^n]_1$. Then, since $\mathcal{H}$ is infinite-dimensional, we deduce that $g(rS_1, \dots, rS_n) \geqslant 0$ for any $r \in [0, 1)$. On the other hand, due to Theorem 1.3, $\lim_{r \to 1} g(rS_1, \dots, rS_n) = g(S_1, \dots, S_n)$ in the operator norm. Hence, $g(S_1, \dots, S_n) \geqslant 0$, and the proof is complete. □

Now, we obtain a noncommutative multivariable version of Herglotz theorem (see [20]).

Theorem 6.3. *Let $f \in (F_n^\infty)^* + F_n^\infty$ and let $u = P[f]$ be its noncommutative Poisson transform. Then u is a free pluriharmonic function on $[B(\mathcal{H})^n]_1$, where $\mathcal{H}$ is a Hilbert space. Moreover, $u \geqslant 0$ on $[B(\mathcal{H})^n]_1$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space, if and only if $f \geqslant 0$.*

Proof. First, notice that, without loss of generality, we can assume that $f = f^*$. Then, one can prove that $f = g^* + g$ for some $g \in F_n^\infty$. Indeed, if $f = h^* + g$ for some $h, g \in F_n^\infty$, the we must have $(g - h)^* = g - h$. Hence, $(g - h)^* 1 = (g - h)1$ and one can easily deduce that $g - h$ is a constant, which proves our assertion. According to Theorem 6.1, $P[g]$ is a free holomorphic function on the open operatorial unit n-ball. On the other hand, due to [39], we have

$$\text{SOT-}\lim_{r \to 1} g_r(S_1, \dots, S_n)^* = L_g^*.$$

Hence, using the properties of the Poisson transform and Theorem 1.3, we deduce that

$$\begin{aligned}\langle P[g^*]x, y\rangle &= \lim_{r\to 1}\langle K_X\big(g_r(S_1,\dots,S_n)^*\otimes I_{\mathcal{H}}\big)K_X x, y\rangle\\ &= \lim_{r\to 1}\langle g_r(X_1,\dots,X_n)^* x, y\rangle\\ &= \langle g(X_1,\dots,X_n)^* x, y\rangle\\ &= \langle P[g]^* x, y\rangle.\end{aligned}$$

Hence, we have $P[g]^* = P[g^*]$. Consequently,

$$u = P[f] = P[g^*] + P[g] = P[g]^* + P[g],$$

which proves that u is a self-adjoint free pluriharmonic function on $[B(\mathcal{H})^n]_1$.

Now, it is clear that if $f \geqslant 0$ then $u = P[f] \geqslant 0$. Conversely, assume that $u(X_1,\dots,X_n) \geqslant 0$ for any $[X_1,\dots,X_n] \in [B(\mathcal{H})^n]_1$. Since $\mathcal{H}$ is an infinite-dimensional Hilbert space $\mathcal{H}$, we deduce that

$$u(rS_1,\dots,rS_n) = g(rS_1,\dots,rS_n)^* + g(rS_1,\dots,rS_n) \geqslant 0, \quad r \in [0,1).$$

Due to Theorem 6.1, we have

$$\text{WOT-}\lim_{r\to 1}\big[g(rS_1,\dots,rS_n)^* + g(rS_1,\dots,rS_n)\big] = L_g^* + L_g \geqslant 0.$$

Under the identification of g with L_g, we deduce $f = g^* + g \geqslant 0$, and complete the proof. □

Here again, we remark that f plays the role of the boundary function from the classical complex analysis.

Our version of the classical Dirichlet extension problem for the unit disc (see [11,20]) is the following extension of Theorem 3.2.

Theorem 6.4. *If $f \in \mathcal{A}_n^* + \mathcal{A}_n$, then $u := P[f]$ is a free pluriharmonic function on the open operatorial unit n-ball such that*:

(i) *u has a continuous extension $\tilde{u}$ to $[B(\mathcal{H})^n]_1^-$ for any Hilbert space $\mathcal{H}$, in the operator norm*;
(ii) *$\tilde{u}(S_1,\dots,S_n) = f$.*

Proof. Without loss of generality, we can assume that f is self-adjoint. As in the proof of Theorem 6.3, one can prove that $f = g^* + g$ for some $g \in \mathcal{A}_n$ and $u := P[f] = P[g]^* + P[g]$ is a self-adjoint pluriharmonic function on the open operatorial unit n-ball. Since $g \in \mathcal{A}_n$, we know that $g_r(S_1,\dots,S_n) \to L_g$ in norm, as $r \to 1$. Consequently,

$$f_r(S_1,\dots,S_n) := g_r(S_1,\dots,S_n)^* + g_r(S_1,\dots,S_n) \to L_f^* + L_f \quad \text{as } r \to 1,$$

in norm. As in the proof of Theorem 6.3, we have

$$u(X_1,\dots,X_n) = f(X_1,\dots,X_n) \quad \text{for } [X_1,\dots,X_n] \in \big[B(\mathcal{H})^n\big]_1.$$

Moreover, $v := P[g]$ is a free holomorphic function such that $v(X_1, \ldots, X_n) = g(X_1, \ldots, X_n)$, for any $[X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$.

For each n-tuple $[Y_1, \ldots, Y_n] \in [B(\mathcal{H})^n]_1^-$, we define

$$\tilde{v}(Y_1, \ldots, Y_n) := \lim_{r \to 1} P_{rY}[g],$$

where $rY := [rY_1, \ldots, rY_n]$. Hence, we have $\tilde{v}(Y_1, \ldots, Y_n) = \lim_{r\to 1} g(rY_1, \ldots, rY_n)$. Now, as in the proof of Theorem 3.2, we deduce that the map $\tilde{v} : [B(\mathcal{H})^n]_1^- \to B(\mathcal{H})$ is a continuous extension of v. Therefore, the map $\tilde{u} := \tilde{v}^* + \tilde{v}$ is a continuous extension of u to $[B(\mathcal{H})^n]_1^-$. To prove (ii), apply part (i) when $\mathcal{H} = F^2(H_n)$ and take into account Theorem 3.2. We obtain

$$\tilde{v}(S_1, \ldots, S_n) = \lim_{r \to 1} g(rS_1, \ldots, rS_n) = g,$$

where we used the identification of g with L_g, and the limit is in the operator norm. Therefore,

$$\tilde{u}(S_1, \ldots, S_n) = \tilde{v}(S_1, \ldots, S_n)^* + \tilde{v}(S_1, \ldots, S_n) = g^* + g = f.$$

This completes the proof. □

Let u and v be two self-adjoint free pluriharmonic functions on $[B(\mathcal{H})^n]_1$. We say that v is the pluriharmonic conjugate of u if $u + iv$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$.

Remark 6.5. The pluriharmonic conjugate of a self-adjoint free pluriharmonic function on $[B(\mathcal{H})^n]_1$ is unique up to an additive real constant.

Proof. Let f be a free holomorphic function on $[B(\mathcal{H})^n]_1$ and $u = \operatorname{Re} f$. Assume that v is a selfadjoint free pluriharmonic function such that $u + iv = g$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$. Hence, we have

$$v = \frac{2g - f - f^*}{2i}. \tag{6.6}$$

Since $v = v^*$, we must have $(g - f = (g - f)^*$, i.e., $\operatorname{Re}(g - f) = 0$. Based on the remarks following Theorem 6.1, we have $g - f = w$, where w is an imaginary complex number. Consequently, relation (6.6), implies $v = \frac{f - f^*}{2i} - iw$. This proves the assertion. □

We remark that if $u = \operatorname{Re} f$ and $f(0)$ is real then $v = \frac{f - f^*}{2i}$ is the unique pluriharmonic conjugate of u such that $v(0) = 0$.

Theorem 6.6. *Let $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ be an n-tuple of operators with joint spectral radius $r(T_1, \ldots, T_n) < 1$. If $f \in H^\infty(B(\mathcal{X})_1^n)$, $u = \operatorname{Re} f$, and $f(0)$ is real, then*

$$\langle f(T_1, \ldots, T_n)x, y\rangle = \big\langle \big(u(S_1, \ldots, S_n) \otimes I_{\mathcal{H}}\big)(1 \otimes x), \big[2C_T(R_1, \ldots, R_n) - I\big](1 \otimes y)\big\rangle$$

for any $x, y \in \mathcal{H}$, where $u(S_1, \ldots, S_n)$ is the boundary function of u.

Proof. Due to Theorem 4.3, we have

$$\begin{aligned}
&\big\langle \big(f(S_1,\dots,S_n)\otimes I_{\mathcal{H}}\big)(1\otimes x), \big[2C_T(R_1,\dots,R_n)-I\big](1\otimes y)\big\rangle\\
&\quad = 2\big\langle \big(f(S_1,\dots,S_n)\otimes I_{\mathcal{H}}\big)(1\otimes x), \big[C_T(R_1,\dots,R_n)\big](1\otimes y)\big\rangle\\
&\qquad - \big\langle \big(f(S_1,\dots,S_n)\otimes I_{\mathcal{H}}\big)(1\otimes x), 1\otimes y\big\rangle\\
&\quad = 2\big\langle f(T_1,\dots,T_n)x, y\big\rangle - f(0)\langle x, y\rangle.
\end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned}
&\big\langle \big(f(S_1,\dots,S_n)^*\otimes I_{\mathcal{H}}\big)(1\otimes x), \big[2C_T(R_1,\dots,R_n)-I\big](1\otimes y)\big\rangle\\
&\quad = \big\langle \big(\overline{f(0)}\otimes I_{\mathcal{H}}\big)(1\otimes x), \big[2C_T(R_1,\dots,R_n)-I\big](1\otimes y)\big\rangle\\
&\quad = \overline{f(0)}\langle x, y\rangle.
\end{aligned}$$

If $f(0)\in\mathbb{R}$, then adding up the above relations, we complete the proof. $\square$

We remark that under the conditions of Theorem 6.6 and using the noncommutative Cauchy transform, one can express the pluriharmonic conjugate of u in terms of u.

In a forthcoming paper [40], we will consider operator-valued Bohr type inequalities for classes of free pluriharmonic functions on the open operatorial unit n-ball with operator-valued coefficients.

7. Hardy spaces of free holomorphic functions

In this section, we define the radial maximal Hardy space $H^p(B(\mathcal{X})_1^n)$, $p\geqslant 1$, and the symmetrized Hardy space $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$, and prove that they are Banach spaces with respect to some appropriate norms. In this setting, we obtain von Neumann type inequalities for n-tuples of operators.

Let F be a free holomorphic function on the open operatorial unit n-ball. The map $\varphi:[0,1)\to B(F^2(H_n))$ defined by $\varphi(r):=F(rS_1,\dots,rS_n)$ is called the *radial boundary function* associated with F. Due to Theorem 1.3, φ is continuous with respect to the operator norm topology of $B(F^2(H_n))$. When $\lim_{r\to 1}\varphi(r)$ exists, in one of the classical topologies of $B(F^2(H_n))$, we call it the *boundary function* of F.

Due to the maximum principle for free holomorphic functions (see Theorem 3.3), we have

$$\big\|\varphi(r)\big\| = \sup\big\|F(X_1,\dots,X_n)\big\|,\quad 0\leqslant r<1,$$

where the supremum is taken over all n tuples of operators $[X_1,\dots,X_n]$ in either one of the following sets $[B(\mathcal{H})^n]_r$, $[B(\mathcal{H})^n]_r^-$, or

$$\big\{[X_1,\dots,X_n]\in B(\mathcal{H})^n:\ \big\|[X_1,\dots,X_n]\big\| = r\big\},$$

where $\mathcal{H}$ is an arbitrary infinite-dimensional Hilbert space. The *radial maximal function* $M_F : [0, 1) \to [0, \infty)$ associated with a free holomorphic function $F \in Hol(B(\mathcal{X})_1^n)$ is defined by

$$M_F(r) := \|\varphi(r)\| = \|F(rS_1, \dots, rS_n)\|.$$

M_F is an increasing continuous function (see the proof of Theorem 3.1). We define the *radial maximal Hardy space* $H^p(B(\mathcal{X})_1^n)$, $p \geqslant 1$, as the set of all free holomorphic functions $F \in Hol(B(\mathcal{X})_1^n)$ such that M_F is in the Lebesgue space $L^p[0, 1]$. Setting

$$\|F\|_p := \|M_F\|_p := \left(\int_0^1 \|F(rS_1, \dots, rS_n)\|^p \, dr \right)^{1/p},$$

it is easy to see that $\|\cdot\|_p$ is a norm on the linear space $H^p(B(\mathcal{X})_1^n)$.

Theorem 7.1. *If $p \geqslant 1$, then the radial maximal Hardy space $H^p(B(\mathcal{X})_1^n)$ is a Banach space.*

Proof. First we prove the result for $p = 1$. Let $\{F_k\}_{k=1}^{\infty} \subset H^1(B(\mathcal{X})_1^n)$ be a sequence such that

$$\sum_{k=1}^{\infty} \|F_k\|_1 \leqslant M < \infty. \tag{7.1}$$

We need to prove that $\sum_{k=1}^{\infty} F_k$ converges in $\|\cdot\|_1$. By (7.1), we have

$$\sum_{k=1}^{m} \int_0^1 \|F_k(rS_1, \dots, rS_n)\| \, dr \leqslant M \quad \text{for any } m \in \mathbb{N}.$$

Using Fatou's lemma, we deduce that the function $\psi(r) := \sum_{k=1}^{\infty} \|F_k(rS_1, \dots, rS_n)\|$ is integrable on $[0, 1]$. Notice that the series $\sum_{k=1}^{\infty} \|F_k(rS_1, \dots, rS_n)\| < \infty$ for any $r \in [0, 1)$. Indeed, assume that there exists $r_0 \in [0, 1)$ such that $\sum_{k=1}^{\infty} \|F_k(r_0S_1, \dots, r_0S_n)\| = \infty$. Since the radial maximal function is increasing, we have

$$\sum_{k=1}^{\infty} \|F_k(rS_1, \dots, rS_n)\| \geqslant \sum_{k=1}^{\infty} \|F_k(r_0S_1, \dots, r_0S_n)\| = \infty$$

for any $r \in [r_0, 1)$. Hence, we deduce that

$$\int_0^1 \sum_{k=1}^{\infty} \|F_k(rS_1, \dots, rS_n)\| \, dr \geqslant (1 - r_0) \sum_{k=1}^{\infty} \|F_k(r_0S_1, \dots, r_0S_n)\| = \infty,$$

which contradicts the fact that ψ is integrable on $[0, 1]$. Therefore, we deduce that the series $\sum_{k=1}^{\infty} \|F_k(rS_1, \dots, rS_n)\|$ is convergent for any $r \in [0, 1)$. Hence, $\sum_{k=1}^{\infty} F_k(rS_1, \dots, rS_n)$

is convergent in the operator norm of $B(F^2(H_n))$ for each $r \in [0, 1)$. For each $m \geqslant 1$, define $g_m := \sum_{k=1}^{m} F_k$. Since $\{g_m\}_{m=1}^{\infty}$ is a sequence of free holomorphic functions such that $\{g_m(rS_1, \ldots, rS_n)\}_{m=1}^{\infty}$ is convergent in norm for each $r \in [0, 1)$, we deduce that $\{g_m\}_{m=1}^{\infty}$ is uniformly convergent on any closed operatorial ball $[B(\mathcal{X})^n]_r^-$, $r \in [0, 1)$. According to our noncommutative Weierstrass type result, Theorem 5.1, there is a free holomorphic function g on the open operatorial unit n-ball such that $\|g_m(rS_1, \ldots, rS_n) - g(rS_1, \ldots, rS_n)\| \to 0$, as $m \to \infty$, and therefore

$$g(rS_1, \ldots, rS_n) = \sum_{k=1}^{\infty} F_k(rS_1, \ldots, rS_n) \quad \text{for any } r \in [0, 1).$$

Moreover, due to the fact that ψ is integrable, we have

$$\int_0^1 \|g(rS_1, \ldots, rS_n)\| \, dr \leqslant \int_0^1 \sum_{k=1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| \, dr < \infty,$$

which shows that $g \in H^1(B(\mathcal{X})_1^n)$. Now, notice that

$$\begin{aligned} \|g - g_m\|_1 &= \int_0^1 \|g(rS_1, \ldots, rS_n) - g_m(rS_1, \ldots, rS_n)\| \, dr \\ &= \int_0^1 \left\| \sum_{k=m+1}^{\infty} F_k(rS_1, \ldots, rS_n) \right\| dr \\ &\leqslant \int_0^1 \sum_{k=m+1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| \, dr. \end{aligned}$$

Since $\sum_{k=1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| < \infty$, we have

$$\lim_{m \to \infty} \sum_{k=m+1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| = 0 \quad \text{for any } r \in [0, 1).$$

On the other hand, $\sum_{k=m+1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| \leqslant \psi(r)$ for any $m \in \mathbb{N}$. Since ψ is integrable on $[0, 1]$, we can apply Lebesgue's dominated convergence theorem and deduce that

$$\lim_{m \to \infty} \int_0^1 \sum_{k=m+1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| \, dr = 0.$$

Now, we deduce that $\|g - g_m\|_1 \to 0$, as $m \to \infty$, which shows that the series $\sum_{k=1}^{\infty} F_k$ is convergent in $\| \cdot \|_1$. This completes the proof when $p = 1$.

Assume now that $p > 1$ and let $\{F_k\}_{k=1}^{\infty} \subset H^p((B(\mathcal{X})_1^n)$ be a sequence such that $\sum_{k=1}^{\infty} \|F\|_p \leqslant M < \infty$. Since $\|F_k\|_1 \leqslant \|F_k\|_p$, we have $\sum_{k=1}^{\infty} \|F\|_1 \leqslant M$. Applying the first part of the proof, we find $g \in H^1(B(\mathcal{X})_1^n)$ such that, for each $r \in [0, 1)$,

$$g(rS_1, \ldots, rS_n) = \sum_{k=1}^{\infty} F_k(rS_1, \ldots, rS_n),$$

where the convergence is in the operator norm of $B(F^2(H_n))$. Moreover, we have

$$\begin{aligned}
\int_0^1 \left\| \sum_{k=1}^{m} F_k(rS_1, \ldots, rS_n) \right\|^p dr &\leqslant \int_0^1 \left(\sum_{k=1}^{m} \left\| F_k(rS_1, \ldots, rS_n) \right\| \right)^p dr \\
&\leqslant \left[\sum_{k=1}^{m} \left(\int_0^1 \left\| F_k(rS_1, \ldots, rS_n) \right\|^p \right)^{1/p} \right]^p \\
&= \left(\sum_{k=1}^{m} \|F_k\|_p \right)^p \leqslant M^p.
\end{aligned}$$

Using Fatou's lemma, we deduce that the function $r \mapsto \| \sum_{k=1}^{\infty} F_k(rS_1, \ldots, rS_n)\|^p$ is integrable on [0, 1] and therefore $g \in H^p(B(\mathcal{X})_1^n)$. Notice also that

$$\|g - g_m\|_p \leqslant \left[\int_0^1 \left(\sum_{k=m+1}^{\infty} \left\| F_k(rS_1, \ldots, rS_n) \right\| \right)^p \right]^{1/p}. \tag{7.2}$$

Since $\sum_{k=m+1}^{\infty} \|F_k(rS_1, \ldots, rS_n)\| \leqslant \psi$ for any $m \in \mathbb{N}$, and

$$\lim_{m \to \infty} \sum_{k=m+1}^{\infty} \left\| F_k(rS_1, \ldots, rS_n) \right\| = 0 \quad \text{for any } r \in [0, 1),$$

we can apply again Lebesgue's dominated convergence theorem and deduce that

$$\lim_{m \to \infty} \left[\int_0^1 \left(\sum_{k=m+1}^{\infty} \left\| F_k(rS_1, \ldots, rS_n) \right\| \right)^p \right]^{1/p} = 0.$$

Hence and using inequality (7.2), we deduce that $\|g - g_m\|_p \to 0$ as $m \to \infty$. Consequently, the series $\sum_{k=1}^{\infty} F_k$ converges in the norm $\| \cdot \|_p$. This completes the proof. □

Proposition 7.2. *Let $p \geqslant 1$.*

(i) *If $f \in H^{\infty}(B(\mathcal{X})_1^n)$, then $\|f\|_1 \leqslant \|f\|_p \leqslant \|f\|_{\infty}$. Moreover,*

$$H^{\infty}\big(B(\mathcal{X})_1^n\big) \subset H^p\big(B(\mathcal{X})_1^n\big) \subset H^1\big(B(\mathcal{X})_1^n\big) \subset Hol\big(B(\mathcal{X})_1^n\big).$$

(ii) *If* $f \in H^\infty(B(\mathcal{X})^n_1)$, *then*

$$\|f\|_\infty = \lim_{p\to\infty}\left(\int_0^1 \|f(rS_1,\ldots,rS_n)\|^p\,dr\right)^{1/p}.$$

(iii) *If* $f = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ *is in* $H^p(B(\mathcal{X})^n_1)$, *then*

$$\left(\sum_{|\alpha|=k} |a_\alpha|^2\right)^{1/2} \leqslant (pk+1)^{1/p}\|f\|_p.$$

Proof. Part (i) follows as in the classical theory of L^p spaces. To prove (ii), define the function $G:[0,1]\to[0,\infty)$ by setting $G(r) := \|f(rS_1,\ldots,rS_n)\|$ if $r\in[0,1)$ and $G(1) := \lim_{r\to 1}\|f(rS_1,\ldots,rS_n)\|$. Due to Theorem 3.1, G is an increasing continuous function and $G(1) = \|f\|_\infty$. Therefore,

$$\lim_{p\to\infty}\left(\int_0^1 \|f(rS_1,\ldots,rS_n)\|^p\,dr\right)^{1/p} = \lim_{p\to\infty}\left(\int_0^1 G(r)^p\right)^{1/p} = \max_{r\in[0,1]} G(r) = G(1) = \|f\|_\infty.$$

To prove (iii), notice that Theorem 2.1 implies

$$r^k\left(\sum_{|\alpha|=k}|a_\alpha|^2\right)^{1/2} \leqslant \|f(rS_1,\ldots,rS_n)\|,\quad r\in[0,1).$$

Integrating over [0, 1], we complete the proof of (iii). □

The next result extends the noncommutative von Neumann inequality from $H^\infty(B(\mathcal{X})^n_1)$ to the radial maximal Hardy space $H^p(B(\mathcal{X})^n_1)$, $p\geqslant 1$.

Theorem 7.3. *If* $T := [T_1,\ldots,T_n] \in [B(\mathcal{H})^n]_1$ *and* $p\geqslant 1$, *then the mapping*

$$\Psi_T : H^p\big(B(\mathcal{X})^n_1\big) \to B(\mathcal{H}) \text{ defined by } \Psi_T(f) := f(T_1,\ldots,T_n)$$

is continuous, where $f(T_1,\ldots,T_n)$ *is defined by the free analytic functional calculus and* $B(\mathcal{H})$ *is considered with the operator norm topology. Moreover,*

$$\|f(T_1,\ldots,T_n)\| \leqslant \frac{1}{(1-\|[T_1,\ldots,T_n]\|)^{1/p}}\|f\|_p$$

for any $f\in H^p(B(\mathcal{X})^n_1)$.

Proof. Assume that $\|[T_1,\ldots,T_n]\| = r_0 < 1$ and let $f\in H^p(B(\mathcal{X})^n_1)$. Since the radial maximal function is increasing and due to Corollary 3.4, we have

$$\|f\|_p \geqslant \left(\int_{r_0}^{1} \|f(rS_1, \dots, rS_n)\|^p \, dr\right)^{1/p} \geqslant (1-r_0)^{1/p} \|f(r_0S_1, \dots, r_0S_n)\|$$
$$\geqslant (1-r_0)^{1/p} \|f(T_1, \dots, T_n)\|.$$

Hence, we deduce the above von Neumann type inequality, which can be used to prove the continuity of Ψ_T. □

We remark that if $f \in H^\infty(B(\mathcal{X})_1^n)$, then one can recover the noncommutative von Neumann inequality [29] for strict row contractions, i.e., $\|f(T_1, \dots, T_n)\| \leqslant \|f\|_\infty$. Indeed, take $p \to \infty$ in the above inequality and use part (ii) of Proposition 7.2.

In the last part of this paper, we introduce a Banach space of analytic functions on the open unit ball of $\mathbb{C}^n$ and obtain a von Neumann type inequality in this setting. We use the standard multi-index notation. Let $\mathbf{p} := (p_1, \dots, p_n)$ be a multi-index in $\mathbb{Z}_+^n$. We denote $|\mathbf{p}| := p_1 + \dots + p_n$ and $\mathbf{p}! := p_1! \cdots p_n!$. If $\lambda := (\lambda_1, \dots, \lambda_n)$, then we set $\lambda^{\mathbf{p}} := \lambda_1^{p_1} \cdots \lambda_n^{p_n}$ and define the symmetrized functional calculus

$$(\lambda^{\mathbf{p}})_{\mathrm{sym}}(S_1, \dots, S_n) := \frac{\mathbf{p}!}{|\mathbf{p}|!} \sum_{\alpha \in \Lambda_{\mathbf{p}}} S_\alpha,$$

where

$$\Lambda_{\mathbf{p}} := \{\alpha \in \mathbb{F}_n^+ \colon \lambda_\alpha = \lambda^{\mathbf{p}} \text{ for any } \lambda \in \mathbb{B}_n\}$$

and $S_1, \dots, S_n$ are the left creation operators on the Fock space $F^2(H_n)$. Notice that card $\Lambda_{\mathbf{p}} = \frac{|\mathbf{p}|!}{\mathbf{p}!}$. Denote by $H_{\mathrm{sym}}(\mathbb{B}_n)$ the set of all analytic functions on $\mathbb{B}_n$ with scalar coefficients

$$f(\lambda_1, \dots, \lambda_n) := \sum_{\mathbf{p} \in \mathbb{Z}_+^{\mathbf{n}}} \lambda^{\mathbf{p}} a_{\mathbf{p}}, \quad a_{\mathbf{p}} \in \mathbb{C},$$

such that

$$\limsup_{k \to \infty} \left(\sum_{\mathbf{p} \in \mathbb{Z}_+^n,\, |\mathbf{p}| = k} \frac{|\mathbf{p}|!}{\mathbf{p}!} |a_{\mathbf{p}}|^2 \right)^{1/2k} \leqslant 1. \tag{7.3}$$

Then

$$f_{\mathrm{sym}}(rS_1, \dots, rS_n) := \sum_{k=0}^{\infty} \sum_{\mathbf{p} \in \mathbb{Z}_+^n,\, |\mathbf{p}| = k} r^k a_{\mathbf{p}} \left[(\lambda^{\mathbf{p}})_{\mathrm{sym}}(S_1, \dots, S_n)\right]$$
$$= \sum_{k=0}^{\infty} \sum_{|\alpha| = k} r^{|\alpha|} c_\alpha S_\alpha,$$

where $c_0 := a_0$ and $c_\alpha := \frac{\mathbf{p}!}{|\mathbf{p}|!} a_{\mathbf{p}}$ for $\mathbf{p} \in \mathbb{Z}_+^n$, $\mathbf{p} \neq (0, \dots, 0)$, and $\alpha \in \Lambda_{\mathbf{p}}$. It is clear that, for each $k = 1, 2, \dots,$ we have

$$\sum_{|\alpha|=k} |c_\alpha|^2 = \sum_{\mathbf{p}\in\mathbb{Z}_+^n,\, |\mathbf{p}|=k} \left(\sum_{\alpha\in\Lambda_\mathbf{p}} |c_\alpha|^2\right) = \sum_{\mathbf{p}\in\mathbb{Z}_+^n,\, |\mathbf{p}|=k} \frac{\mathbf{p}!}{|\mathbf{p}|!} |a_\alpha|^2.$$

Due to Theorem 1.1, condition (7.3) implies that $f_{\mathrm{sym}}(rS_1, \ldots, rS_n)$ is norm convergent for each $r \in [0, 1)$, and $f_{\mathrm{sym}}(Z_1, \ldots, Z_n)$ is a free holomorphic function on the open operatorial unit n-ball.

We define $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$ as the set of all functions $f \in H_{\mathrm{sym}}(\mathbb{B}_n)$ such that

$$\|f\|_{\mathrm{sym}} := \sup_{0\leqslant r<1} \left\| f_{\mathrm{sym}}(rS_1, \ldots, rS_n)\right\| < \infty.$$

Theorem 7.4. $(H^\infty_{\mathrm{sym}}(\mathbb{B}_n), \|\cdot\|_{\mathrm{sym}})$ *is a Banach space.*

Proof. First notice that if $f \in H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$ then $f_{\mathrm{sym}}(rS_1, \ldots, rS_n)$ is norm convergent and $f_{\mathrm{sym}}(Z_1, \ldots, Z_n)$ is a free holomorphic function on the open operatorial unit n-ball. Using Theorem 1.4, it is easy to see that $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$ is a vector space and $\|\cdot\|_{\mathrm{sym}}$ is a norm. Let $\{f_m\}_{m=1}^\infty$ be a Cauchy sequence of functions in $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$. According to Theorem 3.1, $(f_m)_{\mathrm{sym}} \in F_n^\infty$ and $\{(f_m)_{\mathrm{sym}}\}_{m=1}^\infty$ is a Cauchy sequence in $\|\cdot\|_\infty$, the norm of the Banach algebra F_n^∞. Therefore, there exists $g \in F_n^\infty$ such that $\|(f_m)_{\mathrm{sym}} - L_g\|_\infty \to 0$, as $m \to \infty$. If $f(\lambda_1, \ldots, \lambda_n) = \sum_{\mathbf{p}\in\mathbb{Z}_+^\mathbf{n}} a_\mathbf{p}^{(m)} \lambda^\mathbf{p}$, $a_\mathbf{p} \in \mathbb{C}$, then $(f_m)_{\mathrm{sym}}(S_1, \ldots, S_n) = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha^{(m)} S_\alpha$, where $c_\alpha^{(m)} := \frac{|\mathbf{p}|!}{\mathbf{p}!} a_\mathbf{p}^{(m)}$ for $\mathbf{p} \in \mathbb{Z}_+^n$, $\mathbf{p} \neq (0, \ldots, 0)$ and $\alpha \in \Lambda_\mathbf{p}$. If $g = \sum_{\alpha\in\mathbb{F}_n^+} b_\alpha e_\alpha$ is the Fourier representation of g as an element of $F^2(H_n)$, then we have

$$\left|c_\alpha^{(m)} - b_\alpha\right| = \left|\left\langle\left[(f_m)_{\mathrm{sym}}(S_1, \ldots, S_n) - L_g\right]1, 1\right\rangle\right| \leqslant \left\|(f_m)_{\mathrm{sym}} - L_g\right\|_\infty.$$

Taking $m \to \infty$, we deduce that $c_\alpha^{(m)} \to b_\alpha$ for each $\alpha \in \mathbb{F}_n^+$. Since $c_\alpha^{(m)} = c_\beta^{(m)}$ for any $\alpha, \beta \in \Lambda_\mathbf{p}$, we get $b_\alpha = b_\beta$. Setting $h(\lambda_1, \ldots, \lambda_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} b_\alpha \lambda_\alpha$, one can see that h is holomorphic in $\mathbb{B}_n$ and $h_{\mathrm{sym}} = L_g$. Moreover, $\|h\|_{\mathrm{sym}} = \|g\|_\infty < \infty$. This shows that $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$ is a Banach space. □

Now, using Theorem 4.1 in the scalar case, we can deduce the following.

Proposition 7.5. *If* $T := [T_1, \ldots, T_n] \in B(\mathcal{H})^n$ *is a commuting n-tuple of operators with the joint spectral radius* $r(T_1, \ldots, T_n) < 1$ *and* $f(\lambda_1, \ldots, \lambda_n) := \sum_{\mathbf{p}\in\mathbb{Z}_+^\mathbf{n}} a_\mathbf{p} \lambda^\mathbf{p}$ *is in* $H_{\mathrm{sym}}(\mathbb{B}_n)$*, then*

$$f(T_1, \ldots, T_n) := \sum_{k=0}^\infty \sum_{\mathbf{p}\in\mathbb{Z}_+^n,\, |\mathbf{p}|=k} a_\mathbf{p} T^\mathbf{p}$$

is a well-defined operator in $B(\mathcal{H})$*, where the series is convergent in the operator norm topology. Moreover, the map*

$$\Psi_T : H_{\mathrm{sym}}(\mathbb{B}_n) \to B(\mathcal{H}), \quad \Psi_T(f) = f(T_1, \ldots, T_n),$$

is continuous and

$$\left\| f(T_1, \ldots, T_n)\right\| \leqslant M\|f\|_{\mathrm{sym}},$$

where $M = \sum_{k=0}^{\infty} \| \sum_{|\alpha|=k} T_\alpha T_\alpha^* \|^{1/2}$.

In a forthcoming paper [40], we obtain operator-valued Bohr type inequalities for the Banach space $H^\infty_{\mathrm{sym}}(\mathbb{B}_n)$.

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