A Leibniz variety with almost polynomial growth

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Abstract

Let $F$ be a field of characteristic zero. In this paper we study the variety of Leibniz algebras $\tilde{V}_1$ defined by the identity $y_1(y_2y_3)(y_4y_5) \equiv 0$. We give a complete description of the space of multilinear identities in the language of Young diagrams through the representation theory of the symmetric group. As an outcome we show that the variety $\tilde{V}_1$ has almost polynomial growth, i.e., the sequence of codimensions of $\tilde{V}_1$ cannot be bounded by any polynomial function but any proper subvariety of $\tilde{V}_1$ as polynomial growth.

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1. Introduction

A Leibniz algebra $L$ over a field $F$ is a nonassociative algebra with multiplication

$(-,-): L \times L \rightarrow L$,

where $(-,-)$ is a bilinear form satisfying the Leibniz identity

$(x, (y, z)) = ((x, y), z) - ((x, z), y)$.

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In other words, the operator of right multiplication \((-, z)\) is a derivation of the algebra. Notice that in case \((-,-)\) is skew-symmetric, the above identity is equivalent to the classical Jacobi identity.

The free Leibniz algebra was described by Loday and Pirashvili [14] and in [7] the systematic study of polynomial identities of Leibniz algebras was started.

Let \(F\) be a field of characteristic zero and let \(A\) be an \((associative, Lie or Leibniz)\) \(F\)-algebra. It is well known that in characteristic zero all polynomial identities of \(A\) are completely determined by the multilinear ones. Hence let us denote by \(\text{Id}(A)\) the ideal of the free (associative, Lie or Leibniz) algebra of polynomial identities of \(A\) and by \(P_n\) the space of multilinear polynomials of degree \(n\) in the noncommutative variables \(y_1, \ldots, y_n\) over \(F\).

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Recently in [1] the authors studied the variety of Leibniz algebras determined by the identity $x(y(zt)) \equiv 0$. They proved that such variety has many properties similar to those of the variety of abelian-by-nilpotent class 2 Lie algebras. In particular both varieties have overexponential growth of the codimensions and subexponential growth of the colengths.

In this paper we shall extensively study the variety of Leibniz algebras $\tilde{V}_1$ defined by the single polynomial identity

$$y_1(y_2y_3)(y_4y_5) \equiv 0.$$

We shall study in detail the $S_n$-module structure of $P_n(\tilde{V}_1)$ by determining all multiplicities in the $n$th cocharacter of $\tilde{V}_1$. As a reward we shall be able to prove that the variety $\tilde{V}_1$ has almost polynomial growth.

We remark that the variety $\tilde{V}_1$ shares many properties with the solvable variety of Lie algebras defined by the identity $(x_1x_2)(x_3x_4)(x_5x_6) \equiv 0$. In the search for a classification of the varieties of Leibniz algebras of almost polynomial growth, one could be inspired by the analogous varieties of Lie algebras. But so far $\tilde{V}_1$ is the only example explicitly constructed.

2. The algebra $UT_2$

Throughout $F$ will be a field of characteristic zero and $F(Y)$ the free associative algebra on the countable set $Y = \{y_1, y_2, \ldots\}$. Recall that an algebra $A$ is a PI-algebra if it satisfies a non-trivial polynomial identity. Also, if $f$ is a polynomial identity of $A$ we usually write $f \equiv 0$ on $A$.

The set $Id(A) = \{ f \in F(Y) \mid f \equiv 0 \text{ in } A \}$ of all identities of $A$ is a $T$-ideal of $F(Y)$ i.e., an ideal invariant under all endomorphisms of $F(Y)$. If $\mathcal{V}$ is a variety of algebras, $\mathcal{V}$ determines uniquely a $T$-ideal $I = Id(\mathcal{V})$ and, in case $\mathcal{V}$ is generated by the algebra $A$, we write $\mathcal{V} = \text{var}(A) = \text{var}(I)$ and $I = Id(A) = Id(\mathcal{V})$.

Let $I = Id(\mathcal{V})$ be the $T$-ideal of $F(Y)$ of identities of $\mathcal{V}$, then $F(Y)/I$ is the relatively free algebra of $\mathcal{V}$ and $c_n(\mathcal{V})$ measures the space of multilinear polynomials in the first $n$ variables of $F(Y)/I$; hence, if $\mathcal{V} = \text{var}(A)$, $c_n(A) = c_n(\text{var}(A)) = c_n(\mathcal{V})$.

Let $e_{ij}$ be the usual matrix units and let $UT_2 = UT_2(F) = Fe_{11} + Fe_{12} + Fe_{22}$ denote the algebra of $2 \times 2$ upper triangular matrices over $F$. In the theory of associative algebras, $UT_2$ and the infinite dimensional Grassmann algebra $G$ play a basic role. In [11] Kemer characterized the varieties of associative algebras $\mathcal{V}$ having polynomial growth. He showed that $\mathcal{V}$ has such property if and only if $G \notin \mathcal{V}$ and $UT_2 \notin \mathcal{V}$. The sequences of codimensions of the algebras $G$ and $UT_2$ are well known (see [12,13]) and, as a consequence of Kemer’s result, it follows that there exists no variety of associative algebras with intermediate growth between polynomial and exponential. Also, the two algebras $G$ and $UT_2$ generate the only two varieties of associative algebras with almost polynomial growth.

We shall next recall some properties of the algebra $UT_2$. It is well known that the polynomial $[y_1, y_2][y_3, y_4]$ generates $Id(UT_2)$, the $T$-ideal of identities of $UT_2$. (see [15]). The description of the structure of the multilinear part of $UT_2$ follows, for example, from [4].
If $\chi_1$ is an $S_n$-character and $\chi_2$ is an $S_m$-character, let $\chi_1 \otimes \chi_2$ denote their Kronecker product. Recall that such product can be computed by the Littlewood–Richardson rule (see [10, Theorem 2.8.2]). Let us recall the following result which allows to compute the $n$-cocharacter of a product of $T$-ideals.

**Theorem 2.1** (Berele and Regev [4, Theorem 1.1]). Let $A$, $A_1$, $A_2$ be associative PI-algebras such that $Id(A) = Id(A_1)Id(A_2)$. Then

$$\chi_n(A) = \chi_n(A_1) + \chi_n(A_2) + \sum_{j=0}^{n-1} \chi_j(A_1) \otimes \chi_{n-j}(A_2)$$

$$- \sum_{j=0}^{n} \chi_j(A_1) \otimes \chi_{n-j}(A_2).$$

By using Theorem 2.1 it is easy to compute the $S_n$-cocharacter of $Id(UT_2)$. Write

$$\chi_n(UT_2) = \sum_{\lambda\vdash n} m_{\lambda} \chi_{\lambda},$$

(2.1)

where $\chi_{\lambda}$ is the irreducible $S_n$-character associated to $\lambda$ and $m_{\lambda} \geq 0$ is the corresponding multiplicity. For any partition $\lambda\vdash n$ we denote by $h(\lambda)$ the height of the diagram associated to $\lambda$. We have the following:

**Theorem 2.2.** Let $\chi_n(UT_2)=\sum_{\lambda\vdash n} m_{\lambda} \chi_{\lambda}$ be the $n$th cocharacter of $UT_2$. Then $m_{\lambda} = q + 1$ if either

1. $\lambda = (p + q, p)$ for all $p \geq 1$, $q \geq 0$, or
2. $\lambda = (p + q + 1, p + 1, 1)$ for all $p \geq 0$, $q \geq 0$.

In all other cases $m_{\lambda} = 0$, except the case $m_{(n)} = 1$.

**Proof.** Since $\text{dim}_F UT_2 = 3$ any polynomial alternating on four variables vanishes on $UT_2$. It follows that if $m_{\lambda} \neq 0$ then $h(\lambda) \leq 3$. Since $[y_1, y_2][y_3, y_4] \equiv 0$ is a basis of $Id(UT_2)$ then $Id(UT_2(\text{F})) = Id(\text{F})Id(\text{F})$. Since $\chi_n(\text{F}) = \chi_{(n)}$, by applying Theorem 2.1 we obtain

$$\chi_n(UT_2(\text{F})) = 2\chi_{(n)} + \chi_{(1)} \otimes \sum_{j=0}^{n-1} \chi_{(j)} \otimes \chi_{(n-j-1)} - \sum_{j=0}^{n} \chi_{(j)} \otimes \chi_{(n-j)}.$$  

(2.2)

From this we deduce that $m_{(n)} = 2 + n - n - 1 = 1$. Moreover the irreducible character corresponding to $\lambda = (p + q, p)\vdash n$ appears in the right end side of (2.2) only in the sum

$$\chi_{(1)} \otimes \sum_{j=p+1}^{p+q} \chi_{(j)} \otimes \chi_{(n-j-1)} + \sum_{j=p}^{p+q-1} \chi_{(j)} \otimes \chi_{(n-j-1)} - \sum_{j=p}^{p+q} \chi_{(j)} \otimes \chi_{(n-j)}$$

and $m_{(p+q, p)} = ((p + q) - (p - 1) + 1) + ((p + q - 1) - p + 1) - ((p + q) - p + 1) = q + 1.$
From (2.2) it also follows that the irreducible characters corresponding to \( \hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) \parallel n \) appear only if \( \hat{\lambda}_3 = 1 \) and in this case the only sum involving these terms is

\[
\chi_{(1)} \otimes \sum_{j=\hat{\lambda}_2}^{\hat{\lambda}_1} \chi_{(j)} \otimes \chi_{(n-j)}.
\]

It follows that \( m(\hat{\lambda}_1, \hat{\lambda}_2, 1) = \hat{\lambda}_1 - \hat{\lambda}_2 + 1 \) and the proof is complete. \( \square \)

In the next corollary we shall compute the \( n \)th codimension and the \( n \)th colength of \( UT_2 \) for all \( n \geq 1 \). We recall that if \( \chi_n(A) = \sum_{\lambda \vdash r} m_{\lambda} \chi_{\lambda} \) is the decomposition of the \( n \)th cocharacter of \( A \), then the \( n \)th colength of \( A \) is defined as

\[
l_n(A) = \sum_{\lambda \vdash r} m_{\lambda}.
\]

**Corollary 2.3.** (1) \( c_n(UT_2) = 2^{n-1}(n-2) + 2 \).

(2) \( l_n(UT_2(F)) = \frac{1}{2}n^2 + \frac{5}{2}n + 4 \).

**Proof.** Let us consider the set of polynomials in \( P_n \) of the type

\[
y_{i_1} \cdots y_{i_m}[y_{k}, y_{j_1}, \ldots, y_{j_{n-m-1}}],
\]

(2.3)

where \( \{i_1, \ldots, i_m, j_1, \ldots, j_{n-m-1}, k\} = \{1, 2, \ldots, n\} \), and \( i_1 < \cdots < i_m, j_1 < \cdots < j_{n-m-1}, k > j_1, m \neq n-1 \) (see [6]). Here we are using the left normed notation for the Lie commutators \([y_1, y_2], \ldots, y_n] = [y_1, \ldots, y_n] \), where \([a, b] = ab - ba \). It is well known that such polynomials are a basis of \( P_n \) modulo \( P_n \cap Id(UT_2) \).

We now count for any fixed \( n \), the total number of elements (2.3) i.e., the \( n \)th codimension \( c_n(UT_2) \). If \( 0 \leq m \leq n - 2 \) then this number is equal to

\[
\binom{n}{m} (n - m - 1) = \binom{n}{n-m} (n-m-1).
\]

In case \( m = n \) we have exactly one monomial \( y_1 \cdots y_n \). Hence

\[
c_n(UT_2) = \sum_{j=2}^{n} \binom{n}{j} (j - 1) + 1 = \sum_{j=2}^{n} j \binom{n}{j} - \sum_{j=2}^{n} \binom{n}{j} + 1
\]

\[
= n2^{n-1} - \binom{n}{1} - 2^n + \binom{n}{1} + \binom{n}{0} + 1
\]

\[
= n2^{n-1} - 2^n + 2 = 2^{n-1}(n-2) + 2.
\]

This prove (1).
Let us now prove (2). From Theorem 1 we have

\[ l_n(UT_2(F)) = \sum_{\lambda \geq n} m_\lambda \]
\[ = m(n) + \sum_{\lambda_1 + \lambda_2 = n} m(\lambda_1, \lambda_2) + \sum_{\lambda_1 + \lambda_2 = n-1} m(\lambda_1, \lambda_2, 1) \]
\[ = 1 + \sum_{\lambda_1 + \lambda_2 = n} (\lambda_1 - \lambda_2 + 1) + \sum_{\lambda_1 + \lambda_2 = n-1} (\lambda_1 - \lambda_2 + 1) \]
\[ = 1 + \sum_{\lambda_1 = n/2} n (\lambda_1 - (n - \lambda_1) + 1) + \sum_{\lambda_1 = n/2} n (\lambda_1 - (n - 1 - \lambda_1) + 1) \]
\[ = 1 + \frac{1}{4}(n + 2)2 + \frac{1}{4}(n + 4)(n + 2) = \frac{1}{2}n^2 + \frac{5}{4}n + 4. \]

We complete this section by recalling the following well known result [11]. □

**Theorem 2.4.** If \( \mathcal{V} \) is a variety of associative algebras such that \( \mathcal{V} \subset \text{var}(UT_2) \), then \( \mathcal{V} \) has polynomial growth.

### 3. Leibniz algebras

Recall that a Leibniz algebra \( L \) over a field \( F \) is a non associative algebra with multiplication satisfying the Leibniz identity

\[ ((xy)z) = ((xz)y) + (x(yz)), \]

for all \( x, y, z \in L \). If the Leibniz algebra \( L \) satisfies also the condition \( a^2 = aa = 0 \), for all \( a \in L \), then \( L \) is a Lie algebra. The Leibniz identity allows us to express every product of elements of \( L \) as a linear combination of left normed products and we shall tacitly use this fact throughout the paper. Also we shall use the left normed notation and write

\[ (((a_1a_2)a_3)\cdots a_n) = a_1a_2\cdots a_n, \]

for all \( a_1, \ldots, a_n \in L \). Let us observe that from the Leibniz identity it follows that, for all \( a, b \in L \), \( a(b^2) = a(bb) = 0 \).

Let now \( F\{Y\} \) be the free Leibniz algebra on the countable set \( Y = \{ y_1, y_2, \ldots \} \) and let \( \tilde{\mathcal{V}}^{-1} \) be the variety of Leibniz algebras defined by the identity

\[ y_1(y_2y_3)(y_4y_5) \equiv 0. \]

We next want to construct a Leibniz algebra \( U \) lying in the variety \( \tilde{\mathcal{V}}^{-1} \).

Let us denote by \( UT_2^0 \) the algebra of \( 2 \times 2 \) upper triangular matrices with zero multiplication, i.e., for all \( a_1^0, a_2^0 \in UT_2^0 \) then \( a_1^0a_2^0 = 0 \) and let

\[ U = UT_2^0 \oplus UT_2 \]
be the direct sum of the two vector spaces $UT^0_2$ and $UT_2$. We can give to $U$ a structure of Leibniz algebra by defining the following multiplication:

$$(a_1^0 + a_1)(a_2^0 + a_2) = (a_1^0 a_2^0) + [a_1, a_2],$$

where $e_{ij}^0 e_{hl} = \begin{cases} e_{il} & \text{if } j = h \\ 0 & \text{otherwise.} \end{cases}$

Notice that from the multiplication rule of $U$ it follows by induction that for all $a_1^0 + a_1, \ldots, a_n^0 + a_n \in U$,

$$(a_n^0 + a_n)(a_1^0 + a_1) \cdots (a_{n-1}^0 + a_{n-1}) = (((a_1^0 a_1)^0 a_2^0) \cdots a_{n-1})^0 + [a_n, a_1, \ldots, a_{n-1}],$$

where we are using the left normed notation for the Lie commutators $[a_n, a_1, \ldots, a_{n-1}] = [[a_n, a_1], \ldots, a_{n-1}].$

It is easy to check that $U$ is a Leibniz algebra and since $(a_1^0 + a_1)(a_2^0 + a_2)(a_3^0 + a_3) = (a_1^0(a_2^0 + a_3))^0 + [a_1, a_2, a_3]$ we immediately obtain the following:

$$0 = U s(a_1^0(a_2^0 + a_3))^0 + [a_1, a_2, a_3].$$

Recall that $P_n$ is the space of multilinear polynomials in $y_1, \ldots, y_n$ and if $A$ is a Leibniz algebra or a variety of Leibniz algebras, $Id(A)$ is the ideal of $F\{Y\}$ of polynomial identities of $A$. Denote $P_n(A) = P_n/P_n \cap Id(A)$. The symmetric group $S_n$ acts on $P_n$: if $\sigma \in S_n, f(y_1, \ldots, y_n) \in P_n, \sigma f(y_1, \ldots, y_n) = f(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ and this in turn induces a structure of $S_n$-module on $P_n(A)$. Its character denoted $\chi_n(A)$ is the $n$th cocharacter of $A$.

Throughout this paper we assume that the Leibniz algebra $U$ and the Leibniz variety $\tilde{V}_1$ have the following $n$th cocharacters:

$$\chi_n(U) = \sum_{\lambda \vdash n} \tilde{m}_\lambda \chi_\lambda,$$

$$\chi_n(\tilde{V}_1) = \sum_{\lambda \vdash n} \tilde{m}_\lambda' \chi_\lambda,$$

where $\chi_\lambda$ is the irreducible $S_n$-character associated to $\lambda$ and $\tilde{m}_\lambda \geq 0, \tilde{m}_\lambda' \geq 0$ are the corresponding multiplicities. Since $U \in \tilde{V}_1$ we immediately obtain the following:

**Remark 3.1.** For all partitions $\lambda \vdash n$, $\tilde{m}_\lambda \leq \tilde{m}_\lambda'$.

**Proof.** Since $U \in \tilde{V}_1$ then $Id(\tilde{V}_1) \subseteq Id(U)$ hence

$$P_n/(P_n \cap Id(U)) \cong P_n/(P_n \cap Id(\tilde{V}_1)/(P_n \cap Id(U)/P_n \cap Id(\tilde{V}_1))).$$

Then we have an embedding of $FS_n$-modules $P_n/P_n \cap Id(U) \hookrightarrow P_n \cap Id(\tilde{V}_1)$ and this implies that $\tilde{m}_\lambda \leq \tilde{m}_\lambda'$ for all $\lambda \vdash n$. $\square$

We are next going to obtain polynomial identities for $U$ out of associative polynomial identities for $UT_2$. Throughout the paper, unless otherwise stated, all monomials and all polynomials will be left-normed.
Lemma 3.2. The associative polynomial
\[ \sum_{\sigma \in S_{n-1}} a_\sigma y_\sigma(1) \cdots y_\sigma(n-1) \quad (1') \]
is an identity for UT if and only if the left-normed polynomial
\[ \sum_{\sigma \in S_{n-1}} a_\sigma y_{\sigma(1)} \cdots y_{\sigma(n-1)} \quad (2') \]
is an identity for U.

Proof. Suppose that \( \sum a_\sigma y_\sigma(1) \cdots y_\sigma(n-1) \) is not an identity of U. Thus there exist elements \( a_1^0 + a_1, \ldots, a_n^0 + a_n \in U \) such that
\[ \sum_{\sigma} a_\sigma \left( (a_1^0 a_\sigma(1))^0 \cdots a_{\sigma(n-1)}^0 \right) + \sum_{\sigma} a_\sigma [a_n, a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}] \neq 0. \]

Since by the multiplication rule defined in U
\[ \sum_{\sigma} a_\sigma \left( (a_1^0 a_\sigma(1))^0 \cdots a_{\sigma(n-1)}^0 \right) = \left( a_n^0 \left( \sum_{\sigma} a_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} \right) \right)^0, \]
in case such sum is non-zero, we would get that \( \sum a_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} \neq 0 \) and the lemma is proved in this case.

Therefore we may assume that \( \sum a_\sigma [a_n, a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}] \neq 0 \) for some \( a_1, \ldots, a_n \in UT_2 \).

Let \( UT_2^{(-)} \) denote the Lie algebra \( UT_2 \) under the bracket operation \([\ ,\ ]\). If \( \mathcal{A}^2 \) denotes the variety of Lie algebras defined by the identity \([x, y], [z, t] = 0\), then we claim that \( UT_2^{(-)} \) generates \( \mathcal{A}^2 \). In fact \( UT_2^{(-)} \) is not Lie nilpotent and clearly lies in \( \mathcal{A}^2 \). But by [2, 4.7.1] any proper subvariety of \( \mathcal{A}^2 \) is Lie nilpotent and the claim follows.

Let us consistently denote by \( P_n(\mathcal{A}^2) = P_n(UT_2^{(-)}) \) the space of multilinear Lie polynomials in the variables \( y_1, \ldots, y_n \) modulo the identity \([x, y], [z, t] = 0\). It is well known (see [2, 4.8.6]) that under the usual left permutation action of the symmetric group \( S_n \), the space \( P_n(\mathcal{A}^2) \) is an \((n-1)\)-dimensional irreducible module corresponding to the partition \((n-1, 1)\).

Since \( \sum a_\sigma [y_{\sigma(1)}, \ldots, y_{\sigma(n-1)}] \) is not an identity of \( UT_2^{(-)} \) and, so, of \( \mathcal{A}^2 \), this says that there exists \( i \in \{1, \ldots, n\} \) and an evaluation \( a_1 = \cdots = a_{i-1} = a_{i+1} = \cdots = a_n = a \) and \( a_i = b \), with \( a, b \in UT_2^{(-)} \), such that
\[ \sum_{\sigma \in S_{n-1}} a_\sigma [a_n, a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}] \neq 0. \quad (3') \]

Now, if \( i = n \) then \( (3') \) becomes \( \sum a_\sigma [b, a, \ldots, a] \neq 0 \). Thus \( \sum a_\sigma \neq 0 \) and we may substitute \( e_{11} \) for all variables in \((1')\) in order to get the non-zero value \( (\sum a_\sigma) e_{11}^n = (\sum a_\sigma) e_{11} \), as wished.
In case \( i \neq n \), then \((3')\) becomes \((\sum_{\sigma \in \Sigma_{n-1}} \alpha_{\sigma})[a, b, a, \ldots, a] \neq 0\). Then in \((1')\) make the substitution \( y_i = e_{12} \) and \( y_j = e_{22} \), for \( j \neq i \). We obtain
\[
\left( \sum_{\sigma \in \Sigma_{n-1}} \alpha_{\sigma} \right) e_{12} e_{22} \cdots e_{22} = \left( \sum_{\sigma \in \Sigma_{n-1}} \alpha_{\sigma} \right) e_{12} \neq 0
\]
and we are done also in this case.

Conversely if \( f(x_1, \ldots, x_{n-1}) \notin \text{Id}(UT_2) \) then the left-normed polynomial
\[
y_n f(y_1, \ldots, y_{n-1})
\]
does not vanish in \( U \). In fact if \( f(a_1, \ldots, a_{n-1}) \neq 0 \) for some \( a_1, \ldots, a_{n-1} \in UT_2 \), consider the evaluation \( y_n = (e_{11} + e_{22})^0 + 0, y_1 = a_1^0 + a_1, \ldots, y_{n-1} = a_{n-1}^0 + a_{n-1} \) where \( a_1^0 = \cdots = a_{n-1}^0 = 0 \). Then the polynomial \( y_n f(y_1, \ldots, y_{n-1}) \) takes the value \((e_{11} + e_{22})^0 f(a_1, \ldots, a_{n-1})^0 \neq 0 \) and we are done. \( \square \)

In the next lemma we shall find a relation between the multiplicities in \( \chi_\lambda(UT_2) \) and those in \( \chi_\lambda'(\mathcal{V}_1) \). As a consequence we shall derive a relation between the codimensions of \( UT_2 \) and those of \( \mathcal{V}_1 \). For a partition \( \lambda' \) of \( n + 1 \), let \( \lambda^- \) denote the set of partitions of \( n \) obtained by erasing one box from the diagram of \( \lambda \). Notice that one obtains a partition of \( n - 1 \) only by erasing a suitable box from the rim of the diagram of \( \lambda \). Let \( T_\lambda \) be a Young tableau of shape \( \lambda \) and let \( e_{T_\lambda} \) be the corresponding minimal essential idempotent of the group algebra \( FS_n \). Recall that \( e_{T_\lambda} = (\sum_{\sigma \in R_{T_\lambda}} \sigma)(\sum_{\tau \in C_{T_\lambda}} \text{sgn}(\tau)\tau) \), where \( R_{T_\lambda} \) and \( C_{T_\lambda} \) are the subgroups of \( S_n \) consisting of all permutations stabilizing the rows and the columns of \( T_\lambda \), respectively.

**Lemma 3.3.** For all \( n \geq 1 \) we have that \( c_{n+1}(\mathcal{V}_1) = (n + 1)c_n(UT_2) \). Moreover, if the \( n \)-th cocharacter of \( UT_2 \) has the decomposition
\[
\chi_n(UT_2) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda
\]
and the \( n \)-cocharacter \( \chi_n(\mathcal{V}_1) \) of \( \mathcal{V}_1 \) has the decomposition given in \( (3.2) \), we have that
\[
\widetilde{m}'_\lambda(\mathcal{V}_1) \leq \sum_{\mu \in \lambda^-} m_\mu
\]
for all \( \lambda \vdash n + 1 \).

**Proof.** Let \( f_1(y_1, \ldots, y_n), \ldots, f_m(y_1, \ldots, y_n) \) be associative polynomials linearly independent in \( P_n(UT_2) \). We want to prove that the \((n + 1)m\) left-normed polynomials
\[
y_1 f_1(y_1, y_2, \ldots, y_{n+1}), \ldots, y_n f_1(y_1, \ldots, y_{n+1}), \quad i = 1, \ldots, m
\]
are linearly independent in \( P_{n+1}(\mathcal{V}_1) \), where \( \sim \) means that the corresponding element is omitted.
In fact suppose
\[ \sum_{i,j} x_{ij} y_j f_i(y_1, \ldots, \hat{y}_j, \ldots, y_{n+1}) = 0 \]  
(3.3)
in \( P_{n+1}(\hat{\nu}^{-1}) \) with \( x_{i_0 j_0} \neq 0 \), for some \( i_0 \) and \( j_0 \). If we make the substitution \( y_{j_0} = y^2_0 \), since \( a(b^2) = 0 \) holds in any Leibniz algebra, by (3.3) we have that
\[ \sum_{i=1}^m x_{i j_0} y_0^2 f_i(y_1, \ldots, \hat{y}_{j_0}, \ldots, y_{n+1}) = 0 \]
in \( P_{n+2}(\hat{\nu}^{-1}) \). Since \( U \in \hat{\nu}^{-1} \), then also \( y^2_0 \sum_{i=1}^m x_{i j_0} f_i(y_1, \ldots, \hat{y}_{j_0}, \ldots, y_{n+1}) = 0 \) holds in \( P_{n+2}(U) \). By Lemma 3.2, this implies that the associative polynomial
\[ y_0 \sum_{i=1}^m x_{i j_0} f_i(y_1, \ldots, \hat{y}_{j_0}, \ldots, y_{n+1}) \]
is zero in \( P_{n+1}(UT_2) \).

Notice that if \( yf(y_1, \ldots, y_t) \in Id(UT_2) \), then by putting \( y = 1 \) we obtain that \( f(y_1, \ldots, y_t) \in Id(UT_2) \). Applied to the above, this says that
\[ \sum_{i=1}^m x_{i j_0} f_i(y_1, \ldots, \hat{y}_{j_0}, \ldots, y_{n+1}) = 0 \]
in \( P_n(UT_2) \). Since by hypothesis the polynomials \( f_1, \ldots, f_m \) are linearly independent over \( F \) we obtain that \( x_{i j_0} = 0 \), for all \( i \). Hence \( x_{i_0 j_0} = 0 \) and this is a contradiction. Thus \( c_{n+1}(\hat{\nu}^{-1}) \geq (n + 1)c_n(UT_2) \).

Let now \( M_{\hat{\nu}} \) be an irreducible \( S_{n+1} \)-module associated to the partition \( \hat{\nu} + n + 1 \) which appears with non-zero multiplicity in \( P_{n+1}(\hat{\nu}^{-1}) \). By embedding \( S_n \) into \( S_{n+1} \) as the subgroup of all permutations leaving \( n + 1 \) fixed, then we may regard \( M_{\hat{\nu}} \) as an \( S_n \)-module.

By the branching rule of the symmetric group [10] we have
\[ M_{\hat{\nu}} = \bigoplus_{\mu \in \hat{\nu}^\prime} M_{\mu} . \]

Moreover, since \( M_{\mu} \) is irreducible as an \( S_n \)-module, there exists a tableau \( T_{\mu} \) such that \( M_{\mu} \cong y_{n+1} + F S_n e_{T_\mu}(y_1, \ldots, y_n) \). Notice that if for some associative multilinear polynomial \( f(y_1, \ldots, y_n) \), \( e_{T_\mu} f(y_1, \ldots, y_n) \in Id(UT_2) \) for some tableau \( T_\mu \), then, by Lemma 3.2, the left-normed polynomial \( y_{n+1} e_{T_\mu} f(y_1, \ldots, y_n) \in Id(U) \supseteq Id(\hat{\nu}^{-1}) \).

Thus if \( F S_n e_{T_\mu} f(y_1, \ldots, y_n) \) appears in the decomposition of the \( S_n \)-module \( P_n(UT_2) \), then \( y_{n+1} F S_n e_{T_\mu} f(y_1, \ldots, y_n) \) appears in the decomposition of \( P_{n+1}(\hat{\nu}^{-1}) \). It follows that if \( M_{\hat{\nu}} \) appears with multiplicity \( m_{\hat{\nu}} \) in \( P_{n+1}(\hat{\nu}^{-1}) \), i.e., \( \chi_{n+1}(\hat{\nu}^{-1}) = \sum_{\mu \in \hat{\nu}^\prime} m_{\mu} \chi_{\mu} \), and \( \chi_n(UT_2) = \sum_{\mu \in \hat{\nu}^\prime} m_{\mu} \chi_{\mu} \) then
\[ m_{\hat{\nu}}(\hat{\nu}^{-1}) \leq \sum_{\mu \in \hat{\nu}^\prime} m_{\mu} \]  
(3.4)
for all $\lambda \vdash n + 1$. As a consequence,

$$c_{n+1}(\tilde{\chi}_1) = \sum_{\lambda \vdash n+1} \tilde{m}_\lambda \chi_\lambda(1) \leq \sum_{\lambda \vdash n+1} \left( \sum_{\mu \vdash \lambda} m_\mu \right) \chi_\mu(1)$$

$$\leq (n + 1) \sum_{\mu \vdash n} m_\mu \chi_\mu(1) = (n + 1)c_n(UT_2),$$

where the last inequality holds since, for every $\mu \vdash \lambda$, $\chi_\mu(1) \leq (n + 1)c_n(UT_2)$. We have proved that $c_{n+1}(\tilde{\chi}_1) = (n + 1)c_n(UT_2)$. □

4. Some technical lemmas

In this section we shall prove a sequence of combinatorial lemmas giving us the multiplicities in the cocharacter of $\tilde{\chi}_1$ for most partitions.

For $\lambda \vdash n$ let $\lambda^+$ denote the set of partitions of $n + 1$ whose diagrams are obtained from that of $\lambda$ by adding one box. Recall that $\lambda^-$ denotes the set of partitions of $n - 1$ obtained from $\lambda$ by deleting one box. In particular if $\lambda = (\lambda_1, \ldots, \lambda_t)$, for every $i = 1, \ldots, t$, such that $\lambda_i > \lambda_i + 1$, we denote by $\lambda_i^-$ the partition of $n - 1$ whose diagram is obtained from that of $\lambda$ by deleting the rightmost box of the $i$th row.

In the next lemmas we shall construct polynomials corresponding to essential idempotents of the group algebra of $S_n$. More precisely, if $e_{T_\lambda} \in FS_n$ is the essential idempotent corresponding to the tableau $T_\lambda$, we shall identify $e_{T_\lambda}$ with the polynomial $e_{T_\lambda}(y_1, \ldots, y_n) = e_{T_\lambda}y_1 \cdots y_n$ obtained by acting with $e_{T_\lambda}$ on the left normed monomial $y_1 \cdots y_n$. We shall then identify all variables corresponding to each row of the tableau. In order to simplify the notation we shall also use the following convention: a monomial $M$ in which some variables are marked with the same symbol, must be read as the polynomial in which those variables are alternated.

We illustrate this procedure with an example. Let $\lambda = (3, 2, 1, 1) \vdash 7$ and

$$T_\lambda = \begin{array}{ccc}
3 & 2 & 7 \\
4 & 6 \\
5 \\
1
\end{array}$$

Then

$$e_{T_\lambda}(y_1, \ldots, y_7) = \left( \sum_{\sigma \in RT_\lambda} \sigma \right) \sum_{\rho \in S'_4 \tau \in S'_2} (\text{sgn } \rho)(\text{sgn } \tau)y_{\rho(1)}y_{\tau(2)}y_{\rho(3)}y_{\rho(4)}y_{\rho(5)}y_{\tau(6)}y_7,$$

where $S'_4$ and $S'_2$ are the symmetric groups action on the sets $\{1, 3, 5\}$ and $\{2, 6\}$, respectively. We then identify and rename the variables $y_3 = y_2 = y_7 = x_1$, $y_4 = y_6 = x_2$, $y_5 = x_3$, $y_1 = x_4$. 
Hence $e^T_i(y_1, \ldots, y_7)$ becomes a scalar multiple of the polynomial

$$g(x_1, \ldots, x_4) = \sum_{\rho \in S_4} (\text{sgn } \rho)(\text{sgn } \tau)x_{\rho(4)}x_{\tau(1)}x_{\rho(1)}x_{\rho(2)}x_{\rho(3)}x_{\tau(2)}x_1.$$ 

We then write the polynomial $g$ in the form

$$g(x_1, \ldots, x_4) = x_4^2x_1x_2x_3x_2x_1,$$

where "_" and "^_" mean alternation on the corresponding variables.

**Lemma 4.1.** Let $p \geq 0$ and $q \geq 0$. If $\lambda = (p + q + 1, p + 1, 1, 1)$ then $\tilde{m}_\lambda = \tilde{m}_\lambda' = q + 1$.

**Proof.** For every $i = 0, \ldots, q$, we define the tableau $T^{(i)}_{\lambda}$

$$
\begin{array}{cccccc}
  & i + p + 2 & i + 2 & \cdots & i + p + 1 & 2 \cdots i + 1 & i + 2 p + 5 \cdots n \\
  & i + p + 3 & i + p + 5 & \cdots & i + 2 p + 4 \\
  & i + p + 4 \\
 i + p + 1 \\
\end{array}
$$

and we associate to $T^{(i)}_{\lambda}$ the left-normed polynomial

$$B^{(i)}_{p,q}(y_1, y_2, y_3, y_4) = \tilde{y}_4y_1^i \tilde{y}_1 \tilde{y}_2y_3 \tilde{y}_2y_1y_3y_2 \tilde{y}_2y_1y_3y_2y_1q^{-i},$$

where "_" and "^_" mean alternation on the corresponding elements.

Notice that the polynomial $B^{(i)}_{p,q}$ is obtained from the essential idempotent corresponding to the tableau $T^{(i)}_{\lambda}$ by identifying all the elements in each row of $\lambda$.

We start by proving that these polynomials are linearly independent over $F$ modulo $Id(U)$. Suppose not and let $\sum_{i=0}^q \alpha_i B^{(i)}_{p,q} = 0$ (mod $Id(U)$). If $t = \max\{i: \alpha_i \neq 0\}$ let us substitute $y_1$ with $y_1 + y_5$, then we obtain

$$\alpha_t \tilde{y}_4(y_1 + y_5)^i (y_1 + y_5) \tilde{y}_2 \tilde{y}_2(y_1 + y_5)^q - i$$

$$+ \sum_{i < t} \alpha_i \tilde{y}_4(y_1 + y_5)^i (y_1 + y_5) \tilde{y}_2 \tilde{y}_2(y_1 + y_5)q^{-i}$$

$$\times (y_1 + y_5) \tilde{y}_2 \tilde{y}_2 \tilde{y}_2(y_1 + y_5)^q - i = 0.$$ 

Since $|F| = \infty$, all the homogeneous components are still identities for $U$ and we look at the homogeneous component of degree $p + t + 1$ in $y_1$ and of degree $q - t$ in $y_5$. If we make in such component the evaluation $y_1 = e_{11}$, $y_2 = y_5 = e_{22}$, $y_3 = e_{12}$, $y_4 = e_{11}$ we obtain $-\alpha_t e_{11}^0 = 0$ and so $\alpha_t = 0$, a contradiction. We have proved that the $q + 1$ polynomials $B^{(i)}_{p,q}, i = 0, \ldots, q$, are linearly independent modulo $Id(U)$.

Notice that, for every $i$, the complete linearization of $B^{(i)}_{p,q}(y_1, y_2, y_3, y_4)$ is $e^T(i)(y_1, \ldots, y_n)$. Hence, from the above it follows that the polynomials $e^T(i), i = 0, \ldots, q$,
are $FS_n$ independent modulo $Id(U)$. Hence, in the module decomposition of $P_n(U)$, they generate distinct copies of the same irreducible module associated to the partition $\lambda$. This implies that $\tilde{m}_\lambda \geq q + 1$.

On the other hand, by Theorem 2.2 and Lemma 3.3 we have that $\tilde{m}_\lambda' \leq m_{\lambda_1^-} + m_{\lambda_2^-} + m_{\lambda_3^-} = q + 1$. It follows that $\tilde{m}_\lambda' = \tilde{m}_\lambda = q + 1$ and the proof is complete. □

The strategy of the proof of the following lemmas is similar to the one given above, and we reproduce them for convenience of the reader.

**Lemma 4.2.** Let $p \geq 0$ and $q \geq 0$. If $\lambda = (p + q + 2, p + 2, 2)$ then $m_\lambda = m_\lambda' = q + 1$.

**Proof.** For every $i = 0, \ldots, q$, let $T^{(i)}_\lambda$ be the tableau

\[
\begin{array}{cccccccc}
    & & & & & & & \\
    & & & & & & & \\
    & & & & & & & \\
    & & & & & & & \\
    & & & & & & & \\
\end{array}
\]

and let

\[
\tilde{B}^{(i)}_{p,q}(y_1, y_2, y_3, y_4) = \tilde{y}_3 y_1^i \tilde{y}_1 \cdots \tilde{y}_3 \tilde{y}_1 y_1^* y_2 y_3^* y_2 \tilde{y}_2 \cdots \tilde{y}_2 y_1^{q-i}
\]

be the left-normed polynomial associated to $T^{(i)}_\lambda$, where $\tilde{,}^{*}$ mean alternation on the corresponding elements.

We claim that these polynomials are linearly independent over $F$ modulo $Id(U)$. In fact let $\sum_{i=0}^q \alpha_i \tilde{B}^{(i)} = 0$ and assume that $t = \max\{i : \alpha_i \neq 0\}$. Substitute $y_1$ with $y_1 + y_4$, and $y_3$ with $y_3 + y_5$ and consider the homogeneous component of degree 1 in $y_3$ and $y_5$. If we make the substitution $y_1 = e_{11}$, $y_2 = y_4 = e_{22}$, $y_3 = e_{12}$, $y_5 = e_{21}^0$, we obtain $\alpha_i = 0$ and this is a contradiction. We have proved that the $q + 1$ polynomials $\tilde{B}^{(i)}_{p,q}$, $i = 0, \ldots, q$ are linearly independent modulo $Id(U)$. As in the previous lemma, by linearizing these polynomials, we obtain $\tilde{m}_\lambda \geq q + 1$.

Since by Theorem 2.2 and Lemma 3.3, $\tilde{m}_\lambda' \leq m_{\lambda_1^-} + m_{\lambda_2^-} + m_{\lambda_3^-} = q + 1$, we have $\tilde{m}_\lambda' = \tilde{m}_\lambda = q + 1$. □

**Lemma 4.3.** Let $p \geq 0$ and $q \geq 0$. If $\lambda = (p + q + 1, p + 1, 1)$ then

\[
m_\lambda = m'_\lambda = \begin{cases} 
3q + 3 & \text{if } p \neq 0, \\
2q + 1 & \text{if } p = 0.
\end{cases}
\]

**Proof.** Suppose first that $p \neq 0$ and denote by

\[
A^{(i)}_{p,q}(y_1, y_2, y_3) = \tilde{y}_3 y_1^i \tilde{y}_1 \cdots \tilde{y}_1 \tilde{y}_1 y_1^* y_2 y_2^* y_2 \tilde{y}_2 \cdots \tilde{y}_2 y_1^{q-i}, \quad i = 0, \ldots, q,
\]
The left-normed polynomials associated to the following tableaux

\[ T^{(i)}_\lambda : \]

\[
\begin{array}{cccccccc}
    i + p + 2 & i + 2 & \cdots & i + p + 1 & 2 & \cdots & i + 1 & i + 2p + 4 & \cdots & n \\
    i + p + 3 & i + p + 4 & \cdots & i + 2p + 3 & 1 & \\
    1
\end{array}
\]

\[ \tilde{T}^{(i)}_\lambda : \]

\[
\begin{array}{cccccccc}
    i + p + 1 & i + 2 & \cdots & i + p & i + 2p + 3 & 2 & \cdots & i + 1 & i + 2p + 4 & \cdots & q \\
    i + p + 2 & i + p + 4 & \cdots & i + 2p + 2 & 1 & \\
    i + p + 3 & 1
\end{array}
\]

\[ \tilde{T}^{(q+1)}_\lambda : \]

\[
\begin{array}{cccccccc}
    q + p + 2 & q + 3 & \cdots & q + p + 1 & 2 & \cdots & q + 2 \\
    q + p + 3 & q + p + 5 & \cdots & n & 1 & \\
    q + p + 4 & 1
\end{array}
\]

\[ \tilde{z}^{(i)}_\lambda : \]

\[
\begin{array}{cccccccc}
    i + p + 2 & i + 2 & \cdots & i + p + 1 & 2 & \cdots & i + 1 & i + 2p + 5 & \cdots & n \\
    i + p + 3 & i + p + 5 & \cdots & i + 2p + 3 & 1 & \\
    i + p + 4 & 1
\end{array}
\]

respectively.

We claim that these polynomials are linearly independent \((\text{mod } \text{Id}(U))\). In fact let us assume that

\[
\sum_{i=0}^{q} \alpha_i A^{(i)}_{p,q} + \sum_{i=0}^{q+1} \beta_i \tilde{A}^{(i)}_{p,q} + \sum_{i=0}^{q-1} \gamma_i \tilde{z}^{(i)}_\lambda = 0 \pmod{\text{Id}(U)}
\]

and suppose first that for some \(i, \alpha_i \neq 0\). Let \(t = \max\{i : \alpha_i \neq 0\}\). Let us substitute \(y_1\) with \(y_1 + y_4\), and let us consider the homogeneous component of degree \(p + t + 1\) in \(y_1\) and \(q - t\) in \(y_4\) in the new polynomials. If we evaluate \(y_1 = e_{11}, y_2 = e_{12} + e_{22}, y_3 = e_{11}', y_4 = e_{22}\) we obtain \(\alpha_t = 0\), a contradiction. So \(\alpha_t = 0\) for all \(i\). Now let us assume that \(\beta_i \neq 0\) for some \(i\) and let \(t = \max\{i : \beta_i \neq 0\}\).
As before let us substitute $y_1$ with $y_1 + y_4$, and consider the homogeneous component of degree $p + t$ in $y_1$ and $q - t + 1$ in $y_4$. By making the evaluation $y_1 = e_{11}$, $y_2 = e_{12}^0 + e_{22}$, $y_3 = e_{12}$. $y_4 = e_{22}$ we obtain the contradiction $\beta_i = 0$. So $\beta_i = 0$ for all $i$.

Finally, if $t = \max\{i : \gamma_i \neq 0\}$, also in this case we have $\gamma_i = 0$ and, so, as done in the previous lemmas we obtain that $\tilde{m}_t \geq 3q + 3$. Now, by Theorem 2.2 and Lemma 3.3, $\tilde{m}_t \leq m_{x_1} + m_{x_2} + m_{x_3} = q + (q + 2) + (q + 1) = 3q + 3$. Hence $\tilde{m}_t = 3q + 3$ and we are done.

In case $p = 0$ we consider the polynomials

$$A_{p,q}^{(i)}(y_1, y_2, y_3) = y_1 y_1^i y_1 y_2 y_3 y_1^{q-i-1}, \quad i = 0, \ldots, q - 1$$

and

$$\bar{A}_{p,q}^{(i)}(y_1, y_2, y_3) = \bar{y}_3 y_1^i y_1 y_2 y_1^{q-i}, \quad i = 0, \ldots, q.$$  

Also in this case it can be easily proved that $\bar{m}_t = \tilde{m}_t = 2q + 1$. \(\square\)

**Lemma 4.4.** If $\lambda = (p + q, p)$, with $p, q \geq 0$ then $m_\lambda = m'_\lambda = \begin{cases} 2q + 2 & \text{if } p > 1, \\ q + 1 & \text{if } p = 1, \\ 1 & \text{if } p = 0. \end{cases}$

**Proof.** If $\lambda = (n)$ then clearly $m_\lambda = m'_\lambda = 1$. In case $p = 1$ it is also easy to see that the polynomials

$$C_{p,q}^{(i)}(y_1, y_2) = y_1^i [y_1, y_2] y_1^{q-i}, \quad i = 0, \ldots, q - 1$$

are linearly independent (mod $Id(U)$). It will follow that $m_\lambda = m'_\lambda = q + 1$, in this case.

Let us now assume that $p > 1$ and consider the polynomials

$$C_{p,q}^{(i)}(y_1, y_2) = y_1 y_1^i \bar{y}_1 \cdots \bar{y}_1 [y_1, y_2] \bar{y}_2 \cdots \bar{y}_2 y_1^{q-i-1}, \quad i = 0, \ldots, q - 1, \quad q \geq 1$$

and

$$\bar{C}_{p,q}^{(i)}(y_1, y_2) = \begin{cases} \bar{y}_2 y_1^i \bar{y}_2 \cdots \bar{y}_1 [y_1, y_2] \bar{y}_2 \cdots \bar{y}_2 y_1^{q-i}, \quad i = 0, \ldots, q, \\ \bar{y}_2 \bar{y}_1 y_1^i \bar{y}_2 \cdots \bar{y}_1 [y_1, y_2] \bar{y}_2 \cdots \bar{y}_2, \quad i = q + 1. \end{cases}$$

As in the previous lemmas it is possible to prove that these polynomials are $F$-linearly independent (mod $Id(U)$). In fact let assume that

$$\sum_{i=0}^{q-1} \gamma_i C_{p,q}^{(i)} + \sum_{i=0}^{q+1} \beta_i \bar{C}_{p,q}^{(i)} = 0$$

for some $\gamma_i$, $\beta_i \in F$. If for some $i$, $\gamma_i \neq 0$, let $t = \max\{i : \gamma_i \neq 0\}$. If we substitute $y_1$ with $y_1 + y_3$, and we look at the homogeneous component of degree $p + t + 1$ in $y_1$ and $q - t + 1$ in $y_3$, by evaluating $y_1 = e_{11}^0 + e_{11}$, $y_2 = e_{12} + e_{22}$, $y_3 = e_{22}$ we obtain the contradiction...
\(x_i = 0\). So \(x_i = 0\) for all \(i\). In the same way we obtain that \(\beta_i = 0\). Hence it will follow that 
\[\tilde{m}_\lambda \geq 2q + 2.\]

On the other hand by Theorem 2.2 and Lemma 3.3, \(\tilde{m}''_\lambda \leq m_{\lambda_1} + m_{\lambda_2} = q + (q + 2) = 2q + 2\) and \(\tilde{m}_\lambda = \tilde{m}'_\lambda = 2q + 2 \) follows. \(\square\)

5. Some numerical invariant of \(\tilde{\mathcal{V}}_1\)

In this section we shall compute the \(n\)th cocharacter and the \(n\)th colength of \(\tilde{\mathcal{V}}_1\).

**Theorem 5.1.** Let \(\chi_n(\tilde{\mathcal{V}}_1) = \sum_{\lambda \in \mathcal{Y}} \tilde{m}''_\lambda \chi_\lambda\) be the \(n\)th cocharacter of \(\tilde{\mathcal{V}}_1\). Then

\[
\tilde{m}'_\lambda = \begin{cases} 
q + 1 & \text{if } \lambda = (p + q + 1, p + 1, 1), (p + q + 2, p + 2, 2), \\
2q + 1 & \text{or } (q + 1, 1), \\
2q + 2 & \text{if } \lambda = (q + 1, 1, 1), \\
3q + 3 & \text{if } \lambda = (p + q, q), \ p \geq 2, \\
1 & \text{if } \lambda = (n), \\
0 & \text{in all other cases.}
\end{cases}
\]

**Proof.** By Theorem 2.2, \(m_\lambda = 0\) whenever \(h_\lambda\) the height of \(\lambda\), is greater than 3. Hence, since by Lemma 3.3, \(\tilde{m}'_\lambda \leq \sum m_{\lambda''}\), it follows that \(\tilde{m}'_\lambda = 0\) whenever \(h_\lambda > 4\). Moreover a close look at the multiplicities in \(\chi_n(UT_2)\) shows that actually if \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) and \(h_\lambda = 4\), then \(\tilde{m}'_\lambda = 0\) unless \(\lambda_3 = \lambda_4 = 1\). Also, in case \(h_\lambda = 3\), \(\tilde{m}'_\lambda = 0\) if \(\lambda_3 > 2\). At the light of this, now the conclusion follows from Lemmas 4.1, 4.2, 4.3, 4.4. \(\square\)

If the sequence of codimensions, \(c_n(\mathcal{V})\), of a variety \(\mathcal{V}\) is exponentially bounded then one naturally defines the upper exponent and the lower exponent of the variety

\[
\text{Exp}(\mathcal{V}) = \lim_{n \to \infty} \sup \sqrt[n]{c_n(\mathcal{V})}, \quad \text{Exp}(\mathcal{V}) = \lim_{n \to \infty} \inf \sqrt[n]{c_n(\mathcal{V})}.
\]

In case of equality, \(\text{Exp}(\mathcal{V}) = \text{Exp}(\mathcal{V}) = \text{Exp}(\mathcal{V})\) is called the exponent of \(\mathcal{V}\). In the next corollary we shall compute \(\text{Exp}(\tilde{\mathcal{V}}_1)\) and \(l_n(\tilde{\mathcal{V}}_1)\) for all \(n \geq 1\).

**Corollary 5.2.** (1) \(\text{Exp}(\tilde{\mathcal{V}}_1) = 2\).

(2) For all \(n > 2\), \(l_n(\tilde{\mathcal{V}}_1) = \begin{cases} n^2 - \frac{7}{2}n + 6 & \text{if } n \text{ is even}, \\
n^2 - \frac{7}{2}n + \frac{11}{2} & \text{if } n \text{ is odd.}
\end{cases}\)

**Proof.** The first part follows from Lemma 3.3 since \(\text{Exp}(UT_2) = 2\).

The second part of the Corollary is obtained by a direct calculation by making use of Theorem 5.1. \(\square\)

In the sequel we shall make use of the following very useful remark where \(x_i, y_i, z, z_i\), are noncommutative variables.
**Remark 5.3.** (a) For all $n \geq 0$, $z(z_1z_2)y_1 \cdots y_n(z_3z_4) \in Id(\tilde{V}_1)$.

(b) For all $n \geq 0$ and for all permutations $\sigma \in S_n$ we have

$$z y_{\sigma(1)} \cdots y_{\sigma(n)}(z_1z_2) = z y_1 \cdots y_n(z_1z_2) \pmod{Id(\tilde{V}_1)}$$

and

$$z(z_1z_2)y_{\sigma(1)} \cdots y_{\sigma(n)} = z(z_1z_2)y_1 \cdots y_n \pmod{Id(\tilde{V}_1)}.$$  

(c) For all $n, m \geq 0$ and for all permutations $\sigma \in S_n, \tau \in S_m$ we have

$$(z_1z_2x_{\sigma(1)} \cdots x_{\sigma(n)})(z_3z_4y_{\tau(1)} \cdots y_{\tau(m)})$$

$$= (z_1z_2x_1 \cdots x_n)(z_3z_4y_1 \cdots y_m) \pmod{Id(\tilde{V}_1)}.$$  

**Proof.** The first statement follows from the basic relation $zyx = z(xy) + (yx)y$. Let now $w = z y_1 \cdots y_{i-1} y_i y_{i+1} y_i y_{i+2} \cdots y_n(z_1z_2)$. Since $w$ can be written as

$$w = z y_1 \cdots y_{i-1} y_i y_{i+1} \cdots y_n(z_1z_2) + z y_1 \cdots y_{i-1}(y_{i+1}y_i) \cdots y_n(z_1z_2),$$

by (a) we obtain that $w = z y_1 \cdots y_n(z_1z_2) \pmod{Id(\tilde{V}_1)}$. This clearly implies the first part of (b). Similarly we can reorder the variables to the right of $z(z_1z_2)$ proving this way the second part of (b).

Part (c) follows from part (b).  \(\square\)

We conclude this paper by proving that $\tilde{V}_1$ is a variety of Leibniz algebras with almost polynomial growth.

**Lemma 5.4.** Let $\mathcal{V}$ be a subvariety of $\tilde{V}_1$. If $\mathcal{V}$ satisfies an identity of the form

$$z_1z_2x_1 \cdots x_m(z_3z_4y_1 \cdots y_m) \equiv 0,$$

for some $m \geq 1$, then $\mathcal{V}$ has polynomial growth.

**Proof.** Since $\mathcal{V} \subseteq \tilde{V}_1$, for every $n$ and for every partition $\lambda \vdash n, m_\lambda(\mathcal{V}) < m_\lambda(\tilde{V}_1)$. Clearly, if $m_\lambda(\mathcal{V}) \neq 0$ then $\lambda$ has one of the shapes indicated in Lemmas 4.1, 4.2, 4.3, 4.4.

Let $n$ be any integer such that $n \geq 2m + 6$ and take any pair of positive integers $p, q$ such that $p \geq n + 2$ and a partition $\lambda$ as in Lemmas 4.1, 4.2, 4.3, 4.4 i.e., the second row of $\lambda$ has length greater than $m + 1$. Let $D$ be any one of the polynomials $A_{p,q}^{(i)}, B_{p,q}^{(i)}, C_{p,q}^{(i)}$ and $\tilde{C}_{p,q}^{(i)}$. After multilinearizing $D$, since right multiplication is a derivation, we obtain a linear combination of polynomials of the form $(z_1z_2 \cdots y_{11} \cdots y_{1r})(z_3z_4 \cdots y_{21} \cdots y_{2l})$. Since $\mathcal{V}$ satisfies a polynomial of the given type, we obtain that also $D$ is an identity of $\mathcal{V}$.

We have shown that $m_\lambda(\mathcal{V}) = 0$ as soon as the second row of $\lambda$ has length greater than $m + 1$. Recalling that $m_\lambda(\mathcal{V}) < m_\lambda(\tilde{V}_1)$, by Theorem 5.1 we obtain that $m_\lambda(\mathcal{V}) \neq 0$ provided $\lambda$ has at most $m + 3$ boxes below the first row. But this condition says that $\mathcal{V}$ has polynomial growth and we are done.

**Theorem 5.5.** Let $\mathcal{V}$ be a variety of Leibniz algebras and suppose that $\mathcal{V} \subseteq \tilde{V}_1$. Then $\mathcal{V}$ has polynomial growth.
Proof. Since $\mathcal{V} \subset \widetilde{\mathcal{V}}_1$, there exists $n$ and a partition $\lambda + n$ such that $m_\lambda(\mathcal{V}) < m_\lambda(\widetilde{\mathcal{V}}_1)$. Clearly $\lambda$ has one of the shapes indicated in Lemmas 4.1, 4.2, 4.3, 4.4. Suppose first that $\lambda = (p + q + 1, p + 1, 1, 1, 1)$. Then the polynomials $B_{p,q}$ constructed in Lemma 4.1 must be linearly dependent mod $Id(\mathcal{V})$. Recall that for every $i$, 

$$B^{(i)}_{p,q} = \frac{y_1 y_2 \cdots y_{2i+1}}{p} y_1^{2i+1}$$

and let $\sum_i x_i B^{(i)}_{p,q} = 0 \pmod{Id(\mathcal{V})}$, for some coefficients $x_i \in F$ not all zero.

Let us make the substitutions $y_1 = z^2$ and $y_3 = z_3 z_4$, then from the identity $\alpha(b^2) = 0$ we obtain

$$\sum x_i \sum_{\sigma \in S_2} z^2 y_1^{\sigma(1)} y_1^{\sigma(2)} (z_3 z_4) y_2^{\sigma(1)} y_2^{\sigma(2)} y_1^{q-i}$$

$$- \sum x_i \sum_{\sigma \in S_2} z^2 y_1^{\sigma(1)} y_1^{\sigma(2)} (z_3 z_4) y_2^{\sigma(1)} y_2^{\sigma(2)} y_1^{q-i}$$

$$+ \sum x_i \sum_{\sigma \in S_2} z^2 y_1^{\sigma(1)} y_1^{\sigma(2)} (z_3 z_4) y_2^{\sigma(1)} y_2^{\sigma(2)} y_1^{q-i} = 0.$$ 

By part (b) of Remark 5.3, it follows that

$$\sum x_i z^2 y_1^{\sigma(1)} y_1^{\sigma(2)} (z_3 z_4) y_2^{\sigma(1)} y_2^{\sigma(2)} y_1^{q-i} = 0. \quad (5.1)$$

Let $s$ be the least integer such that $x_s \neq 0$.

Substitute $z = z_1 z_2 y_1 \cdots y_1 (p+q-s+1) + y_1, y_2 = y_21 + \cdots + y_2(p+1)$ and $y_1 = y_1 + y_2(p+2) + \cdots + y_2(p+q-s+1)$.

Look at the multilinear part on all variables except $y_1$ and alternate $y_1 j$ with $y_2 j$ for $j = 1, \ldots, (p + q - s + 1)$. By Remark 5.3 the relation (5.1) becomes

$$x_s z_1 z_2 y_1^{p+s+2} \frac{y_1 y_1 (p+q-s+1) (z_3 z_4) y_22 (p+q-s+1)}{p+q-s+1} = 0.$$ 

Since right multiplication is a derivation we can rewrite this element in the following form:

$$(z_1 z_2 y_1^{p+s+2} \frac{y_1 y_1 (p+q-s+1) (z_3 z_4) y_22 (p+q-s+1)}{p+q-s+1}) = 0.$$ 

Let us substitute $z_4 = y_2(p+q+3), z_3 = z_3 z_4 y_2(p+q-s+2) \cdots y_2(p+q+2)$ and $y_1 = y_1(p+q-s+2) + \cdots + y_1(p+q+3)$. Take the multilinear part and alternate it by the pairs $y_1 j, y_2 j$ for $j = (p + q - s + 2), \ldots, (2p + q + 3)$.

Finally by Remark 5.3 we get

$$(z_1 z_2 \frac{y_1 y_1 (p+q-s+1)}{z_3 z_4 \frac{y_22 (p+q-s+1)}{m}}) = 0 \pmod{Id(\mathcal{V})}. \quad (5.2)$$
for $m = 2p + q + 3$. But then, by Lemma 5.4, $\mathcal{V}'$ has polynomial growth and we are done in this case.

A proof similar to the one given above, works also in case $\lambda$ has one of the other allowed shapes.

For instance, if $\lambda = (p + q + 2, p + 2, 2)$, then the polynomials

$$\tilde{B}^{(i)}_{p,q}(y_1, y_2, y_3, y_4) = \tilde{y}_3 y_1 \tilde{y}_1 \cdots \tilde{y}_1 y_1 y_2 y_3 \tilde{y}_2 \cdots \tilde{y}_2 y_1^{q-i}$$

constructed in Lemma 4.2 must be linearly dependent mod $Id(\mathcal{V})$. Let $\sum_i a_i \tilde{B}^{(i)}_{p,q} = 0$ (mod $Id(\tilde{V}_1)$), for some not all zero coefficients $a_j$ and make the substitution $y_3 = z^2 + y_3$.

Then we obtain

$$\sum_i a_i z^2 y_1 y_1 \cdots \tilde{y}_1 y_1 y_2 y_3 \tilde{y}_2 \cdots \tilde{y}_2 y_1^{q-i} = 0.$$ 

If we now substitute $y_3$ with $z_3 z_4$, we have

$$\sum_i a_i z^2 y_1 y_1 \cdots \tilde{y}_1 (z_3 z_4) \tilde{y}_2 \cdots \tilde{y}_2 y_1^{q-i} = 0.$$ 

This is a relation similar to (5.1) and we proceed as in the previous case. $\square$

References