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THE SEMIMARTINGALE DECOMPOSITION OF ONE-DIMENSIONAL QUASIDIFFUSIONS WITH NATURAL SCALE

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Quasidiffusions (with natural scale) are semimartingales obtained as time changed Wiener processes. Examples are diffusions and birth- and death-processes. In general, quasidiffusions are not continuous but they are skip-free. In this note we determine the continuous and the purely discontinuous martingale part of all such quasidiffusions.

quasidiffusions * semimartingales * local times

1. Introduction

Let $W=(W_t, \mathscr{F}_t, P_x)$ $(t\geq 0, x\in R)$ be a standard Wiener process on the real line R and $l^W(t,x)$ its (continuous) local time normalized such that

$$\int_0^t f(W_s)\,\mathrm{d}s = 2\int_R l^W(t,x)\,\mathrm{d}x \quad (f \text{ bounded}, t\geq 0).$$

It is well known that $l^W(\cdot,x)$ increases at t if and only if $W_t=x$. Assume m to be a nondecreasing extended real-valued function on R and introduce the closed set $E_m := \{x\in R \mid m(x-\varepsilon)<m(x+\varepsilon)\ \forall\varepsilon>0\}$. Then $R\backslash E_m$ is the union of mutually disjoint open intervals $I_k=(a_k,b_k)$, the "gaps" of E_m. Thereby k runs through a subset K of the set N of all non-negative integers. We shall suppose $0\in K$ (resp. $1\in K$) if and only if $b_0:=\inf E_m>-\infty$ ($a_1:=\sup E_m<\infty$ resp.) and $I_0:=(-\infty,b_0)$, $I_1:=(a_1,\infty)$, $K':=K\backslash\{0,1\}$. Define

$$S_t := \int_R l^W(t,x)m(\mathrm{d}x) \quad (t\geq 0)$$

and

$$T_t := \inf\{u>0 \mid S_u>t\} \quad (t\geq 0).$$

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Then $T := (T_t, t \geq 0)$ is a strictly increasing, right-continuous family of $\mathscr{F}$-stopping times with $T_0 = 0$ a.s. This means T is a change of time with respect to $\mathscr{F}$. Define

$$X_t := W_{T_t}, \quad \mathscr{G}_t := \mathscr{F}_{T_t} \qquad (t \geq 0). \tag{1.1}$$

Then $X = (X_t, \mathscr{G}_t, P_x)$ $(t \geq 0, x \in E_m)$ turns out to be a standard Markov process on E_m. We call it a *quasidiffusion with speed measure m* (and canonical scale $p(x) = x$). In general X is not continuous. But it is *skip-free* in the following sense:

$$(\min(X_{t-}, X_t), \max(X_{t-}, X_t)) \cap E_m = \emptyset \quad (t > 0).$$

Thus, if $X_t \neq X_{t-}$, then $\min(X_{t-}, X_t) = a_k$ and $\max(X_{t-}, X_t) = b_k$ for some $k \in K'$.

In particular, if m is strictly increasing, then X is a diffusion, and if m increases in isolated points only, then X is a birth-and-death-process.

Now suppose $0 \in E_m$ and put $P = P_0$. In the following we restrict ourselves to P.

As a time changed Wiener process $(X_t, \mathscr{G}_t)$ is a semimartingale. We shall construct the decomposition of X into its continuous martingale part M^c, the purely discontinuous martingale part M^d and a continuous locally bounded variation process A.

The decomposition is more explicit than the results from [7], where general time changed martingales are considered.

2. Notations and results

Denote by $\mathscr{M}_{loc}(\mathscr{G})$ the set of all local $\mathscr{G}$-martingales, by $\mathscr{M}^2_{loc}(\mathscr{G})$ the set of all locally square integrable local $\mathscr{G}$-martingales (all with respect to P). Let $\mathbb{1}_B(\cdot)$ be the indicator function of the set B.

2.1. We shall start with some notions necessary for the formulations of the theorem below. Let $A_{k,i}$ (resp. $B_{k,i}$) be the time of the i-th jump of X from a_k to b_k (resp. from b_k to a_k) if this jump occurs, let it be equal to ∞ otherwise. For $k \in K'$ put

$$A_k(t) := \sum_{i=1}^{\infty} \mathbb{1}_{[A_{k,i},\infty)}(t), \quad B_k(t) := \sum_{i=1}^{\infty} \mathbb{1}_{[B_{k,i},\infty)}(t) \qquad (t \geq 0).$$

Define

$$l(t,x) := l^W(T_t, x) \quad (t \geq 0,\ x \in E_m)$$

to be the local time of X. Then $l(\cdot,\cdot)$ is continuous and it holds a.s. that

$$\int_0^t f(X_s)\,\mathrm{d}s = \int_{E_m} l(t,x) f(x) m(\mathrm{d}x) \quad (f \text{ bounded},\ t \geq 0). \tag{2.1}$$

Moreover, $l(\cdot, x)$ increases at t if and only if $T_t < T_\infty := \lim_{s\uparrow\infty} T_s$ and $X_t = x$ or $X_{t-} = x$. By $U := (U_t)_{t\geq 0}$ we denote the continuous local martingale

$$U_t := \int_0^t \mathbb{1}_{E_m}(W_s)\,\mathrm{d}W_s \quad (t \geq 0).$$

2.2. Theorem. *The quasidiffusion* $(X_t, \mathcal{G}_t)$ *defined by* (1.1) *is a semimartingale and admits the representation* $X = M^c + M^d + A$, *where*

(a) $(M_t^c, \mathcal{G}_t)$ *is the continuous local martingale given by*

$$M_t^c := U_{T_t} = \int_0^{T_t} \mathbb{1}_{E_m}(W_s)\, \mathrm{d} W_s$$

and having the characteristic

$$\langle M^c \rangle_t = T_t^c := T_t - \sum_{s \leq t} \Delta T_s \quad (t \geq 0)$$

with $\Delta T_s := T_s - T_{s-}$ $(s > 0)$,

(b) $(M_t^d, \mathcal{G}_t)$ *is the purely discontinuous locally square integrable local martingale defined by*

$$M_t^d := \sum_{k \in K'} ([(b_k - a_k)A_k(t) - l(t, a_k)] - [(b_k - a_k)B_k(t) - l(t, b_k)]) \quad (t \geq 0),$$

(c) $(A_t, \mathcal{G}_t)$ *is the adapted continuous process with locally integrable variation given by*

$$A_t := l(t, b_0) - l(t, a_1) \quad (t \geq 0).$$

We have $l(t, b_0) \equiv 0$ if and only if b_0 is inaccessible or absorbing (i.e. $|b_0| + |\int_{b_0}^0 m(x)\, \mathrm{d}x| = \infty$ or $|m(b_0-)| = +\infty$). This holds analogously for $l(\cdot, a_1)$. Thus the Theorem implies:

2.3. Corollary. *The quasidiffusion* X *is a local martingale if and only if* $b_0 = \inf E_m$ *and* $a_1 = \sup E_m$ *are inaccessible or absorbing boundaries of the state space* E_m.

As a counterpart to (2.1) we get

2.4. Corollary. *If* f *is bounded, then*

$$\int_0^t f(X_s)\, \mathrm{d}\langle M^c \rangle_s = 2 \int_{E_m} f(x) l(t, x)\, \mathrm{d}x \quad (t \geq 0). \tag{2.2}$$

One could ask if the continuous local martingale M^c turns out be a diffusion. But this is not the case. It is shown by

2.5. Corollary. *Assume* $E_m = [0, 1] \cup [2, 3]$ *and* $\mathrm{d}m(x) = h(x)\, \mathrm{d}x$ *with* $h(x) = \sigma_1^{-2} \mathbb{1}_{(0,1)}(x) + \sigma_2^{-2} \mathbb{1}_{(2,3)}(x)$ *for some* $\sigma_1, \sigma_2 > 0$, $\sigma_1 \neq \sigma_2$. *Put* $\sigma(x) = \sigma_1 \mathbb{1}_{[0,1]}(x) + \sigma_2 \mathbb{1}_{[2,3]}(x)$. *Then there exists a Wiener process* $\tilde{W}$ *adapted to* $\mathcal{G}$ *such that*

$$M_t^c = \int_0^t \sigma(X_s)\, \mathrm{d}\tilde{W}_s \quad (t \geq 0).$$

In particular, M^c *is not Markovian.*

An analogous integral representation for M^c under weak assumptions on m only is given in [6].

3. Proofs

We shall prove the Theorem step by step and start with the following

3.1. Lemma. *There exist a locally square integrable local $\mathcal{G}$-martingale M and a continuous $\mathcal{G}$-adapted process A with locally integrable variation such that $M_0 = A_0 = 0$ and $X = M + A$ hold.*

Proof. For every $n \geq 1$ define $D_n := \inf\{t > 0 \mid |X_t| \geq n\}$. Then the stopped process $X^{D_n} := (X_{D_n \wedge t}, \mathcal{G}_{D_n \wedge t})$ is a bounded semimartingale. This follows from the skip-freeness of X. Indeed, we have $|X_t| < n$ $(t < D_n)$ and

$$c_n := \sup(E_m \cap (-\infty, -n)) \leq X_{D_n} \leq \inf(E_m \cap (n, \infty)) =: d_n$$

with $c_n := -n$ if $E_m \subset [-n, \infty)$, $d_n := n$ if $E_m \subset (-\infty, n]$. Moreover, from the skip-freeness of X it follows that the only jumps of X^{D_n} are the jumps over intervals I_k with $I_k \cap [-n, n] \neq \emptyset$ $(k \in K')$. Consequently, the jumps of X^{D_n} are bounded. This implies (see [2, Chapter VII.2]) that X^{D_n} is a special semimartingale. Since the set of all special semimartingales is closed under localization, X is a special semimartingale. This means that X is the sum of a local martingale $(M_t, \mathcal{G}_t)$ with $M_0 = 0$ and a predictable $\mathcal{G}$-adapted process $(A_t, \mathcal{G}_t)_{t \geq 0}$ having locally integrable variation and satisfying $A_0 = 0$.

As a standard process X is quasi-left-continuous and therefore A is continuous.

Put $C_n := \inf\{t > 0 \mid |X_t| + |A_t| \geq n\}$. Then the stopped process M^{C_n} is a bounded martingale. This follows from the skip-freeness of X and the continuity of A. (Compare the argument for the boundedness of X^{D_n} above.) Now the local square integrability of M is obvious. □

The purely discontinuous part M^d of M is equal to the sum of the compensated jumps of M (see [2, p. 367]). Because of the continuity of A the jumps of M and X coincide, and therefore in order to determine M^d it remains to calculate the compensators of the processes $A_k(\cdot)$, $B_k(\cdot)$ $(k \in K')$. This will be done as the next step. Let $A_{k,i}^W$ be the time at which the i-th upcrossing of W over I_k is finished $(k \in K', i \geq 1)$.

3.2. Lemma. *For every $k \in K'$ the process*

$$A_k^W(t) := \sum_{i=1}^{\infty} \mathbb{1}_{[l^W(A_{k,i}^W, a_k), \infty]}(t) \quad (t \geq 0)$$

is a Poisson process with intensity $\lambda_k := (b_k - a_k)^{-1}$.

Proof. $l^W(A_{k,1}^W, a_k)$ is exponentially distributed with the parameter λ_k (see [4, Chapter 2.8]). Now use the independence of the variables

$$l^W(A_{k,i+1}^W, a_k) - l^W(A_{k,i}^W, a_k) = l^W(A_{k,1}^W \circ \Theta_{A_{k,i}^W}, a_k) \quad (i \geq 0)$$

with $A_{k,0}^W = 0$ and the strong Markov property of W. □

3.3. Lemma. *For every $k \in K'$ we have*

$$A_k(t) = A_k^W(l(t, a_k)) \quad (t \geq 0).$$

Proof. Assume $k \in K'$. Firstly remark that

$$A_k(t) = \sum_{i=1}^{i_k} \mathbb{1}_{[A_{k,i},\infty)}(t) \quad (t \geq 0)$$

with $i_k := \sup\{i \geq 1 \mid A_{k,i} < \infty\}$.

It is easy to see that $\{A_{k,i} = \infty\} = \{T_\infty < A_{k,i}^W\}$ holds. Put $\sigma := \inf\{t > T_\infty \mid W_t = a_k\}$ on $\{T_\infty < \infty\}$. By virtue of $W_{T_\infty} = b_0$ or $= a_1$ on $\{T_\infty < \infty\}$ we have $T_\infty < \sigma < A_{k,i}^W$ on $\{A_{k,i} = \infty\}$. Since $l^W(\cdot, a_k)$ increases at σ we have

$$l^W(T_\infty, a_k) < l^W(A_{k,i}^W, a_k) \quad \text{on } \{A_{k,i} = \infty\}.$$

This implies

$$\mathbb{1}_{[l^W(A_{k,i}^W,\, a_k),\infty)}(l^W(T_t, a_k)) = 0 \quad (i > i_k,\ t \geq 0). \tag{3.1}$$

Now assume $i \leq i_k$. Then $X_{A_{k,i}-} = a_k$ and $X_{A_{k,i}} = b_k$. Thus $l(\cdot, a_k)$ increases at $A_{k,i}$ and $l(A_{k,i}, a_k) = l(A_{k,i} + \eta, a_k)$ for some $\eta > 0$ because of the right continuity of X. Consequently we have $l(t, a_k) < l(A_{k,i}, a_k)$ if and only if $t < A_{k,i}$. In particular it follows that

$$A_k(t) = \sum_{i=1}^{i_k} \mathbb{1}_{[l(A_{k,i}, a_k),\infty)}(l(t, a_k)) \quad (t \geq 0).$$

By virtue of $A_{k,i}^W = T_{A_{k,i}}$ $(i \leq i_k)$ and (3.1) we get the assertion. □

3.4. Lemma. $(A_k(t) - \lambda_k l(t, a_k))_{t \geq 0} \in \mathcal{M}_{\mathrm{loc}}(\mathcal{G})$ $(k \in K')$.

Proof. Fix $k \in K'$ and put

$$L(t) := \inf\{s > 0 \mid l^W(s, a_k) > t\} \quad (t \geq 0).$$

Then $L(\cdot)$ is a change of time for $\mathcal{F}$. Define $\mathcal{H}_t := \mathcal{F}_{L(t)}$ $(t \geq 0)$. By virtue of

$$\{l^W(A_{k,i}^W, a_k) \leq t\} = \{L(t) \geq A_{k,i}^W\} \quad (t \geq 0)$$

we have

$$A_k^W(t) = \sum_{i=1}^{\infty} \mathbb{1}_{[A_{k,i}^W,\infty)}(L(t)) \quad (t \geq 0).$$

Thus $A_k^W(\cdot)$ is $\mathcal{H}$-adapted. Moreover, $l^W(\cdot, a_k)$ forms a continuous change of time with respect to H. Thus, from Lemma 3.2 and [5, p. 26], it follows that

$$A_k^W(l^W(t, a_k)) - \lambda_k l^W(t, a_k) \quad (t \geq 0)$$

forms a local martingale with respect to $\mathcal{H}_{l^W(\cdot, a_k)}$. Because of $L(l^W(t, a_k)) \geq t$ we have

$$\mathcal{F}_t \subseteq \mathcal{H}_{l^W(t, a_k)} \quad (t \geq 0).$$

Consequently, (T_t) is a time change also with respect to $\mathscr{H}_{l^W(\cdot,a_k)}$. We know that $l^W(\cdot, a_k)$ is constant on (T_{t-}, T_t) $(t>0)$, i.e. (T_t) is $l^W(\cdot, a_k)$-continuous in the sense of [5]. Therefore, from [5, p. 26], it follows that

$$A_k^W(l^W(T_t, a_k)) - \lambda_k l^W(T_t, a_k) = A_k(t) - \lambda_k l(t, a_k) \quad (t \geq 0)$$

is a local $\mathscr{H}_{l(\cdot,a_k)}$-martingale. Now use $\mathscr{G}_t \subseteq \mathscr{H}_{l(t,a_k)}$ $(t \geq 0)$ and the $\mathscr{G}$-adaptedness of $A_k(\cdot) - \lambda_k l(\cdot, a_k)$ to get the assertion. □

Analogously to Lemma 3.4 one can show that

$$(B_k(t) - \lambda_k l(t, b_k))_{t\geq 0} \in \mathscr{M}_{\text{loc}}(\mathscr{G}) \quad (k \in K').$$

Now part (b) of the Theorem is obvious.

3.5. Lemma. *$U_T := (U_{T_t}, \mathscr{G}_t)_{t\geq 0}$ is a continuous local martingale having the characteristic*

$$\langle U_T \rangle_t = \langle U \rangle_{T_t} = T_t^c.$$

Proof. We have $W_s \in R \backslash E_m$ on (T_{t-}, T_t) $(t>0)$. Thus $\langle U \rangle$ and therefore also U is constant on (T_{t-}, T_t) for every $t>0$ (see e.g. [3]). This implies U-continuity of the change of time T, and consequently U_T is a continuous local martingale with respect to $\mathscr{G}$ (see [5]). By the same arguments T is $(U^2 - \langle U \rangle)$-continuous, and therefore U_T^2-$\langle U \rangle_T$ is a continuous local martingale with respect to $\mathscr{G}$. This implies $\langle U \rangle_T = \langle U_T \rangle$ and by using

$$\bigcup_{0<u\leq t} (T_{u-}, T_u) = \{s \leq T_t \mid W_s \in R \backslash E_m\}$$

we obtain the equation

$$\langle U_T \rangle_t = \langle U \rangle_{T_t} = \int_0^{T_t} \mathbb{1}_{E_m}(W_s)\, \mathrm{d}s = T_t - \int_0^{T_t} \mathbb{1}_{R\backslash E_m}(W_s)\, \mathrm{d}s$$
$$= T_t - \sum_{u \leq t} (T_u - T_{u-}) =: T_t^c \quad (t \geq 0). \qquad \square$$

3.6. Consider the function φ_k $(k \in K)$ and φ defined by

$$\varphi_k(x) = \int_0^x \mathbb{1}_{I_k}(u)\, \mathrm{d}u \quad (k \in K)$$

and

$$\varphi(x) = \sum_{k\in K} \varphi_k(x) = \int_0^x \mathbb{1}_{R\backslash E_m}(u)\, \mathrm{d}u \quad (x \in R). \tag{3.2}$$

Using Tanaka's formula we can derive (put $l^W(t, x) = 0$ if $|x| = \infty$)

$$\varphi(W_t) = \int_0^t \mathbb{1}_{R\backslash E_m}(W_s)\, \mathrm{d}W_s + \sum_{k\in K'} (l^W(t, a_k) - l^W(t, b_k))$$
$$- l^W(t, b_0) + l^W(t, a_1) \quad (t \geq 0). \tag{3.3}$$

Remark that

$$\varphi_k(X_t)=(b_k-a_k)(A_k(t)-B_k(t)) \quad (k\in K', t\geq 0)$$

and

$$\varphi_k(X_t)\equiv 0 \quad (k=0,1;\ t\geq 0).$$

Thus we get from (3.2) and (3.3), for $t\geq 0$,

$$\sum_{k\in K'}(b_k-a_k)(A_k(t)-B_k(t))=\varphi(X_t)=\varphi(W_{T_t})$$

$$=W_{T_t}-U_{T_t}+\sum_{k\in K'}(l(t,a_k)-l(t,b_k))$$

$$-l(t,b_0)+l(t,a_1).$$

This means (use part (b) of the Theorem)

$$W_{T_t}-U_{T_t}=M_t^{\mathrm{d}}+l(t,b_0)-l(t,a_1) \quad (t\geq 0).$$

Now (a) and (c) of the Theorem follow immediately. □

3.7. To prove (2.2) we remark (see point 3.5) that

$$\int_0^t f(X_s)\,\mathrm{d}\langle U_T\rangle_s=2\int_0^t f(X_s)\int_{E_m} l^W(\mathrm{d}T_s,x)\,\mathrm{d}x$$

$$=2\int_0^{T_t} f(W_s)\int_{E_m} l^W(\mathrm{d}s,x)\,\mathrm{d}x$$

$$=2\int_{E_m}\int_0^{T_t} f(W_s)l^W(\mathrm{d}s,x)\,\mathrm{d}x$$

$$=2\int_{E_m} f(x)l(t,x)\,\mathrm{d}x \quad (t\geq 0).$$

3.8. Now we shall prove Corollary 2.5. It holds that

$$t=S_{T_t}=\int_0^{T_t} h(W_s)\,\mathrm{d}s=\int_0^{T_t} h(W_{T_{S_s}})\,\mathrm{d}s \quad (t\geq 0).$$

(We have used that $T_{S_s}\geq s$ $(s\geq 0)$, $W_u\notin E_m$ for $u\in(s,T_{S_s})$ and therefore $h(W_s)=h(W_{T_{S_s}})$ $(s\geq 0)$.) Thus we get

$$t=\int_0^t h(X_u)\,\mathrm{d}T_u=\int_0^t h(X_u)\,\mathrm{d}T_u^{\mathrm{c}}+\sum_{s\leq t} h(X_s)\,\Delta T_s$$

$$=\int_0^t h(X_u)\,\mathrm{d}T_u^{\mathrm{c}} \quad (t\geq 0)$$

because of $X_s\in\{0,1,2,3\}$ if $\Delta T_s>0$ and therefore $h(X_s)\,\Delta T_s\equiv 0$. Put

$$\tilde{h}(x):=\sigma_1^{-2}\mathbb{1}_{[0,1]}(x)+\sigma_2^{-2}\mathbb{1}_{[2,3]}(x) \quad (x\in R).$$

Then

$$\int_0^t (\tilde{h}-h)(X_u)\,\mathrm{d}T_u^{\mathrm{c}} = 2\int_{\{\tilde{h}\neq h\}} l(t,x)\,\mathrm{d}x = 0$$

and therefore

$$\int_0^t \tilde{h}(X_u)\,\mathrm{d}T_u^{\mathrm{c}} = t \quad (t\geq 0).$$

We obtain

$$T_t^{\mathrm{c}} = \int_0^t \frac{\mathrm{d}u}{\tilde{h}(X_u)} = \int_0^t \sigma^2(X_u)\,\mathrm{d}u.$$

Now introduce a continuous local $\mathscr{G}$-martingale $\tilde{W}$ by

$$\tilde{W}_t = \int_0^t (\tilde{h}(X_u))^{1/2}\,\mathrm{d}U_{T_u} \quad (t\geq 0).$$

This is a Wiener process. Indeed, we have

$$\langle \tilde{W}\rangle_t = \int_0^t \tilde{h}(X_u)\,\mathrm{d}T_u^{\mathrm{c}} = t.$$

Now, by standard arguments it follows that

$$\mathrm{d}M_t^{\mathrm{c}} = \mathrm{d}U_{T_t} = \sigma(X_t)\,\mathrm{d}\tilde{W}_t. \qquad \square$$

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