Finite Groups Whose Sylow Subgroups Are Abelian

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1. INTRODUCTION

In this paper we study the structure of A-groups, finite groups all of whose Sylow subgroups are abelian. We use the recent results of Walter [7], classifying the simple groups with abelian Sylow 2-subgroups, to extend the work of Taunt [6], who studied the structure of solvable A-groups.

We prove the following:

THEOREM. Let G be an A-group. Then there exist subgroups H, S, and K, of G satisfying:

i) \( HSK = G, \ |H||S||K| = |G|. \)

ii) \( H \leq G, \ K \leq N(S). \) (i.e. \( HSK \) is a triple semi-direct product)

iii) H and K are solvable, S is semi-simple.

Furthermore, by adding appropriate conditions we insure that S, K, and SK are determined up to conjugacy (and H is uniquely determined) and, if \( N \trianglelefteq G \), then \( N = (N \cap H)(N \cap S)(N \cap K). \)

2. NOTATION

Our notation is standard and follows, for the most part, that of Huppert [4].

The following items may be unusual:

Let \( G \) be a finite group, then,

\( G^{(c)} \) denotes \( \cap G^{(i)} \), the intersection of the members of the derived series for \( G \) (i.e. the "last" member of the derived series).

\( \text{Sol}(G) \) denotes the maximal normal solvable subgroup of \( G \).

Unless otherwise stated, all groups we consider are finite.
3. $A$-Groups

**Definition 3.1.** An $A$-group is a finite group, all of whose Sylow subgroups are abelian.

**Theorem 3.2.** (Walter) Let $G$ be a simple, non-abelian, $A$-group. Then $G$ is isomorphic to $J(11)$, the Janko group (see Janko [5]), or $G$ is isomorphic to $L_2(q)$ for $q > 3$ and $q = 0, 3, \text{ or } 5 \pmod{8}$.

**Proof.** By Walter [7] a simple, non-abelian, group with abelian Sylow 2-subgroup is isomorphic to $J(11), L_2(q)$ (for $q > 3$ and $q = 0, 3, \text{ or } 5 \pmod{8}$), or $G$ is of Ree type. Since groups of Ree type have non-abelian Sylow 3-subgroups (see Ward [8]), the result follows.

**Lemma 3.3.** Let $G$ be a simple, non-abelian, $A$-group. Let $\alpha$ be an $r$-automorphism of $G$ for some prime $r$ with $r \mid |G|$. Assume $\alpha$ centralizes a Sylow $r$-subgroup of $G$. Then $\alpha$ induces an inner automorphism of $G$.

**Proof.** By Theorem 3.2 we may assume $G$ is $J(11)$ or $L_2(q)$. For $J(11)$ the result is trivial since, by Janko [5], the outer automorphism group of $J(11)$ is 1. So we may assume $G = L_2(q)$, $q > 3$, $q = 0, 3, \text{ or } 5 \pmod{8}$, and let $q = p^n$, for some prime $p$.

From Gorenstein [3, p. 462] we see that the automorphism group of $L_2(q)$ is the semi-direct product $\langle \beta_0 \rangle \rtimes \text{PGL}(2, q)$, where $\beta_0$ is induced by a field automorphism of $GF(q)$ of order $n$ and the elements of $\text{PGL}(2, q)$ act by conjugation. Thus we shall assume:

$$\alpha : X \rightarrow B^{-1}X^\beta B$$

for some $B \in \text{PGL}(2, q), \beta \in \langle \beta_0 \rangle$.

However, for $r \neq 2$, or for $r = 2 = p$, any Sylow $r$-subgroup of $\langle \beta_0 \rangle \rtimes \text{PGL}(2, q)$ lies in $\langle \beta_0 \rangle \rtimes L_2(q)$. So we may assume (in these cases) that $B \in L_2(q)$. Furthermore, if $\langle \delta \rangle$ is a Sylow $r$-subgroup of $\langle \beta_0 \rangle$ and $\langle \delta \rangle$ normalizes $R$, a Sylow $r$-subgroup of $L_2(q)$, then some $L_2(q)$ conjugate of $\alpha$ lies in $\langle \delta \rangle \rtimes R$, hence must normalize, and by hypothesis centralize, $R$. So we may assume, without loss of generality, $\alpha \in \langle \delta \rangle \leq \langle \beta_0 \rangle$, and

$$\alpha : X \rightarrow X^\beta \quad (\text{in case } r \neq 2 \text{ or } r = 2 = p).$$

During the remainder of this proof we will perform certain matrix calculations. The matrices used—although written in $\text{GL}(2, q)$—will represent cosets of $Z(\text{GL}(2, q))$, i.e. elements of $\text{PGL}(2, q)$ or of $L_2(q)$. 
A. \( r = p \)

A Sylow \( r \)-subgroup of \( L_2(q) \) is given by:

\[
R = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad x \in GF(q).
\]

\( \alpha \) clearly normalizes, hence must centralize \( R \). Thus

\[
\begin{bmatrix} 1 & x^\beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.
\]

Hence \( x^\beta = x \) for all \( x \in GF(q) \). Hence \( \beta = 1 \).

B. \( r \neq p, r \neq 2, r \mid q - 1 \)

Let \( |G|_r = r^m \). Then \( r^m \mid q - 1 \) and there is \( x \in GF(q) \) satisfying \( x^{r^m} = 1 \), \( x^{r^{m-1}} \neq 1 \). A Sylow \( r \)-subgroup of \( L_2(q) \) is given by:

\[
R = \begin{bmatrix} x^i & 0 \\ 0 & x^{-i} \end{bmatrix}, \quad 0 \leq i \leq r^m - 1.
\]

There exists an integer \( s \) such that, for all \( y \in GF(q) \) \( y^\beta = y^{p^s} \). Hence \( \alpha \) normalizes, and, thus, centralizes \( R \). Thus

\[
\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}^\beta = \begin{bmatrix} x^\beta & 0 \\ 0 & x^{-\beta} \end{bmatrix}.
\]

In \( L_2(q) \) this means \( x^\beta = \pm x \). If \( x^\beta = -x \) then \( x^{\beta^2} = x \) and, since \( \beta \) is an \( r \)-element with \( r \) odd, \( x^\beta = x \). Hence we may assume \( x^\beta = x \), i.e. \( x^\beta = x \), or \( x^{\beta^{r-1}} = 1 \). Let \( \beta \) have order \( r^k \). We have:

\[
\begin{align*}
r^m & \mid p^s - 1, \quad n \mid sr^k, \quad n \mid sr^{k-1}, \quad r^m \mid p^n - 1, \quad r^{m+1} \mid p^n - 1.
\end{align*}
\]

Thus there are integers \( a, b, \) and \( c \), satisfying:

\[
br^m + 1 = p^a, \quad an = sr^k, \quad (a, r) = 1, \quad cr^m + 1 = p^n, \quad (c, r) = 1.
\]

Hence, \( p^{an} = (cr^m + 1)^a = d_1r^m + 1 \), for some integer \( d_1 \) with \( (d_1, r) = 1 \).

But \( p^{an} = p^{sr^k} = (br^m + 1)^x = d_2r^m + 1 \), for some integer \( d_2 \) with \( (d_2, r) = r \), unless \( r^k = 1 \). Hence \( r^k = 1 \), and \( \beta = 1 \).

C. \( r \neq p, r \neq 2, r \mid q + 1 \)

Since \( GF(q) \subseteq GF(q^2) \), \( L_2(q) \subseteq L_2(q^2) \). Since \( |L_2(q)|_r = |L_2(q^2)|_r \), a Sylow \( r \)-subgroup of \( L_2(q) \) is a Sylow \( r \)-subgroup of \( L_2(q^2) \). Since \( \beta \) can be extended to a field automorphism, \( \beta^* \), of \( GF(q^2) \) satisfying \( t \mid \beta \), \( \beta^* \) where \( t = 1 \) or \( 2 \), we may regard \( \alpha^* \) as an automorphism of \( L_2(q^2) \) satisfying our hypotheses. But, now, \( r \mid q^2 - 1 \), so we may apply the proof of \( \beta \) to assert \( \beta^{*t} = 1 \). But for either \( t = 1 \) or \( t = 2 \) this implies \( \beta = 1 \), as desired.
D. \( r = 2, p \neq 2 \)

Since \( n \) must be odd \( (q = p^n = 3 \text{ or } 5 \pmod{8}) \), \( \alpha : X \to B^{-1}XB \) for some \( B \in \text{PGL}(2, q) \). But if \( \alpha \), hence \( B \), is a 2-element centralizing a Sylow 2-subgroup of \( L_2(q) \), and if \( B \notin L_2(q) \), then \( \text{PGL}(2, q) \) contains an abelian 2-subgroup of order 8. But the Sylow 2-subgroups of \( \text{PGL}(2, q) \) are dihedral of order 8 (see Carter and Fong [1]). Hence \( B \in L_2(q) \), as desired.

Remark. The structure of \( L_2(q) \) is discussed in Huppert [4, pp. 191–214].

\( L_2(q) = \text{PSL}(2, q) \).

**Lemma 3.4.** Let \( G = G_1 \times G_2 \times \cdots \times G_k \), with each \( G_i \) a simple, non-abelian, \( A \)-group. Let \( \alpha \) be an \( r \)-automorphism of \( G \) for some prime \( r \) with \( r \mid |G_i| \), all \( i \). Assume that \( \alpha \) centralizes a Sylow \( r \)-subgroup of \( G \). Then \( \alpha \) induces an inner automorphism of \( G \).

**Proof.** By the Krull-Schmidt theorem (see Huppert [4, I Satz 12.3]), \( \alpha \) induces a permutation of the \( G_i \). If this permutation were non-trivial \( \alpha \) would not centralize a Sylow \( r \)-subgroup of \( G \). Hence, for all \( i \), \( G_i^\alpha = G_i \).

By Lemma 3.3 \( \alpha \) induces an inner automorphism on each \( G_i \), hence on \( G \).

**Lemma 3.5.** Let \( G \) be an \( A \)-group. Let \( \text{Sol}(G) \) be the maximal normal subgroup of \( G \). Then \( G/\text{Sol}(G) \) is an extension of a semi-simple group by a (solvable) group of odd order.

**Proof.** Without loss of generality assume \( \text{Sol}(G) = 1 \). Any minimal normal subgroup of \( G \) is semi-simple. Let \( N \) be the maximal semi-simple normal subgroup of \( G \) (i.e. \( N = \text{socle of } G \)). Then, by Feit and Thompson [2] and Lemma 3.4, any 2-element of \( G \) induces an inner automorphism of \( N \). Thus, if \( G/N \) has even order, \( 1 < C(N) \leq G \). But then \( C(N) \cap N = 1 \) would contradict \( N = \text{socle of } G \).

**Corollary 3.6.** Let \( G \) be an \( A \)-group. Then \( G^{(\infty)}/\text{Sol}(G^{(\infty)}) \) is semi-simple.

**Proof.** By Lemma 3.5, the result is clear.

We shall eventually prove that \( G \) splits over \( G^{(\infty)} \) and that \( G^{(\infty)} \) splits over \( \text{Sol}(G^{(\infty)}) \).

4. \( A \)-Pairs

**Definition 4.1.** Let \( G \) be a normal subgroup of \( G_0 \). If for every prime \( p \), with \( p \mid |G| \), a Sylow \( p \)-subgroup of \( G_0 \) is abelian, then we call \((G_0, G)\) an \( A \)-pair.

**Remark.** If \((G_0, G)\) is an \( A \)-pair, then \( G \) is an \( A \)-group.
Lemma 4.2. Let \((G, N)\) be an A-pair. Assume that \(N\) is a homogeneous semi-simple group (i.e. a direct product of isomorphic, non-abelian, simple groups). Then there exists a complement, \(K\), of \(N\) in \(G\), with \(C(N) \leq K\), and all such complements are conjugate.

Proof. We proceed by induction on \(|G|\). If \(C(N) \neq 1\), \((G/C(N), \, NC(N)/C(N))\) satisfies our hypotheses, and, by induction, we get a complement \(K/C(N)\). Clearly, \(KN = G\), and \(K \cap N \leq K \cap NC(N) \cap N \leq C(N) \cap N = 1\). So we may assume \(C(N) = 1\). Hence, by Lemma 3.4 \(|G : N|, |N| = 1\). Application of the Schur-Zassenhaus theorem (see Huppert [4, I Hauptsatz 18.1, 18.2]) yields the desired result. (Note that \(C_{G/C(N)}(NC(N)/C(N)) = 1\).

Lemma 4.3. Let \((G_0, G)\) be an A-pair. If \(G^{(w)}\) is semi-simple (i.e. \(\text{Sol}(G^{(w)}) = 1\)) and non-trivial, then there exists a non-trivial, homogeneous semi-simple subgroup \(N\) of \(G\), with \(N \leq G_0\). (Hence, by Lemma 4.3, \(N\) is complemented in \(G_0\)).

Proof. Choose \(N\) to be a minimal normal subgroup of \(G_0\) contained in \(G^{(w)}\).

The usefulness of the following transfer theorem was brought to our attention by M. Isaacs.

Theorem 4.4. Let \(G\) be a group with abelian Sylow \(p\)-subgroup. Then \(p \nmid |G' \cap Z(G)|\).

Proof. See Huppert [4, IV Satz 2.2].

Corollary 4.5. Let \(G\) be an A-group. Then \(G' \cap Z(G) = 1\). In particular, if \(G' = G\), \(Z(G) = 1\).

Corollary 4.6. Let \(H \leq G\), \(H\) a solvable A-group. Let \(N\) be a relative system normalizer, in \(G\), of a Sylow system of \(H\). Then \(H'N = G\), \(H' \cap N = 1\).

Proof. By the Frattini argument \(HN = G\). Hence, it will suffice to show \(H'N_0 = H\), \(H' \cap N_0 = 1\) where \(N_0 = N \cap H\) is a system normalizer of \(H\). Now apply Huppert [4, VI Satz 14.4].

Remark. Corollaries 4.5 and 4.6 where proved for \(G\) solvable by Taunt [6]. By Corollary 4.6 a system normalizer of an A-group is abelian.

Lemma 4.7. Let \((G, H)\) be an A-pair with \(H \leq Z(G)\). Assume \(G/H\) is semi-simple. Then \(G'H = G\), \(G' \cap H = 1\) (hence \(G = G' \times H\)).

Proof. By Theorem 4.4 \(G' \cap H = 1\). And \(G/H = (G/H)' = G'H/H\); thus \(G'H = G\).
Lemma 4.8. Let \((G, H)\) be an A-pair with \(H\) abelian and \(G/H\) semi-simple. Then \(H\) is complemented in \(G\) and all its complements are conjugate.

Proof. We shall proceed by induction on \(|G|\).

Let \(p\) be a prime with \(p \mid |H|\). Let \(H_p\) be a Sylow \(p\)-subgroup of \(H\). The A-pair \((G/H_p, H/H_p)\) satisfies our hypotheses and, hence, \(H/H_p\) has a complement, \(L/H_p\), in \(G/H_p\) and all such complements are conjugate. If \(H_p \neq H\) then the A-pair \((L, H_p)\) satisfies our hypotheses and, by induction, \(H_p\) has a complement \(K\) in \(L\) and all such complements are conjugate. Then \(K\) will be a complement to \(H\) in \(G\). Let \(K_0\) be another complement to \(H\) in \(G\). Then \(K_0H_p/H_p\) is a complement to \(H/H_p\) in \(G/H_p\). Hence \(K_0H_p\) is conjugate to \(L\) and \(K_0\) is, therefore, conjugate to a complement of \(H_p\) in \(L\) (noting \(H_p \leq G\)); and, thus, \(K_0\) is conjugate to \(K\). Consequently, we may assume \(H = H_p\).

If \(H \neq C(H)\) then, by Lemma 4.7, \(C(H) = H \times S\), where \(S = C(H)^c \leq G\). The A-pair \((G/S, HS/S)\) satisfies our hypotheses and, by induction, we find a complement \(K/S\) of \(HS/S\) in \(G/S\). Thus \(KII = G\); and \(K \cap H \leq K \cap HS \cap H \leq S \cap H = 1\). Hence, \(K\) is a complement to \(H\) in \(G\). Let \(K_0\) be another such complement. Then \(C(H) = H(C(H) \cap K_0) = H \times (C(H) \cap K_0)\). Thus \(S = C(H)^c = (C(H) \cap K_0)^c\), so \(S \leq K_0\). By induction, \(K_0S\) is conjugate to \(K/S\), hence \(K_0\) is conjugate to \(K\). Consequently, we may assume \(H = C(H)\).

In this case the hypotheses imply \((|G : H|, |H|) = 1\). The result now follows from the Schur-Zassenhaus theorem.

Theorem 4.9. Let \((G_0, G)\) be an A-pair, with \(G\) non-abelian. Then there exists a non trivial subgroup, \(N\), of \(G\), normal in \(G_0\), satisfying:

i) \(N\) is either a (abelian) \(p\)-group for some prime \(p\) or \(N\) is a homogeneous semi-simple group.

ii) \(N\) is complemented in \(G_0\).

Furthermore, we may choose a complement \(K\) of \(N\) such that for any prime \(p\), with \(p \mid |N|\), every \(p\)-element of \(K\) centralizes \(N\). And, if \(N\) is semi-simple, \(C(N) \leq K\).

Proof. If \(G^{(\infty)} = 1\), let the derived length of \(G\) be \(k(\neq 1)\). Let \(M = G^{(k-1)}\). Then, by Corollary 4.6, \(M\) is complemented in \(G_0\) (by a relative system normalizer of \(G^{(k-2)}\)). Any non-trivial Sylow subgroup of \(M\) will serve as \(N\).

If \(G^{(\infty)} \neq 1\), but \(\text{Sol}(G^{(\infty)}) = 1\), the result follows from Lemma 4.3.

If \(L = \text{Sol}(G^{(\infty)})\) has derived length \(k > 1\), then \(L^{(k-1)}\) is complemented in \(G_0\) (by a relative system normalizer of \(L^{(k-2)}\)). Again, any non-trivial Sylow subgroup of \(L^{(k-1)}\) will serve as \(N\).
If \( L = \text{Sol}(G^{(\omega)}) \) is abelian and non-trivial, then Lemma 4.8 applied to the \( A \)-pair \( (G^{(\omega)}, L) \) implies that \( L \) has a complement, \( K \), in \( G^{(\omega)} \) and that all such complements are conjugate. By the Frattini argument \( G_0 = N(K)L \).

It will suffice to show that \( N_L(K) = 1 \) (for then a non-trivial Sylow subgroup of \( L \) will serve as \( N \)). But \( N_L(K) = C_L(K) \subseteq Z(LK) = Z(G^{(\omega)}) \). But \( Z(G^{(\omega)}) = 1 \), by Corollary 4.5.

5. Structure Theorems

**Theorem 5.1.** Let \( (G_0, G) \) be an \( A \)-pair. Let \( H = \text{Sol}(G^{(\omega)}) \). Then there exist subgroups \( S \), of \( G \), and \( K \), of \( G_0 \), satisfying:

i) \( HS = G^{(\omega)} \);

ii) \( HSK = G_0 \), \( |H||S||K| = |G_0| \);

iii) \( K \leq N(S) \);

iv) \( SK \) normalizes a Sylow system of \( H \);

v) If \( M \leq S \), \( M \leq SK \), then \( C_{MK}(M) \leq K \).

Furthermore if \( S_1 \), \( K_1 \) satisfy conditions i)—v) then there exists \( x \in G_0 \) such that \( S_1^x = S \), \( K_1^x = K \).

**Remark.** Condition i), together with Lemmas 3.5 and 3.6, implies that \( S \) is semi-simple and \( K \cap G \) is solvable.

**Proof.** We proceed by induction on \( |G_0| \).

First, let us assume \( H \neq 1 \). By Theorem 4.9 there is a non-trivial \( p \)-group \( N \leq G_0 \), \( N \leq H \), which is complemented in \( G_0 \), say \( G_0 = NT \), \( N \cap T = 1 \).

By our inductive hypothesis \( T = H_1SK \), where \( H_1 = \text{Sol}(T^{(\omega)}) \), \( (T_1 = T \cap G) \), and \( H_1 \), \( S \), \( K \) satisfy i)—v). We readily find that \( G_0 = HSK \) is the desired factorization.

Now we may assume \( H = 1 \). We must have \( S = G^{(\omega)} \). If \( S = 1 \) the result is trivial, so let us assume \( S \neq 1 \). In this case Theorem 4.9 asserts that there is a minimal normal subgroup, \( N \), of \( G_0 \), such that \( N \leq S \) and \( N \) is complemented in \( G_0 \) by a group \( T \) with \( C(N) \leq T \). By induction \( T = S_0K \); and \( S = N \times S_0 \) (where \( S_0 = (T \cap G)^{(\omega)} = C_\delta(N) \)). Now \( G_0 = SK \), and \( S \), \( K \) satisfy conditions i)—iv). To verify v) let \( M \leq S \), \( M \leq SK \). If \( M \cap N = 1 \), then \( M \leq S_0 \) and we are done by induction. So we may assume \( M = N \times N_0 \) for some \( N_0 \leq G_0 \). Now \( C_{MK}(M) \leq C_{MK}(N_0) = NC_{N_0}(N_0) \leq NK \) (by induction). But \( C_{MK}(M) \leq C(N) \leq T = S_0K \). Thus \( C_{MK}(M) \leq NK \cap S_0K = K \).

Now assume \( S_1 \), \( K_1 \) satisfy conditions i)—v). Then there exist abelian subgroups \( H_0 \) and \( H_1 \) of \( H \) such that \( H_0SK \) and \( H_1S_1K_1 \) are relative system
normalizers, in $G_0$, of Sylow systems of $H$. Hence, there is $y \in H$ such that $(H_0 S K)^y = H_0 S_1 K_1$ and $H_0^y = H_1$. Thus, without loss of generality we may assume $H = H_0$ is abelian, and $H S K = H S_1 K_1 = G_0$. Now $H S = H S_1 = G^{(c)}$ and, by Lemma 4.8 there is $x \in H$ such that $S_x = S_1$. Thus we may assume $S = S_1$, $H S K = H S K_1 = G_0$. But, as in the proof of Theorem 4.9, $S K = S K_1 = G_0$. So we may assume $G_0 = S K = S K_1$. Let $N$ be a minimal normal subgroup of $G_0$, contained in $S$. Let $S = N \times N_0$, $N_0 \subseteq G_0$. Then $C(N) = N C_{N_0}(N) \subseteq N_0 K$. Similarly $C(N) \subseteq N_0 K_1$. By Lemma 4.2 $N_0 K_1$ is conjugate to $N_0 K$ (by an element of $N$, which fixes $N_0$). Hence, by induction, $K$ is conjugate to $K_1$ (by an element of $N_0$) and the theorem follows.

**Remark.** If we examine the inductive nature of the proofs of Theorems 4.9 and 5.1 we find that, if $G$ is an $A$-group, then

$$G = H_1 \cdot H_2 \cdot \cdots \cdot H_r \cdot S_1 \cdot S_2 \cdot \cdots \cdot S_t \cdot K_1 \cdot K_2 \cdot \cdots \cdot K_s,$$

where each $H_i$ and $K_i$ is a $p$-group, for some (varying) prime $p$, and each $S_t$ is a homogeneous semi-simple group. Furthermore each subgroup in the factorization normalizes each of the preceding ones and:

$$H = H_1 \cdot H_2 \cdot \cdots \cdot H_r,$$

$$S = S_1 \times S_2 \times \cdots \times S_t,$$

$$K = K_1 \cdot K_2 \cdot \cdots \cdot K_s,$$

$$|H| = |H_1| \cdot |H_2| \cdot \cdots \cdot |H_r|,$$

$$|S| = |S_1| \cdot |S_2| \cdot \cdots \cdot |S_t|,$$

$$|K| = |K_1| \cdot |K_2| \cdot \cdots \cdot |K_s|,$$

($H, S, K$ as in Theorem 5.1).

This generalizes Taunt's "Basis Theorem" [6, Theorem 7.1].

**Theorem 5.2.** Let $(G_0, G)$ be an $A$-pair, $H = \text{Sol}(G^{(c)})$. Let $G_0 = H S K$, where $S$ and $K$ satisfy conditions i)--v) of Theorem 5.1. Let $N \subseteq G_0$. Then

$$N = (N \cap H)(N \cap S)(N \cap K).$$

**Proof.** We proceed by induction on $|G_0|$.

A. First let us assume $H = 1$. Without loss of generality we may assume $S \neq 1$. If $N \cap S = 1$, $N \subseteq C(S)$ and, by condition v), $N \subseteq K$, as desired. Let $S_1 = N \cap S$ and let $S_0 \times S_1 = S$. Then $N \subseteq C(S_0) = S_1 C_{S_0 K}(S_0) \subseteq S_1 K$. Thus $N = S_1(N \cap K) = (N \cap S)(N \cap K)$.

B. Assume $K = 1$. Without loss of generality we may assume $H \neq 1$, $S \neq 1$. Let $N_1 = H \cap N$. If $N_1 \neq 1$ consider $G_0/N_1 = H/N_1 \cdot S N_1/N_1$. By induction $N/N_1 = (N \cap S N_1)/N_1$, and, thus, $N = (N \cap S)N_1$ as desired. So we may assume $H \cap N = 1$. 

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Let $H_1 \leq H$ be chosen such that $H_1S$ is a relative system normalizer, in $G_0$, of a Sylow system of $H$. Since $N \leq C(H)$, $N \leq H_1S$. Now $NH_1 = H_2(NH_1 \cap S)$. Hence $N$ and $NH_1 \cap S$ are complements of $H_1$ in $NH_1$ and, by Lemma 4.8, $N = NH_1 \cap S$. Thus $N \leq S$, as desired.

C. Assume $H \neq 1$, $S \neq 1$, $K \neq 1$. Let $N_1 = N \cap H$. If $N_1 \neq 1$ consider $G_0/N_1 = H/N_1 \cdot SN_1/N_1 \cdot KN_1/N_1$. By induction

$$N = (N \cap SN_1)(N \cap KN_1) = (N \cap H)(N \cap S)(N \cap K).$$

Thus, we may assume $H \cap N = 1$. By part A, it will suffice to show $N \leq SK$.

Let $H_1 \leq H$ be chosen such that $H_1SK$ is a relative system normalizer, in $G_0$, of a Sylow system of $H$. Since $N \leq C(H)$, $N \leq H_1SK$. But $[S, N] \leq N \cap H_1S = (by \ part \ B.) (N \cap H_1)(N \cap S) = N \cap S$. Thus $N \leq N_{H_1SK}(S)$. But as in the proof of Theorem 4.9, $N_{H_1SK}(S) = SK$. Thus $N \leq SK$, as desired.

**Remark.** If $G$ is an $A$-group, and $N \leq G$, and

$$G = H_1 \cdot H_2 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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