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Finite Groups Whose Sylow Subgroups Are Abelian

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1. INTRODUCTION

In this paper we study the structure of A-groups, finite groups all of whose Sylow subgroups are abelian. We use the recent results of Walter [7], classifying the simple groups with abelian Sylow 2-subgroups, to extend the work of Taunt [6], who studied the structure of solvable A-groups.

We prove the following:

THEOREM. *Let G be an A-group. Then there exist subgroups H, S, and K, of G satisfying*:

i) $HSK = G$, $|H||S||K| = |G|$.

ii) $H \trianglelefteq G$, $\quad K \leqslant N(S)$.

(i.e. HSK is a triple semi-direct product)

iii) *H and K are solvable, S is semi-simple.*

Furthermore, by adding appropriate conditions we insure that S, K, and SK are determined up to conjugacy (and H is uniquely determined) and, if $N \trianglelefteq G$, then $N = (N \cap H)(N \cap S)(N \cap K)$.

2. NOTATION

Our notation is standard and follows, for the most part, that of Huppert [4]. The following items may be unusual:

Let G be a finite group, then,

$G^{(\infty)}$ denotes $\cap\, G^{(i)}$, the intersection of the members of the derived series for G (*i.e.* the "last" member of the derived series).

Sol(G) denotes the maximal normal solvable subgroup of G.

Unless otherwise stated, all groups we consider are finite.

3. A-Groups

Definition 3.1. An A-group is a finite group, all of whose Sylow subgroups are abelian.

Theorem 3.2. (Walter) *Let G be a simple, non-abelian, A-group. Then G is isomorphic to $J(11)$, the Janko group (see Janko [5]), or G is isomorphic to $L_2(q)$ for $q > 3$ and $q \equiv 0,3$, or 5 (mod 8).*

Proof. By Walter [7] a simple, non-abelian, group with abelian Sylow 2-subgroup is isomorphic to $J(11)$, $L_2(q)$ (for $q > 3$ and $q \equiv 0,3$ or 5 (mod 8)), or G is of Ree type. Since groups of Ree type have non-abelian Sylow 3-subgroups (see Ward [8]), the result follows.

Lemma 3.3. *Let G be a simple, non-abelian, A-group. Let α be an r-automorphism of G for some prime r with $r \mid |G|$. Assume α centralizes a Sylow r-subgroup of G. Then α induces an inner automorphism of G.*

Proof. By Theorem 3.2 we may assume G is $J(11)$ or $L_2(q)$. For $J(11)$ the result is trivial since, by Janko [5], the outer automorphism group of $J(11)$ is 1. So we may assume $G = L_2(q)$, $q > 3$, $q \equiv 0,3$ or 5 (mod 8), and let $q = p^n$, for some prime p.

From Gorenstein [3, p. 462] we see that the automorphism group of $L_2(q)$ is the semi-direct product $\langle\beta_0\rangle$ PGL(2, q), where β_0 is induced by a field automorphism of $GF(q)$ of order n and the elements of PGL(2, q) act by conjugation. Thus we shall assume:

$$\alpha : X \to B^{-1}X^{\beta}B$$

for some $B \in \mathrm{PGL}(2, q)$, $\beta \in \langle\beta_0\rangle$.

However, for $r \neq 2$, or for $r = 2 = p$, any Sylow r-subgroup of $\langle\beta_0\rangle$ PGL(2, q) lies in $\langle\beta_0\rangle L_2(q)$. So we may assume (in these cases) that $B \in L_2(q)$. Furthermore, if $\langle\delta\rangle$ is a Sylow r-subgroup of $\langle\beta_0\rangle$ and $\langle\delta\rangle$ normalizes R, a Sylow r-subgroup of $L_2(q)$, then some $L_2(q)$ conjugate of α lies in $\langle\delta\rangle$ R, hence must normalize, and by hypothesis centralize, R. So we may assume, without loss of generality, $\alpha \in \langle\delta\rangle \leqslant \langle\beta_0\rangle$, and

$$\alpha : X \to X^{\beta} \text{ (in case } r \neq 2 \text{ or } r = 2 = p).$$

During the remainder of this proof we will perform certain matrix calculations. The matrices used—although written in GL(2, q)—will represent cosets of $Z(\mathrm{GL}(2, q))$, *i.e.* elements of PGL(2, q) or of $L_2(q)$.

A. $r = p$

A Sylow r-subgroup of $L_2(q)$ is given by:

$$R = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \middle| \, x \in GF(q) \right\}.$$

α clearly normalizes, hence must centralize R. Thus

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{\beta} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Hence $x^{\beta} = x$ for all $x \in \mathrm{GF}(q)$. Hence $\beta = 1$.

B. $r \neq p, r \neq 2, r \mid q - 1$

Let $|G|_r = r^m$. Then $r^m \mid q - 1$ and there is $x \in \mathrm{GF}(q)$ satisfying $x^{r^m} = 1$, $x^{r^{m-1}} \neq 1$. A Sylow r-subgroup of $L_2(q)$ is given by:

$$R = \left\{ \begin{bmatrix} x^i & 0 \\ 0 & x^{-i} \end{bmatrix} \middle| \, 0 \leqslant i \leqslant r^m - 1 \right\}.$$

There exists an integer s such that, for all $y \in \mathrm{GF}(q)$ $y^{\beta} = y^{p^s}$. Hence α normalizes, and, thus, centralizes R. Thus

$$\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}^{\beta} = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}.$$

In $L_2(q)$ this means $x^{\beta} = \pm x$. If $x^{\beta} = -x$ then $x^{\beta^2} = x$ and, since β is a r-element with r odd, $x^{\beta} = x$. Hence we may assume $x^{\beta} = x$, i.e. $x^{p^s} = x$, or $x^{p^{s-1}} = 1$. Let β have order r^k. We have:

$$r^m \mid p^s - 1, \quad n \mid sr^k, \quad n \nmid sr^{k-1}, \quad r^m \mid p^n - 1, \quad r^{m+1} \nmid p^n - 1.$$

Thus there are integers a, b, and c, satisfying:

$$br^m + 1 = p^s, \quad an = sr^k, \quad (a, r) = 1, \quad cr^m + 1 = p^n, \quad (c, r) = 1.$$

Hence, $p^{an} = (cr^m + 1)^a = d_1 r^m + 1$, for some integer d_1 with $(d_1, r) = 1$. But $p^{an} = p^{sr^k} = (br^m + 1)^{r^k} = d_2 r^m + 1$, for some integer d_2 with $(d_2, r) = r$, unless $r^k = 1$. Hence $r^k = 1$, and $\beta = 1$.

C. $r \neq p, r \neq 2, r \mid q + 1$

Since $\mathrm{GF}(q) \leqslant \mathrm{GF}(q^2)$, $L_2(q) \leqslant L_2(q^2)$. Since $|L_2(q)|_r = |L_2(q^2)|_r$, a Sylow r-subgroup of $L_2(q)$ is a Sylow r-subgroup of $L_2(q^2)$. Since β can be extended to a field automorphism, β^*, of $\mathrm{GF}(q^2)$ satisfying $t\,|\langle\beta\rangle| = |\langle\beta^*\rangle|$ where $t = 1$ or 2, we may regard α^t as an automorphism of $L_2(q^2)$ satisfying our hypotheses. But, now, $r \mid q^2 - 1$, so we may apply the proof of B. to assert $\beta^{*t} = 1$. But for either $t = 1$ or $t = 2$ this implies $\beta = 1$, as desired.

D. $r = 2, p \neq 2$

Since n must be odd ($q = p^n \equiv 3$ or $5 \pmod 8$)), $\alpha : X \to B^{-1}XB$ for some $B \in \mathrm{PGL}(2, q)$. But if α, hence B, is a 2-element centralizing a Sylow 2-subgroup of $L_2(q)$, and if $B \notin L_2(q)$, then $\mathrm{PGL}(2, q)$ contains an abelian 2-subgroup of order 8. But the Sylow 2-subgroups of $\mathrm{PGL}(2, q)$ are dihedral of order 8 (see Carter and Fong [1]). Hence $B \in L_2(q)$, as desired.

Remark. The structure of $L_2(q)$ is discussed in Huppert [4, pp. 191–214]. $L_2(q) = \mathrm{PSL}(2, q)$.

LEMMA 3.4. *Let $G = G_1 \times G_2 \times \cdots \times G_k$, with each G_i a simple, non-abelian, A-group. Let α be an r-automorphism of G for some prime r with $r \mid |G_i|$, all i. Assume that α centralizes a Sylow r-subgroup of G. Then α induces an inner automorphism of G.*

Proof. By the Krull-Schmidt theorem (see Huppert [4, I *Satz* 12.3]), α induces a permutation of the G_i. If this permutation were non-trivial α would not centralize a Sylow r-subgroup of G. Hence, for all i, $G_i^\alpha = G_i$. By Lemma 3.3 α induces an inner automorphism on each G_i, hence on G.

LEMMA 3.5. *Let G be an A-group. Let* $\mathrm{Sol}(G)$ *be the maximal normal subgroup of G. Then $G/\mathrm{Sol}(G)$ is an extension of a semi-simple group by a (solvable) group of odd order.*

Proof. Without loss of generality assume $\mathrm{Sol}(G) = 1$. Any minimal normal subgroup of G is semi-simple. Let N be the maximal semi-simple normal subgroup of G (*i.e.* $N =$ sockel of G). Then, by Feit and Thompson [2] and Lemma 3.4, any 2-element of G induces an inner automorphism of N. Thus, if G/N has even order, $1 < C(N) \trianglelefteq G$. But then $C(N) \cap N = 1$ would contradict $N =$ sockel of G.

COROLLARY 3.6. *Let G be an A-group. Then $G^{(\infty)}/\mathrm{Sol}(G^{(\infty)}$ is semi-simple.*

Proof. By Lemma 3.5, the result is clear.

We shall eventually prove that G splits over $G^{(\infty)}$ and that $G^{(\infty)}$ splits over $\mathrm{Sol}(G^{(\infty)})$.

4. A-PAIRS

DEFINITION 4.1. Let G be a normal subgroup of G_0. If for every prime p, with $p \mid |G|$, a Sylow p-subgroup of G_0 is abelian, then we call (G_0, G) an A-pair.

Remark. If (G_0, G) is an A-pair, then G is an A-group.

LEMMA 4.2. *Let (G, N) be an A-pair. Assume that N is a homogeneous semi-simple group (i.e. a direct product of isomorphic, non-abelian, simple groups). Then there exists a complement, K, of N in G, with $C(N) \leqslant K$, and all such complements are conjugate.*

Proof. We proceed by induction on $|G|$. If $C(N) \neq 1$, $(G/C(N), NC(N)/C(N))$ satisfies our hypotheses, and, by induction, we get a complement $K/C(N)$. Clearly, $KN = G$, and $K \cap N \leqslant K \cap NC(N) \cap N \leqslant C(N) \cap N = 1$. So we may assume $C(N) = 1$. Hence, by Lemma 3.4 $(|G : N|, |N|) = 1$. Application of the Schur-Zassenhaus theorem (see Huppert [4, I *Hauptsatz* 18.1, 18.2]) yields the desired result. (Note that $C_{G/C(N)}(NC(N)/C(N)) = 1$.)

LEMMA 4.3. *Let (G_0, G) be an A-pair. If $G^{(\infty)}$ is semi-simple (i.e. $\mathrm{Sol}(G^{(\infty)}) = 1$) and non-trivial, then there exists a non-trivial, homogeneous semi-simple subgroup N of G, with $N \trianglelefteq G_0$. (Hence, by Lemma 4.3, N is complemented in G_0).*

Proof. Choose N to be a minimal normal subgroup of G_0 contained in $G^{(\infty)}$.

The usefulness of the following transfer theorem was brought to our attention by M. Isaacs.

THEOREM 4.4. *Let G be a group with abelian Sylow p-subgroup. Then $p \nmid |G' \cap Z(G)|$.*

Proof. See Huppert [4, IV *Satz* 2.2].

COROLLARY 4.5. *Let G be an A-group. Then $G' \cap Z(G) = 1$. In particular, if $G' = G$, $Z(G) = 1$.*

COROLLARY 4.6. *Let $H \trianglelefteq G$, H a solvable A-group. Let N be a relative system normalizer, in G, of a Sylow system of H. Then $H'N = G$, $H' \cap N = 1$.*

Proof. By the Frattini argument $HN = G$. Hence, it will suffice to show $H'N_0 = H$, $H' \cap N_0 = 1$ where $N_0 = N \cap H$ is a system normalizer of H. Now apply Huppert [4, VI *Satz* 14.4].

Remark. Corollaries 4.5 and 4.6 where proved for G solvable by Taunt [6]. By Corollary 4.6 a system normalizer of an A-group is abelian.

LEMMA 4.7. *Let (G, H) be an A-pair with $H \leqslant Z(G)$. Assume G/H is semi-simple. Then $G'H = G$, $G' \cap H = 1$ (hence $G = G' \times H$).*

Proof. By Theorem 4.4 $G' \cap H = 1$. And $G/H = (G/H)' = G'H/H$; thus $G'H = G$.

LEMMA 4.8. *Let (G, H) be an A-pair with H abelian and G/H semi-simple. Then H is complemented in G and all its complements are conjugate.*

Proof. We shall proceed by induction on $|G|$.

Let p be a prime with $p \mid |H|$. Let H_p be a Sylow p-subgroup of H. The A-pair $(G/H_p, H/H_p)$ satisfies our hypotheses and, hence, H/H_p has a complement, L/H_p, in G/H_p and all such complements are conjugate. If $H_p \neq H$ then the A-pair (L, H_p) satisfies our hypotheses and, by induction, H_p has a complement K in L and all such complements are conjugate. Then K will be a complement to H in G. Let K_0 be another complement to H in G. Then K_0H_p/H_p is a complement to H/H_p in G/H_p. Hence K_0H_p is conjugate to L and K_0 is, therefore, conjugate to a complement of H_p in L (noting $H_p \trianglelefteq G$); and, thus, K_0 is conjugate to K. Consequently, we may assume $H = H_p$.

If $H \neq C(H)$ then, by Lemma 4.7, $C(H) = H \times S$, where $S = C(H)' \trianglelefteq G$. The A-pair $(G/S, HS/S)$ satisfies our hypotheses and, by induction, we find a complement K/S of HS/S in G/S. Thus $KH = G$; and $K \cap H \leqslant K \cap HS \cap H \leqslant S \cap H = 1$. Hence, K is a complement for H in G. Let K_0 be another such complement. Then $C(H) = H(C(H) \cap K_0) = H \times (C(H) \cap K_0)$. Thus $S = C(H)' = (C(H) \cap K_0)'$, so $S \leqslant K_0$. By induction, K_0/S is conjugate to K/S, hence K_0 is conjugate to K. Consequently, we may assume $H = C(H)$.

In this case the hypotheses imply $(|G : H|, |H|) = 1$. The result now follows from the Schur-Zassenhaus theorem.

THEOREM 4.9. *Let (G_0, G) be an A-pair, with G non-abelian. Then there exists a non trivial subgroup, N, of G, normal in G_0, satisfying:*

i) *N is either a (abelian) p-group for some prime p or N is a homogeneous semi-simple group.*

ii) *N is complemented in G_0.*

Furthermore, we may choose a complement K of N such that for any prime p, with $p \mid |N|$, every p-element of K centralizes N. And, if N is semi-simple, $C(N) \leqslant K$.

Proof. If $G^{(\infty)} = 1$, let the derived length of G be $k (\neq 1)$. Let $M = G^{(k-1)}$. Then, by Corollary 4.6, M is complemented in G_0 (by a relative system normalizer of $G^{(k-2)}$). Any non-trivial Sylow subgroup of M will serve as N.

If $G^{(\infty)} \neq 1$, but $\mathrm{Sol}(G^{(\infty)}) = 1$, the result follows from Lemma 4.3.

If $L = \mathrm{Sol}(G^{(\infty)})$ has derived length $k > 1$, then $L^{(k-1)}$ is complemented in G_0 (by a relative system normalizer of $L^{(k-2)}$). Again, any non-trivial Sylow subgroup of $L^{(k-1)}$ will serve as N.

If $L = \mathrm{Sol}(G^{(\infty)})$ is abelian and non-trivial, then Lemma 4.8 applied to the A-pair $(G^{(\infty)}, L)$ implies that L has a complement, K, in $G^{(\infty)}$ and that all such complements are conjugate. By the Frattini argument $G_0 = N(K)L$. It will suffice to show that $N_L(K) = 1$ (for then a non-trivial Sylow subgroup of L will serve as N). But $N_L(K) = C_L(K) \leqslant Z(LK) = Z(G^{(\infty)})$. But $Z(G^{(\infty)}) = 1$, by Corollary 4.5.

5. Structure Theorems

Theorem 5.1. *Let (G_0, G) be an A-pair. Let $H = \mathrm{Sol}(G^{(\infty)})$. Then there exist subgroups S, of G, and K, of G_0, satisfying:*

i) $HS = G^{(\infty)}$;

ii) $HSK = G_0$, $|H||S||K| = |G_0|$;

iii) $K \leqslant N(S)$;

iv) *SK normalizes a Sylow system of H;*

v) *If $M \leqslant S$, $M \trianglelefteq SK$, then $C_{MK}(M) \leqslant K$.*

Furthermore if S_1, K_1 satisfy conditions i)—v) *then there exists $x \in G_0$ such that $S_1^x = S$, $K_1^x = K$.*

Remark. Condition i), together with Lemmas 3.5 and 3.6, implies that S is semi-simple and $K \cap G$ is solvable.

Proof. We proceed by induction on $|G_0|$.

First, let us assume $H \neq 1$. By Theorem 4.9 there is a non-trivial p-group $N \trianglelefteq G_0$, $N \leqslant H$, which is complemented in G_0, say $G_0 = NT$, $N \cap T = 1$. By our inductive hypothesis $T = H_1SK$, where $H_1 = \mathrm{Sol}(T_1^{(\infty)})$, $(T_1 = T \cap G)$, and H_1, S, K satisfy i)–v). We readily find that $G_0 = HSK$ is the desired factorization.

Now we may assume $H = 1$. We must have $S = G^{(\infty)}$. If $S = 1$ the result is trivial, so let us assume $S \neq 1$. In this case Theorem 4.9 asserts that there is a minimal normal subgroup, N, of G_0, such that $N \leqslant S$ and N is complemented in G_0 by a group T with $C(N) \leqslant T$. By induction $T = S_0K$; and $S = N \times S_0$ (where $S_0 = (T \cap G)^{(\infty)} = C_S(N)$). Now $G_0 = SK$, and S, K satisfy conditions i)–iv). To verify v) let $M \leqslant S$, $M \trianglelefteq SK$. If $M \cap N = 1$, then $M \leqslant S_0$ and we are done by induction. So we may assume $M = N \times N_0$ for some $N_0 \leqslant G_0$. Now $C_{MK}(M) \leqslant C_{MK}(N_0) = NC_{N_0K}(N_0) \leqslant NK$ (by induction). But $C_{MK}(M) \leqslant C(N) \leqslant T = S_0K$. Thus $C_{MK}(M) \leqslant NK \cap S_0K = K$.

Now assume S_1, K_1 satisfy conditions i)–v). Then there exist abelian subgroups H_0 and H_1 of H such that H_0SK and $H_1S_1K_1$ are relative system normalizers, in G_0, of Sylow systems of H. Hence, there is $y \in H$ such that $(H_0SK)^y = H_1S_1K_1$ and $H_0{}^y = H_1$. Thus, without loss of generality we may assume $H = H_0$ is abelian, and $HSK = HS_1K_1 = G_0$. Now $HS = HS_1 = G^{(\infty)}$ and, by Lemma 4.8 there is $x \in H$ such that $S^x = S_1$. Thus we may assume $S = S_1$, $HSK = HSK_1 = G_0$. But, as in the proof of Theorem 4.9, $SK = SK_1 = N(S)$. So we may assume $G_0 = SK = SK_1$. Let N be a minimal normal subgroup of G_0, contained in S. Let $S = N \times N_0$, $N_0 \trianglelefteq G_0$. Then $C(N) = N_0C_{NK}(N) \leqslant N_0K$. Similarly $C(N) \leqslant N_0K_1$. By Lemma 4.2 N_0K_1 is conjugate to N_0K (by an element of N, which fixes N_0). Hence, by induction, K is conjugate to K_1 (by an element of N_0) and the theorem follows.

Remark. If we examine the inductive nature of the proofs of Theorems 4.9 and 5.1 we find that, if G is an A-group, then

$$G = H_1 \cdot H_2 \cdot \cdots \cdot H_r \cdot S_1 \cdot S_2 \cdot \cdots \cdot S_t \cdot K_1 \cdot K_2 \cdot \cdots \cdot K_s,$$

where each H_i and K_j is a p-group, for some (varying) prime p, and each S_k is a homogeneous semi-simple group. Furthermore each subgroup in the factorization normalizes each of the preceeding ones and:

$$H = H_1 \cdot H_2 \cdot \cdots \cdot H_r, \qquad |H| = |H_1| \cdot |H_2| \cdot \cdots \cdot |H_r|,$$
$$S = S_1 \times S_2 \times \cdots \times S_t$$
$$K = K_1 \cdot K_2 \cdot \cdots \cdot K_s, \qquad |K| = |K_1| \cdot |K_2| \cdot \cdots \cdot |K_s|,$$

(H, S, K as in Theorem 5.1).

This generalizes Taunt's "Basis Theorem" [6, Theorem 7.1].

THEOREM 5.2. *Let (G_0, G) be an A-pair, $H =$ Sol$(G^{(\infty)})$. Let $G_0 = HSK$, where S and K satisfy conditions* i)–v) *of Theorem* 5.1. *Let $N \trianglelefteq G_0$. Then*

$$N = (N \cap H)(N \cap S)(N \cap K).$$

Proof. We proceed by induction on $|G_0|$.

A. First let us assume $H = 1$. Without loss of generality we may assume $S \neq 1$. If $N \cap S = 1$, $N \leqslant C(S)$ and, by condition v), $N \leqslant K$, as desired. Let $S_1 = N \cap S$ and let $S_0 \times S_1 = S$. Then $N \leqslant C(S_0) = S_1C_{S_0K}(S_0) \leqslant S_1K$. Thus $N = S_1(N \cap K) = (N \cap S)(N \cap K)$.

B. Assume $K = 1$. Without loss of generality we may assume $H \neq 1$, $S \neq 1$. Let $N_1 = H \cap N$. If $N_1 \neq 1$ consider $G_0/N_1 = H/N_1 \cdot SN_1/N_1$. By induction $N/N_1 = (N \cap SN_1)/N_1$, and, thus, $N = (N \cap S)N_1$ as desired. So we may assume $H \cap N = 1$.

Let $H_1 \leqslant H$ be chosen such that H_1S is a relative system normalizer, in G_0, of a Sylow system of H. Since $N \leqslant C(H)$, $N \leqslant H_1S$. Now $NH_1 = H_1(NH_1 \cap S)$. Hence N and $NH_1 \cap S$ are complements of H_1 in NH_1 and, by Lemma 4.8, $N = NH_1 \cap S$. Thus $N \leqslant S$, as desired.

C. Assume $H \neq 1$, $S \neq 1$, $K \neq 1$. Let $N_1 = N \cap H$. If $N_1 \neq 1$ consider $G_0/N_1 = H/N_1 \cdot SN_1/N_1 \cdot KN_1/N_1$. By induction

$$N = (N \cap SN_1)(N \cap KN_1) = (N \cap H)(N \cap S)(N \cap K).$$

Thus, we may assume $H \cap N = 1$. By part A. it will suffice to show $N \leqslant SK$.

Let $H_1 \leqslant H$ be chosen such that H_1SK is a relative system normalizer, in G_0, of a Sylow system of H. Since $N \leqslant C(H)$, $N \leqslant H_1SK$. But $[S, N] \leqslant N \cap H_1S =$ (by part B.) $(N \cap H_1)(N \cap S) = N \cap S$. Thus $N \leqslant N_{H_1SK}(S)$. But as in the proof of Theorem 4.9, $N_{H_1SK}(S) = SK$. Thus $N \leqslant SK$, as desired.

Remark. If G is an A-group, and $N \trianglelefteq G$, and

$$G = H_1 \cdot H_2 \cdot \cdots \cdot H_r \cdot S_1 \cdot S_2 \cdot \cdots \cdot S_t \cdot K_1 \cdot K_2 \cdot \cdots \cdot K_s$$

(as discussed in the previous remark) then Theorem 5.2, together with a result of Taunt [6, Theorem 5.1], shows that:

$$N = (N \cap H_1) \cdot \cdots \cdot (N \cap H_r) \cdot (N \cap S_1) \cdot \cdots \cdot (N \cap S_t) \cdot (N \cap K_1) \cdot \cdots \cdot (N \cap K_s).$$

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