European Journal of Combinatorics 32 (2011) 861–869

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On the circuit-spectrum of binary matroids

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ARTICLE INFO

Article history:
Available online 22 February 2011

ABSTRACT

Murty, in 1971, characterized the connected binary matroids with all circuits having the same size. We characterize the connected binary matroids with circuits of two different sizes, where the largest size is odd. As a consequence of this result we obtain both Murty's result and other results on binary matroids with circuits of only two sizes. We also show that it will be difficult to complete the general case of this problem.

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1. Introduction

We consider the problem of determining the binary matroids with circuits of two different sizes. Murty [11] solved the motivating instance of this problem below, where the matroids considered contain circuits of a single size. The *circuit-spectrum* of a matroid M is $\text{spec}(M) = \{|C| : C \in \mathcal{C}(M)\}$. A *k-subdivision* of a matroid is obtained by replacing each element by a series class of size k.

Theorem 1.1. *Let M be a connected binary matroid. For $c \in \mathbb{Z}^+$, $\text{spec}(M) = \{c\}$ if and only if M is isomorphic to one of the following matroids:*

(i) *a c-subdivision of $U_{0,1}$,*
(ii) *a k-subdivision of $U_{1,n}$, where $c = 2k$ and $n \geq 3$,*
(iii) *an l-subdivision of $PG(r, 2)^*$, where $c = 2^r l$ and $r \geq 2$,*
(iv) *an l-subdivision of $AG(r + 1, 2)^*$, where $c = 2^r l$ and $r \geq 2$.*

It is difficult to characterize the matroids having a particular circuit-spectrum set even when the set is small and the matroids belong to an interesting class. For example, characterizing the non-binary matroids M with $|\text{spec}(M)| = 1$ would involve solving questions from design theory (see results of Edmonds et al. [4,5,13]). Authors including Cordovil et al. [3,10] constructed all matroids M

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doi:10.1016/j.ejc.2011.01.012

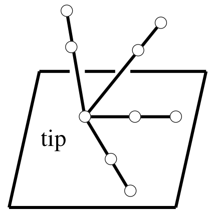

Fig. 1. Some of the lines of the binary spike S_4.

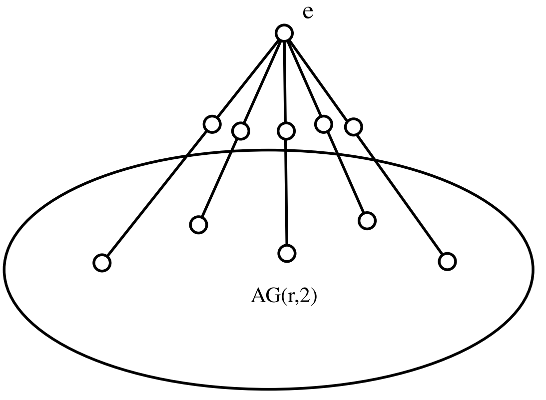

Fig. 2. $B(r, 2)$.

with spec$(M) \subseteq \{1, 2, 3, 4, 5\}$, and constructed all 3-connected binary matroids M with spec$(M) \subseteq \{3, 4, 5, 6, 7\}$.

We next give some terminology before stating the main results of the paper. This terminology mostly follows [12]. If M is a matroid, then the sets of circuits, hyperplanes, and series classes of M are denoted by $\mathcal{C}(M)$, $\mathcal{H}(M)$, and $\mathcal{S}(M)$, respectively. The *series-connection* of matroids M and N is denoted by $S(M, N)$ [12, Section 7.1]. The binary projective geometry and binary affine geometry of rank $r + 1$ are denoted by $PG(r, 2)$ and $AG(r, 2)$, respectively. For an integer exceeding two, the binary spike of rank n, denoted by S_n, is the vector matroid of the matrix consisting of all binary columns of length n with exactly one, $n - 1$, or n ones. The *tip* (*cotip*) of S_n (S_n^*) corresponds to the column of all ones. The Fano-matroid is S_3. The four 3-point lines of the binary spike S_4 are shown in Fig. 1.

The matroid $B(r, 2)$ is constructed as follows. Add a point e of projective space to $AG(r, 2)$, where e is outside the $r + 1$-dimensional subspace determined by $AG(r, 2)$. Add a point of projective space to each line joining e to a point of $AG(r, 2)$. The resulting matroid is $B(r, 2)$ (see Fig. 2). Equivalently, $B(r, 2)$ may be constructed by adding a single new point e of $PG(r + 1, 2)$ to $AG(r + 1, 2)$. The dual matroid $B(r, 2)^*$ is constructed as follows. Let $C_0, C_1, C_2, \ldots, C_r$ be a minimal set of circuits that spans the cycle space of $AG(r + 1, 2)^*$ with $C_2 = E(M) - C_1$ for some integer r exceeding one. If e is a new element of projective space, then $B(r, 2)^*$ is the matroid whose ground set is $E(AG(r + 1, 2)) \cup e$ and whose cycle space is spanned by $C_0 \cup e, C_1, C_2, \ldots, C_n$. The element e mentioned above is called the *tip* (*cotip*) of $B(r, 2)$ $(B(r, 2^*))$.

The first main result both generalizes Theorem 1.1 and has applications that provide information on the matroids with circuit-spectrum size two. A *deletable series class* in a connected matroid is one whose deletion from the matroid results in a connected matroid.

Theorem 1.2. *Let $c, d \in \mathbb{Z}^+$. Let T be a deletable series class in a connected binary matroid with corank exceeding one. If each circuit of M that avoids T has c elements and each circuit of M that contains T has d elements, then every series class of M different from T has l elements and the cosimplification of M is isomorphic to one of the following matroids.*

(i) $U_{1,n}$, *for some* $n \geq 3$, *where* $c = 2l$ *and* $d = l + |T|$.
(ii) S_n^*, *for some* $n \geq 4$, *where* T *is a subdivision of the cotip*, $c = 4l$, *and* $d = nl + |T|$.
(iii) $PG(r, 2)^*$, *for some* $r \geq 2$, *where* $c = 2^r l$ *and* $d = (2^r - 1)l + |T|$.
(iv) $AG(r + 1, 2)^*$, *for some* $r \geq 2$, *where* $c = 2^r l$ *and* $d = (2^r - 1)l + |T|$.
(v) $B(r, 2)^*$, *for some* $r \geq 3$, *where* T *is a subdivision of the cotip*, $c = 2^r l$, *and* $d = 2^r l + |T|$.

We next obtain Murty's result as a consequence of this result.

Proof of Theorem 1.1. If M is a circuit, then the result follows. If M has corank exceeding one, then it follows from [12, Section 10.2, Exercise 2] that M has a deletable series class. The result follows from Theorem 1.2 by taking $c = d$ because cases (ii) and (v) cannot occur (in these two cases $c < d$). □

The second main result provides a complete characterization of the connected binary matroids with a circuit-spectrum of size two in the special case that the largest circuit size is odd.

Theorem 1.3. *Let* $c, d \in \mathbb{Z}^+$ *with* $c < d$ *and* d *odd. Let* M *be a connected binary matroid. Then* $\text{spec}(M) = \{c, d\}$ *if and only if there are connected binary matroids* $M_0, M_1, \ldots, M_n$ *for some* $n \in \mathbb{Z}^+$ *such that the following hold.*

(i) $E(M_i) \cap E(M_j) = \{e\}$, *for distinct* i *and* j *in* $\{0, 1, \ldots, n\}$.
(ii) $E(M_0)$ *is a circuit of* M_0.
(iii) *For* $i \in \{1, 2, \ldots, n\}$, $\{e\}$ *is a series class of* M_i, *all other series classes of* M_i *have size* l_i, *and the cosimplification of* M_i *is isomorphic to one of the following matroids.*
  (a) U_{1,n_i}, *for some* $n_i \geq 3$, *where* $c = 2l_i$.
  (b) $PG(r_i, 2)^*$, *for some* $r_i \geq 2$, *where* $c = 2^{r_i} l_i$.
  (c) $AG(r_i, 2)^*$, *for some* $r_i \geq 3$, *where* $c = 2^{r_i - 1} l_i$.
  (d) $S_{n_i}^*$, *for some* $n_i \geq 4$, *and* e *is the cotip, where* $c = 4l_i$.
  (e) $B(r_i, 2)^*$, *for some* $r_i \geq 3$, *and* e *is the cotip, where* $c = 2^{r_i} l_i$.
(iv) $d = |E(M_0)| - 1 + d_1 + d_2 + \cdots + d_n > c$, *where* $d_i = \frac{c}{2}$ *when* (iii) (a) *holds*, $d_i = (2^{r_i} - 1)l_i$ *when* (iii) (b) *holds*, $d_i = (2^{r_i - 1} - 1)l_i$ *when* (iii) (c) *holds*, $d_i = n_i l_i$ *when* (iii) (d) *holds, and* $d_i = c$ *when* (iii) (e) *holds.*
(v) $M = S(M_0, M_1, \ldots, M_n)/e$.

We next note an attractive corollary of Theorem 1.3.

Corollary 1.4. *Let* M *be a* 3*-connected binary matroid with largest circuit size odd. Then* $|\text{spec}(M)| \leq 2$ *if and only if* M *is isomorphic to one of the following matroids.*

(i) $U_{0,1}$ *or* $U_{2,3}$.
(ii) S_{2n}^*, *for some* $n \geq 2$.
(iii) $B(r, 2)^*$, *for some* $r \geq 2$.

In Section 2 of the paper we give a useful lemma that is of independent interest as well as an application of this lemma. In Section 3 we give a construction that is used in the proof of the main results. In Section 4 we prove Theorem 1.2, while in Section 5 we prove Theorem 1.3. Finally, an example showing the difficulty in extending our results to a complete characterization of the connected binary matroids with circuits of two different sizes is given in Section 6.

2. A useful result

We use the following result which may be known.

Lemma 2.1. *Let* M *be a cosimple matroid with an assigned weight* $x_e \in \mathbb{R}$ *to each* $e \in E(M)$. *If* $\sum_{e \in D} x_e = 0$, *for every circuit* D *of* M, *then* $x_e = 0$ *for every* $e \in E(M)$.

Proof. Let M be a minimal counterexample. Then there exists $e \in E(M)$ such that $x_e \neq 0$. Choose $f \in E(M)$. We first show that for each series class S of $M \setminus f$,

$$\sum_{g \in S} x_g = 0. \tag{1}$$

Let N be the cosimplification of $M \setminus f$. For each element g of N, we define y_g to be equal to $\sum_{h \in S} x_h$, where S is the series class of $M \setminus f$ containing g. Note that $\sum_{g \in D} y_g = 0$ for every circuit D of N. The result holds for N by the choice of M. Therefore $y_g = 0$ for every $g \in E(N)$. Thus (1) follows.

If $f \neq e$, then (1) implies that $\sum_{g \in S} x_g = 0$ when S is the series class of $M \setminus f$ that contains e. Hence $|S| \geq 2$ because $x_e \neq 0$. Therefore $L^* = S \cup f$ is a coline of M such that $|L^*| \geq 3$ and $e \in L^*$. There exists $f' \in L^*$ such that $\sum_{g \in L^* - f'} x_g \neq 0$ because $x_e \neq 0$. We arrive at a contradiction to (1) when applied to f' instead of f because $L^* - f'$ is a series class of $M \setminus f'$. □

The following beautiful result of Basterfield and Kelly [1] and Greene [9] is a consequence of Lemma 2.1. Note that their result is more general in that they characterize the corresponding extremal matroids.

Corollary 2.2. *If M is a simple matroid, then $|\mathcal{H}(M)| \geq |E(M)|$.*

Proof. Observe that M^* is a cosimple matroid such that $|\mathcal{H}(M)| = |\mathcal{C}(M^*)|$. Consider the system having a variable x_e for each element $e \in E(M)$ and an equation for each circuit C of M^*:

$$\sum_{e \in C} x_e = 0.$$

By Lemma 2.1, this system has only the trivial solution. Therefore the number of equations $|\mathcal{C}(M^*)|$ is greater or equal to the number of variables $|E(M)|$. □

Many papers have been written that provide partial characterization of the matroids that satisfy $|\mathcal{H}(M)| = |E(M)| + d$ for $d \in \mathbb{Z}^+$ (see, for example, [2,8]). Dentice [6,7] characterized these matroids for $d \in \{1, 2\}$.

3. A construction

We say that a binary matroid N is *even* when $|S|$ is even for every series class S. Such a matroid has no coloops. Let N be an even binary matroid. For a set X disjoint from $E(N)$, we say that M is an X-extension of N provided $E(M) = X \cup E(N)$ and the cycle space of M is spanned by $\mathcal{C}(N)$ together with $X \cup (\cup_{S \in \mathcal{S}(N)} X_S)$, where a set $X_S \subseteq S$ such that $2|X_S| = |S|$ is chosen arbitrarily for every $S \in \mathcal{S}(N)$. Observe that all X-extensions of N are isomorphic. Moreover, for each $S \in \mathcal{S}(N)$, both X_S and $S - X_S$ are series classes of M. We next describe the dual construction: start with $N^* \setminus (\cup_{S \in \mathcal{S}(N)} X_S)$; add a parallel class X outside the space determined by $N^* \setminus (\cup_{S \in \mathcal{S}(N)} X_S)$; for each line that contains the parallel class X and the parallel class $S - X_S$ add a parallel class X_S. Note that there is just one binary matroid obtained this way.

Lemma 3.1. *Let N be an even binary matroid. If M is an X-extension of N, then $|C| = |X| + \frac{|E(M)|}{2}$, for every circuit $C \in \mathcal{C}(M) - \mathcal{C}(N)$.*

Proof. If $D = X \cup (\cup_{S \in \mathcal{S}(N)} X_S)$, then there is a collection of disjoint cycles $D'_1, D'_2, \ldots, D'_k$ of N such that $C = D \triangle D'_1 \triangle D'_2 \triangle \cdots \triangle D'_k$ because the cycle space of M is spanned by D and the cycle space of N. Let $D' = D'_1 \triangle D'_2 \triangle \cdots \triangle D'_k$. Hence $C - X = [\cup_{S \in \mathcal{S}(N): S \not\subseteq D'} X_S] \cup [\cup_{S \in \mathcal{S}(N): S \subseteq D'} (S - X_S)]$ and so

$$|C| = |X| + \sum_{S \in \mathcal{S}(N): S \not\subseteq D'} |X_S| + \sum_{S \in \mathcal{S}(N): S \subseteq D'} |S - X_S|.$$

As $|X_S| = |S - X_S|$, it follows that

$$|C| = |X| + \sum_{S \in \mathcal{S}(N)} |X_S| = |X| + \sum_{S \in \mathcal{S}(N)} \frac{|S|}{2} = |X| + \frac{|E(M)|}{2}. \quad \square$$

The straightforward proof of the next result is omitted.

Lemma 3.2. *Let N be a binary even matroid having a circuit-spectrum of size one and M be an X-extension of N. Then the following hold.*

(i) *If N is an l-subdivision of $U_{0,1}$, then M is obtained from an $\frac{l}{2}$-subdivision of $U_{1,3}$ by replacing a series class by X, where X is a series class.*

(ii) *If N is an l-subdivision of $U_{1,n}$, for some $n \geq 3$, then M is obtained from an $\frac{l}{2}$-subdivision of S_n^* by replacing the series class corresponding to the cotip by X, where X is a series class.*
(iii) *If N is an l-subdivision of $PG(r, 2)^*$, for some $r \geq 2$, then M is obtained from an $\frac{l}{2}$-subdivision of $PG(r+1, 2)^*$ by replacing a series class by X, where X is a series class.*
(iv) *If N is an l-subdivision of $AG(r+1, 2)^*$, for some $r \geq 2$, then M is obtained from an $\frac{l}{2}$-subdivision of $B(r+1, 2)^*$ by replacing the series class corresponding to the cotip by X, where X is a series class.*

Observe that, when $n = 3$ in (ii) and $r = 2$ in (iii), we obtain the same matroid whose cosimplification is F_7^* (therefore we may suppose that $n \geq 4$ in (ii)).

4. A proof of Theorem 1.2

Lemma 4.1. *Let $c, d \in \mathbb{Z}^+$. Suppose that T is a series class of a connected binary matroid M with corank exceeding one such that each circuit of M that avoids T has c elements and each circuit of M that contains T has d elements. If C and D are circuits of M that avoid and meet T, respectively, and $C \cap D \neq \emptyset$, then $C \triangle D$ is a circuit of M. Moreover, $|C \cap D| = \frac{c}{2}$.*

Proof. There are pairwise disjoint circuits $D_1, D_2, \ldots, D_n$ of M such that $C \triangle D = D_1 \cup D_2 \cup \cdots \cup D_n$. As T is a series class of M and $T \subseteq D - C$, it follows that T is contained in exactly one D_i, say D_1. Then $|D_1| = d$ and $|D_2| = |D_3| = \cdots = |D_n| = c$. Thus

$$c + d > |C \triangle D| = |D_1| + |D_2| + \cdots + |D_n| = d + (n-1)c.$$

Therefore $n = 1$ and $C \triangle D$ is a circuit of M.

To prove the second part of this lemma, observe that

$$2|C \cap D| = |C| + |D| - |C \triangle D| = c + d - d = c$$

and the result follows. □

We now prove the first main result of the paper.

Proof of Theorem 1.2. We argue by contradiction. Suppose this result is not true and choose a counterexample (M, T) such that $|E(M)|$ is minimum. Consider the connected matroid $N = M \setminus T$. It follows again from [12, Section 10.2, Exercise 2] that N is a circuit or N has a deletable series class T'. In the latter case the choice of (M, T) implies that the result holds for the pair (N, T'). Therefore N is a circuit or N is a $\frac{c}{2}$-subdivision of $U_{1,n}$, for some $n \geq 3$, or N is a $\frac{c}{2^r}$-subdivision of $PG(r, 2)^*$ or $AG(r+1, 2)^*$, for some $r \geq 2$. (Note that Murty's result holds from this observation when $c = d$.) If N is a circuit, then (i) follows for $n = 3$. Choose a circuit C of M such that $T \subseteq C$.

First, assume that N is a $\frac{c}{2}$-subdivision of $U_{1,n}$, for some $n \geq 3$. Let $S_1, S_2, \ldots, S_n$ be the series classes of N. The cycle space of M is spanned by the cycle space of N together with a circuit C of M such that $T \subseteq C$. For $i \in \{1, 2, \ldots, n\}$, let $X_i = S_i \cap C$. We may reorder $S_1, S_2, \ldots, S_n$ so that

$$|X_1| \geq |X_2| \geq \cdots \geq |X_n|.$$

Observe that

$$d = |C| = |T| + |X_1| + |X_2| + \cdots + |X_n|. \tag{2}$$

Suppose that $X_2 = X_3 = \cdots = X_n = \emptyset$. As M is connected, it follows that $X_1 \neq \emptyset$. By Lemma 4.1, $|X_1| = |C \cap (S_1 \cup S_2)| = \frac{c}{2}$ and so $X_1 = S_1$. Therefore $T \cup S_1, S_1 \cup S_2, S_2 \cup S_3, \ldots, S_{n-1} \cup S_n$ span both the cycle spaces of M and H, where H is a subdivision of $U_{1,n+1}$ having series classes $T, S_1, S_2, \ldots, S_n$. Thus $M = H$. Hence (i) follows.

Now we assume that it is not true that $X_2 = X_3 = \cdots = X_n = \emptyset$. There is an m such that $X_m \neq \emptyset$ and, when $m \neq n$, $X_{m+1} = \emptyset$. It follows from the assumption that $m \geq 2$. Choose a 2-element subset $\{i, j\}$ of $\{1, 2, \ldots, n\}$ such that $i \leq m$. By Lemma 4.1,

$$|X_i| + |X_j| = |X_i \cup X_j| = \frac{c}{2} \tag{3}$$

because $X_i \cup X_j = (S_i \cup S_j) \cap C$.

Now, we show that $m = n$. If $m < n$, then, by (3) applied to $\{i, j\} = \{m, m+1\}$, $|X_m| = \frac{c}{2}$ and so $|X_i| \geq \frac{c}{2}$, for every $i \leq m$; a contradiction to (3) (recall that $m \geq 2$). Hence $m = n$. By (3) applied to each 2-element subset of $\{1, 2, \ldots, n\}$, we conclude that $|X_1| = |X_2| = \cdots = |X_n| = \frac{c}{4}$. We have (ii), when $n \geq 4$, and (iii), when $n = 3$. (That is, M is a T-extension of N.)

Now, suppose that N is a $\frac{c}{2^r}$-subdivision of $PG(r, 2)^*$ or $AG(r+1, 2)^*$, for some $r \geq 2$. We consider two subcases. First, assume that $C \cap D = \emptyset$, for some $D \in \mathcal{C}(N)$. In this subcase we establish that (iv) occurs. Now we prove that D is the unique circuit of N avoiding C. Assume that $D'' \neq D$ is a circuit of N such that $D'' \cap C = \emptyset$. Hence $|D \cap D''| = \frac{c}{2}$ because N is isomorphic to $\frac{c}{2^r}$-subdivision of $PG(r, 2)^*$ or $AG(r+1, 2)^*$, for some $r \geq 2$. Let D' be a circuit of M such that $D' \cap (D \cap D'') \neq \emptyset$ and $D' \cap C \neq \emptyset$. Such a circuit D' exists because in either $PG(r, 2)^*$ or $AG(r+1, 2)^*$ we can choose any two elements and construct a circuit containing both, and $C \cap E(N) \neq \emptyset$. Then the structure of the circuits of $PG(r, 2)^*$ and $AG(r+1, 2)^*$ imply that $|D' \cap D| = |D' \cap D''| = \frac{c}{2}$ and $|D' \cap D \cap D''| = \frac{c}{4}$. Hence $|D' - (D \cup D'')| = \frac{c}{4}$. We arrive at a contradiction because, by Lemma 4.1, $|D' \cap C| = \frac{c}{2}$. Thus D is the unique circuit of N avoiding C. Next, we prove that $E(N) - D \subseteq C$. Consider any series class S of N such that $S \cap D = \emptyset$. It is enough to show that $S \subseteq C$. Let D' be a circuit of N such that $S \subseteq D'$ and $D' \cap D \neq \emptyset$. So $|D \cap D'| = \frac{c}{2}$. But $C \cap D' \neq \emptyset$ because $D \neq D'$ and so, by Lemma 4.1, $|C \cap D'| = \frac{c}{2}$. As $|D'| = c$ and $C \cap D'$, $D \cap D'$ are disjoint subsets of D' with cardinality $\frac{c}{2}$, it follows that $\{C \cap D', D \cap D'\}$ partition D' and so $C \cap D' = D' - D$. In particular, $S \subseteq C$. Thus $E(N) - D \subseteq C$. Hence $C = [E(N) - D] \cup T$, and $E(N) - D$ is not a circuit of N. Therefore N is isomorphic to a $\frac{c}{2^r}$-subdivision of $PG(r, 2)^*$ and so M is obtained from a matroid isomorphic to a $\frac{c}{2^r}$-subdivision of $AG(r+1, 2)^*$ by replacing a series class by T. Thus case (iv) occurs.

Now, we consider the second case. Set $X_S = C \cap S$, for every $S \in \mathcal{S}(N)$. As $C \triangle D$ is a circuit of M, for every $D \in \mathcal{C}(N)$, it follows that

$$0 = |C| - |C \triangle D| = \sum_{S \in \mathcal{S}(N): S \subseteq D} [|X_S| - |S - X_S|].$$

By Lemma 2.1, $|X_S| = |S - X_S|$, for every $S \in \mathcal{S}(N)$. Therefore M is a T-extension of N. Thus Lemma 3.2 implies that (iii) or (v) occurs. □

5. Odd circumference case

We determine the connected binary matroids having a spectrum of size two in this section in the special case that the largest circuit size is odd. First we give two interesting lemmas.

Lemma 5.1. *Let $c, d \in \mathbb{Z}^+$ with $c < d$ and d odd. If M is a connected binary matroid with* spec$(M) = \{c, d\}$*, then c is even.*

Proof. The matroid M has corank exceeding one as it has at least two circuits. If X is a subset of $E(M)$ such that $M|X$ is a coloopless connected matroid with corank two, then $M|X$ is a subdivision of $U_{1,3}$. So $M|X$ must contain an even circuit. Therefore c is even. □

Lemma 5.2. *Let $c, d \in \mathbb{Z}^+$ with $c < d$ and d odd. Suppose M is a connected binary matroid with* spec$(M) = \{c, d\}$*. If C_1 and C_2 are circuits of M such that $C_1 \cap C_2 \neq \emptyset$, then:*

(i) *$C_1 \triangle C_2$ is a minimum size circuit of M when $|C_1| = |C_2| = c$.*
(ii) *$C_1 \triangle C_2$ is a maximum size circuit of M when $\{|C_1|, |C_2|\} = \{c, d\}$.*
(iii) *$C_1 \triangle C_2$ is a union of pairwise disjoint minimum size circuits of M when $|C_1| = |C_2| = d$.*

Proof. There are pairwise disjoint circuits $D_1, D_2, \ldots, D_n$ of M such that $C_1 \triangle C_2 = D_1 \cup D_2 \cup \cdots \cup D_n$. Thus

$$|C_1| + |C_2| - 2|C_1 \cap C_2| = |C_1 \triangle C_2| = |D_1| + |D_2| + \cdots + |D_n|. \tag{4}$$

If $|C_1| = |C_2| = c$, then, by (4),

$$2c > |D_1| + |D_2| + \cdots + |D_n| \geq nc.$$

Hence $n = 1$ and so $C_1 \triangle C_2$ is a circuit of M. Observe that $|C_1 \triangle C_2|$ is even by (4). Therefore $|C_1 \triangle C_2| = c$. Hence (i) holds.

If $|C_1| = c$ and $|C_2| = d$, then, by (4), $|C_1 \triangle C_2|$ is odd. By (4), there is $i \in \{1, 2, \ldots, n\}$ such that $|D_i|$ is odd, say $|D_1| = d$. Again, by (4),

$$c + d > c + d - 2|C_1 \cap C_2| = |D_1| + |D_2| + \cdots + |D_n| \geq d + (n-1)c.$$

Hence $n = 1$ and (ii) follows.

If $|C_1| = |C_2| = d$, then, by (4), $|C_1 \triangle C_2|$ is even and an even number of D_i has maximum size. Again, by (4),

$$2d > d + d - 2|C_1 \cap C_2| = |D_1| + |D_2| + \cdots + |D_n|$$

and so at most one D_i have maximum size. Thus no D_i has maximum size and (iii) follows. □

Lemma 5.3. *Let $c, d \in \mathbb{Z}^+$ with $c < d$ and d odd. Suppose M is a connected binary matroid with* $\mathrm{spec}(M) = \{c, d\}$. *If $\mathcal{C}$ is the family of minimum size circuits of M, then:*

(i) *There is a binary matroid N over $E(M)$ such that $\mathcal{C}(N) = \mathcal{C}$.*
(ii) *The cycle space of M is spanned by the cycle space of N together with any maximum size circuit of M.*

Moreover, if $N_1, N_2, \ldots, N_n$ are the connected components of N with non-zero corank and N_0 is the set of coloops, then there are matroids $M_0, M_1, \ldots, M_n$ such that:

(iii) *$E(M_i) \cap E(M_j) = \{e\}$, for every 2-element subset $\{i, j\}$ of $\{0, 1, \ldots, n\}$;*
(iv) *$N_i = M_i \setminus e$, for every $i \in \{1, 2, \ldots, n\}$;*
(vi) *$E(M_0) = L^* \cup e \in \mathcal{C}(M_0)$, where L^* is the set of coloops of N;*
(vii) *for $i \in \{1, 2, \ldots, n\}$, $\{e\}$ is a series class of M_i, all the other series class of M_i has size l_i and the cosimplification of M_i is isomorphic to*
   (a) *U_{1,n_i}, for some $n_i \geq 3$, where $c = 2l_i$; or*
   (b) *$PG(r_i, 2)^*$, for some $r_i \geq 2$, where $c = 2^{r_i} l_i$; or*
   (c) *$AG(r_i, 2)^*$, for some $r_i \geq 3$, where $c = 2^{r_i - 1} l_i$; or*
   (d) *$S^*_{n_i}$, for some $n_i \geq 4$, and e is the tip of S_{n_i}, where $c = 4l_i$;*
   (e) *$B(r_i, 2)^*$, for some $r_i \geq 3$, and e corresponds to the cotip, where $c = 2^{r_i} l_i$; and*

(viii) *$M = S(M_0, M_1, \ldots, M_n)/e$.*

Proof. Observe that (i) is a consequence of Lemma 5.2(i). Now, we establish (ii). Fix a maximum size circuit C of M. We need to prove that $\{C\} \cup \mathcal{C}$ spans the cycle space of M. It is enough to show that $\{C\} \cup \mathcal{C}$ spans any maximum size circuit D of M. If $C \cap D \neq \emptyset$, then the result follows from Lemma 5.2(iii). We may assume that $C \cap D = \emptyset$. There is a circuit $C'$of M meeting both C and D because M is connected.

If C' is even, then $C \triangle C'$ is an maximum size circuit of M, by Lemma 5.2(ii). But $C \triangle C'$ is spanned by $\{C\} \cup \mathcal{C}$. By Lemma 5.2(iii), $(C \triangle C') \triangle D$ is spanned by $\mathcal{C}$. Hence D is spanned by $\{C\} \cup \mathcal{C}$.

If C' is odd, then, by Lemma 5.2(iii), both $C \triangle C'$ and $C' \triangle D$ are spanned by $\mathcal{C}$. Therefore $C \triangle D = (C \triangle C') \triangle (C' \triangle D)$ is spanned by $\mathcal{C}$. Thus D is spanned by $\{C\} \cup \mathcal{C}$. Thus (ii) holds.

Let K_i be the binary matroid whose ground set is $E(N_i) \cup C \cup e$ and whose cycle space is spanned by the circuits of N_i together with $C \cup e$. Set $M_i = K_i/[C - E(N_i)]$. Observe that $M = S(M_0, M_1, \ldots, M_n)/e$. Note that the cycle space of $S(M_0, M_1, \ldots, M_n)$ is spanned by $\mathcal{C}(N_1) \cup \cdots \cup \mathcal{C}(N_n) = \mathcal{C}$ together with $\cup_{i=0}^{n}[(C \cap E(N_i)) \cup e] = C \cup e$. As every circuit of N_i has size c and every circuit of M_i that contains e must have the same size, it follows, by Theorem 1.2, that M_i is as described in (vii). The result follows. □

Proof of Theorem 1.3. The proof of the reverse direction of the theorem statement is straightforward. The forward direction of the theorem statement follows from Lemma 5.3. □

6. An example

We next give some examples to show that the problem of finding all connected binary matroids with a circuit-spectrum of size two is very complex. In particular, the following proposition together with Theorem 1.1 allows us to construct many such matroids.

Proposition 6.1. *If* $c, d \in \mathbb{Z}^+$, *then there exists a connected binary matroid with* $\mathrm{spec}(M) = \{2c, d|E(Z)|\}$, *where* Z *is a connected binary matroid with* $\mathrm{spec}(M) = \{c\}$.

Proof. Let N be a coloopless simple binary matroid with $\mathrm{spec}(N) = \{c\}$ whose connected components $N_1, N_2, \ldots, N_k$ have the same size. Suppose H is a connected binary matroid on $\{1, 2, \ldots, k\}$ with $\mathrm{spec}(H) = \{d\}$. We next construct the matroid M. Add an element e' in series with each element e of N to obtain the matroid N'. Each circuit of N' has size $2c$. For a subset A of $\{1, 2, \ldots, k\}$, we set $X_A = \cup_{i\in A} E(N_i)$. Note that $X_A \triangle X_B = X_{A\triangle B}$, where A and B are subsets of $\{1, 2, \ldots, k\}$.

Let M be the binary matroid whose cycle space is spanned by $\mathcal{C}(N') \cup \{X_C : C \in \mathcal{C}(H)\}$. If C is a circuit of M, then there are circuits $C_1, C_2, \ldots, C_m$ of N' and circuits $D_1, D_2, \ldots, D_n$ of H such that

$$C = (C_1 \triangle C_2 \triangle \cdots \triangle C_m) \triangle (X_{D_1} \triangle X_{D_2} \cdots \triangle X_{D_n}).$$

This identity can be rewritten as

$$C = (C_1 \triangle C_2 \triangle \cdots \triangle C_m) \triangle (X_{D_1\triangle D_2\triangle\cdots\triangle D_n}).$$

If $D_1 \triangle D_2 \triangle \cdots \triangle D_n = \emptyset$, then C is a circuit of N'. In this case, C has $2c$ elements. We may assume that $D_1 \triangle D_2 \triangle \cdots \triangle D_n \neq \emptyset$. The symmetric difference $D_1 \triangle D_2 \triangle \cdots \triangle D_n$ is a union of disjoint circuits of H. However, since for all $i \in \{1, 2, \ldots, m\}$, $C_i \subseteq N'_j$ for some $j \in \{1, 2, \ldots, k\}$, if $(C_1 \triangle C_2 \triangle \cdots \triangle C_m) \triangle X_{D_1\triangle D_2\triangle\cdots\triangle D_n}$ is a circuit, then $D_1 \triangle D_2 \triangle \cdots \triangle D_n$ must also be a circuit as otherwise $(C_1 \triangle C_2 \triangle \cdots \triangle C_m) \triangle X_{D_1\triangle D_2\triangle\cdots\triangle D_n}$ would be a union of disjoint circuits by the definition of the cycle space of M. Denote $D_1 \triangle D_2 \triangle \cdots \triangle D_n$ by D. Then

$$C = (C_1 \triangle C_2 \triangle \cdots \triangle C_m) \triangle X_D. \tag{5}$$

Choose $C_1, C_2, \ldots, C_m$ so that this identity holds and m is as small as possible. Then $C_1, C_2, \ldots, C_m$ are pairwise disjoint by the choice of m.

We show, by induction on n, that

$$(C_1 \triangle C_2 \triangle \cdots \triangle C_n) \triangle X_D$$

meets each series class of N'_i in exactly half of its elements, when $i \in D$, and avoids each series class of N'_i, when $i \notin D$. In particular, the cardinality of this set is equal to

$$\sum_{i\in D} |E(N_i)| = |D||E(N_1)| = d|E(N_1)|,$$

where the first equality follows by (5). Taking $m = n$, we obtain the cardinality of C.

If $n = 0$, then the result holds. Suppose that the result holds for $n - 1$. Assume that C_n is a circuit of N_i. Note that $i \in D$ because $C_i \cap X_D \neq \emptyset$. As $i \in D$ and every series class S of N'_i is contained in C_n or avoids C_n, it follows that

$$S \cap [(C_1 \triangle \cdots \triangle C_n) \triangle X_D] = S \cap [(C_1 \triangle \cdots \triangle C_{n-1}) \triangle X_D],$$

when S avoids C_n, and

$$S \cap [(C_1 \triangle \cdots \triangle C_n) \triangle X_D] = S - (S \cap [(C_1 \triangle \cdots \triangle C_{n-1}) \triangle X_D]),$$

when S meets C_n. The result follows by induction because $|S| = 2|S \cap (C_1 \triangle \ldots \triangle C_{n-1})|$. □

The statement of Proposition 6.1 is particularly attractive if, for example, $c = 2^{r-1}l$ and the components N_i chosen in the proof are all isomorphic to either an l-subdivision of $AG(r, 2)^*$, where $r \geq 3$, or are isomorphic to a $2^{r-2}l$-subdivision of $U_{1,4}$. In this case $|E(N_1)| = 2c$ so that the matroid M produced will have $\mathrm{spec}(M) = \{2c, 2cd\}$.

Note that $M(K_{3,n})$, for $n \geq 3$, is a 3-connected graphic matroid having circuits whose sizes are equal to 4 and 6 only.

We end by giving another example that shows that the problem in general is complex. Let $\{\mathcal{C}_1, \mathcal{C}_2\}$ be a partition of the circuits of F_7^*. Then there are even integers c_1 and c_2 satisfying $c_1 < c_2$ and a subdivision of F_7^* such that $|C| = c_i$ for every circuit C that is a subdivision of a circuit in $\mathcal{C}_i$. This may be true for the dual of other projective geometries, but to prove this we need to find positive integers c_1 and c_2 such that the solutions for the system are positive integers:

$$\sum_{e \in C_i} x_e = c_i,$$

for every $C_i \in \mathcal{C}_i$ and $i \in \{1, 2\}$. Note that the system has integer solutions when c_1 and c_2 are multiples of the determinant of the system; however these integers may not be positive.

Acknowledgements

Lemos's research was partially supported by CNPq Grant No. 302195/02-5 and FAPESP/CNPq Grant No. 2003/09925-5. The authors thank the anonymous referee for numerous helpful comments.

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