NORTH-HOLLAND

Perturbation Analysis of the Least Squares Solution in Hilbert Spaces

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ABSTRACT

Let H_1, H_2 be two Hilbert spaces over the same field, and let $T : H_1 \to H_2$ be a bounded linear operator with closed range. We give a complete description of the perturbation analysis for the least squares solution to the operator equation $Tx = y$, where $x \in H_1$, $y \in H_2$.

1. INTRODUCTION

Let H_1 and H_2 be two Hilbert spaces, let $T : H_1 \to H_2$ be a bounded linear operator with closed range, and let $y \in H_2$, $x \in H_1$. Consider the

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minimum norm least squares problem

$$\min \|x\| \qquad \text{subject to} \quad \|y - Tx\| = \min_{z \in H_1} \|y - Tz\|, \tag{1}$$

where $\|\cdot\|$ is the norm of H_1 or H_2 induced by its inner product $(\cdot, \cdot)$.

The problem (1) has many applications (cf. [1, 4]). Error estimate for the perturbation of (1) in the finite dimensional case has been discussed in literature such as [5], [6], [7], and [8]. In the infinite dimensional case, error estimate of (1) is mentioned in [4, 2]. The authors of [2] presented an error estimate of (1) when T is injective or surjective, or the perturbation of T does not change the null space or the range of T.

In this paper, we will give an error estimate of the problem (1) for T when the perturbation of T is type I or type II (for definitions, see Section 3), which is a general condition that the perturbation of T satisfy. This means that the problem of error estimate in Hilbert spaces for the perturbation of the problem (1) has been completely solved.

2. PRELIMINARIES

Throughout this paper we assume that H_1, H_2 are Hilbert spaces over the same field. Let $L(H_1, H_2)$ denote the Banach space of all bounded linear operators $T: H_1 \to H_2$ with the operator norm $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$.

Let $T \in L(H_1, H_2)$. We denote the range and null space of T by $R(T)$ and $\mathrm{Ker}(T)$, respectively. According to [4], $T \in L(H_1, H_2)$ with $R(T)$ closed has a generalized inverse T^+, namely, T^+ is the unique solution for the four Moore-Penrose equations

$$TT^+T = T, \qquad T^+TT^+ = T^+, \qquad (TT^+)^* = TT^+, \qquad (T^+T)^* = T^+T, \tag{2}$$

in which T^* denotes the adjoint operator of T.

For $T \in L(H_1, H_2)$, the reduced minimum module of T, denoted $r(T)$, is as follows:

$$r(T) = \inf\{\|Tx\| : \mathrm{dist}(x, \mathrm{Ker}\, T) = 1\}, \tag{3}$$

where $\mathrm{dist}(x, \mathrm{Ker}\, T) = \min_{y \in \mathrm{Ker}\, T} \|x - y\|$.

According to [3], $R(T)$ is closed if and only if $r(T) > 0$, and by [2, Lemma 4.1], $\|T^+\| = r(T)^{-1}$ if $r(T) > 0$.

Let X be a Banach space, and let $V(X)$ denote the set of all closed subspaces of X. Define a function $\delta : V(X) \times V(X) \to R^+$ as follows. For any two elements $M, N \in V(X)$, we set (cf. [3])

$$\delta(M, N) = \sup\{\operatorname{dist}(u, N) : \|u\| = 1,\ u \in M\}. \tag{4}$$

LEMMA 2.1. *Let p, q be the projections (i.e. $p^2 = p$, $q^2 = q$) of X onto M, N, respectively. Then*

$$\delta(M, N) \leqslant \|p - q\|. \tag{5}$$

Proof. For any $u \in M$ with $\|u\| = 1$, we have

$$\operatorname{dist}(u, N) \leqslant \|u - qu\| = \|(p - q)u\| \leqslant \|p - q\|.$$

This shows that $\delta(M, N) \leqslant \|p - q\|$. ∎

LEMMA 2.2. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and $\tilde{T} = T + \delta T \in L(H_1, H_2)$ with $R(\tilde{T})$ closed. Then*

$$r(T) \cdot \delta\big(R(T), R(\tilde{T})\big) \leqslant \|\delta T\|. \tag{6}$$

Proof. For any $u \in R(T)$ with $\|u\| = 1$, take $x \in H_1$ such that $u = Tx$. Then $x \neq 0$ and for any $z \in \operatorname{Ker} T$,

$$\begin{aligned}\operatorname{dist}\big(u, R(\tilde{T})\big) &\leqslant \big\|u - \tilde{T}(x - z)\big\| = \big\|T(x - z) - \tilde{T}(x - z)\big\| \\ &\leqslant \|\delta T\|\,\|x - z\|\end{aligned}$$

This means that $\operatorname{dist}(u, R(\tilde{T})) \leqslant \|\delta T\| \operatorname{dist}(x, \operatorname{Ker} T)$. Since

$$1 = \|u\| = \|Tx\| \geqslant r(T) \operatorname{dist}(x, \operatorname{Ker} T),$$

it follows that

$$r(T) \cdot \delta\big(R(T), R(\tilde{T})\big) \leqslant \|\delta T\|.$$ ∎

LEMMA 2.3. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$. Then*

$$r(T)\delta(\operatorname{Ker}\tilde{T}, \operatorname{Ker}T) \leqslant \|\delta T\|. \tag{7}$$

$$r(\tilde{T}) \geqslant r(T)\left[1 - \delta(\operatorname{Ker}T, \operatorname{Ker}\tilde{T})\right] - \|\delta T\|. \tag{8}$$

Proof. (7): For any $u \in \operatorname{Ker}\tilde{T}$ with $\|u\| = 1$, we have $Tu = (\tilde{T} - \delta T)u = -(\delta T)u$. Thus

$$\|\delta T\| \geqslant \|(\delta T)u\| = \|Tu\| \geqslant r(T)\operatorname{dist}(u, \operatorname{Ker}T),$$

so

$$r(T)\delta(\operatorname{Ker}\tilde{T}, \operatorname{Ker}T) \leqslant \|\delta T\|.$$

(8): According to the definition of $r(\tilde{T})$, we can choose $x_n \in (\operatorname{Ker}\tilde{T})^{\perp}$ with $\|x_n\| = 1$ such that $\|\tilde{T}x_n\| \to r(\tilde{T})$ for $n \to \infty$. Then we can choose $y_n \in \operatorname{Ker}T$ such that $\operatorname{dist}(x_n, \operatorname{Ker}T) = \|x_n - y_n\|$, and choose $\tilde{y}_n \in \operatorname{Ker}\tilde{T}$ such that $\operatorname{dist}(y_n, \operatorname{Ker}\tilde{T}) = \|y_n - \tilde{y}_n\|$. Therefore we have

$$\begin{aligned}
\|\tilde{T}x_n\| &= \|Tx_n + (\delta T)x_n\| \geqslant \|Tx_n\| - \|\delta T\| \\
&\geqslant r(T)\operatorname{dist}(x_n, \operatorname{Ker}T) - \|\delta T\| \\
&= r(T)\|x_n - y_n\| - \|\delta T\| \\
&\geqslant r(T)\left[\|x_n - \tilde{y}_n\| - \|y_n - \tilde{y}_n\|\right] - \|\delta T\| \\
&\geqslant r(T)\left[\operatorname{dist}\left(x_n, \operatorname{Ker}\tilde{T}\right) - \operatorname{dist}\left(y_n, \operatorname{Ker}\tilde{T}\right)\right] - \|\delta T\| \\
&\geqslant r(T)\left[1 - \delta(\operatorname{Ker}T, \operatorname{Ker}\tilde{T})\right] - \|\delta T\|,
\end{aligned} \tag{9}$$

in which we have used the fact that $\operatorname{dist}(x_n, \operatorname{Ker}T) = \|x_n - y_n\|$ and $\|x_n\| = 1$, $\|y_n\| \leqslant 1\ \forall n$. So if $y_n = 0$, then

$$0 = \operatorname{dist}\left(y_n, \operatorname{Ker}\tilde{T}\right) \leqslant \delta(\operatorname{Ker}T, \operatorname{Ker}\tilde{T}); \tag{10a}$$

if $y_n \neq 0$, then

$$\begin{aligned}\operatorname{dist}\left(y_n, \operatorname{Ker} \tilde{T}\right) &= \|y_n\| \operatorname{dist}\left(\frac{y_n}{\|y_n\|}, \operatorname{Ker} \tilde{T}\right) \\ &\leqslant \|y_n\| \delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T}) \\ &\leqslant \delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T}). \end{aligned} \tag{10b}$$

From the definition of $\{x_n\}$, letting $n \to \infty$ in (9), we get

$$r(\tilde{T}) \geqslant r(T)\left[1 - \delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T})\right] - \|\delta T\|. \quad \blacksquare$$

COROLLARY 2.1. *Let* $T \in L(H_1, H_2)$ *with closed range and let* $\tilde{T} = T + \delta T \in L(H_1, H_2)$ *such that* $\operatorname{Ker} T = \operatorname{Ker} \tilde{T}$. *Then*

$$\left|r(T) - r(\tilde{T})\right| \leqslant \|\delta T\|. \tag{11}$$

Proof. When $\operatorname{Ker} T = \operatorname{Ker} \tilde{T}$, then the inequality (8) becomes

$$r(T) - r(\tilde{T}) \leqslant \|\delta T\|.$$

By interchanging the roles of $r(T)$ and $r(\tilde{T})$ we obtain

$$r(\tilde{T}) - r(T) \leqslant \|\delta T\|,$$

so that

$$\left|r(T) - r(\tilde{T})\right| \leqslant \|\delta T\|. \quad \blacksquare$$

REMARK. Corollary 2.1 is the same as [2, Lemma 4.2], which is a special case of Lemma 2.3.

3. THE ESTIMATE OF $\|\tilde{T}^+\|$ FOR THE PERTURBATION OPERATOR $\tilde{T}$

Suppose $T \in L(H_1, H_2)$ with $R(T)$ closed and $\tilde{T} = T + \delta T \in L(H_1, H_2)$. In this section we will derive the bound for $\|\tilde{T}^+\|$ with respect to $\|T^+\|$ and $\|\delta T\|$. We first define

DEFINITION 3.1. Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ be the perturbation version of T.

$\tilde{T}$ is called a type I perturbation of T if

$$\overline{R(\tilde{T})} \cap R(T)^{\perp} = \{0\}, \tag{12}$$

in which $\overline{R(\tilde{T})}$ is the closure of $R(\tilde{T})$. $\tilde{T}$ is called a type II perturbation of T if

$$\operatorname{Ker} T \cap (\operatorname{Ker} \tilde{T})^{\perp} = \{0\}. \tag{13}$$

REMARK. From Definition 3.1, if $R(\delta T) \subseteq R(T)$, then $\tilde{T}$ is a type I perturbation of T. If $\operatorname{Ker} T \subseteq \operatorname{Ker} \delta T$, then $\tilde{T}$ is a type II perturbation of T. These special cases have been discussed in [2]. It is easy to construct an example from Corollary 3.1 below such that $\tilde{T}$ is a type I perturbation of T but $R(\delta T) \not\subseteq R(T)$.

LEMMA 3.1. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$. We have*

(i) *If $\delta(R(T), \overline{R(\tilde{T})}) < 1$, then $\underline{R(T)} \cap R(\tilde{T})^{\perp} = \{0\}$;*

(ii) *If $\delta(R(\tilde{T}), R(T)) < 1$, then $\overline{R(\tilde{T})} \cap R(T)^{\perp} = \{0\}$;*

(iii) (a) *$\tilde{T}$ is a type I perturbation of T if and only if $\tilde{T}^*$ is a type II perturbation of T^*;* (b) *$\tilde{T}$ is a type II perturbation of T if and only if $\tilde{T}^*$ is a type I perturbation of T^*.*

Proof. (i): If $R(T) \cap R(\tilde{T})^{\perp} \neq 0$, we choose $u \in R(T) \cap R(\tilde{T})^{\perp}$ with $\|u\| = 1$. Then $\delta(R(T), \overline{R(\tilde{T})}) \geqslant \operatorname{dist}(u, \overline{R(\tilde{T})}) = \|u\| = 1$. This contradicts the assumption.

(ii): Using the same method as in the proof of (i), we obtain (ii).

(iii): Note that $\operatorname{Ker} A = R(A^*)^{\perp}$ and $(\operatorname{Ker} A)^{\perp} = \overline{R(A^*)}$ for any $A \in L(H_1, H_2)$ (cf. [3]). Then we can prove statements (a) and (b) easily. ∎

COROLLARY 3.1. *Let $T \in L(\mathscr{C}^n, \mathscr{C}^m)$ and $\tilde{T} = T + \delta T \in L(\mathscr{C}^n, \mathscr{C}^m)$. If $\|T^+\| \, \|\delta T\| < 1$ and $\operatorname{rank} \tilde{T} = \operatorname{rank} T$, then $\tilde{T}$ is a type I perturbation of T.*

Proof. When $\operatorname{rank} \tilde{T} = \operatorname{rank} T = 0$, the statement is trivial. Now assume that $\operatorname{rank} \tilde{T} = \operatorname{rank} T > 0$. Obviously $R(\tilde{T})$ is closed. From (6) in Lemma 2.2,

$$\delta\big(R(T), R(\tilde{T})\big) \leqslant r(T)^{-1}\|\delta T\| = \|T^+\| \, \|\delta T\| < 1;$$

thus $R(T) \cap R(\tilde{T})^{\perp} = \{0\}$. Since $[R(\tilde{T}) \cap R(T)^{\perp}]^{\perp} \supseteq R(\tilde{T})^{\perp} + R(T)$, it follows that

$$m \geqslant \dim\left[R(\tilde{T}) \cap R(T)^{\perp}\right]^{\perp} \geqslant \dim\left[R(\tilde{T})^{\perp} + R(T)\right]$$

$$= \dim R(\tilde{T})^{\perp} + \dim R(T) = m - \dim R(\tilde{T}) + \dim R(T)$$

$$= m - \operatorname{rank} \tilde{T} + \operatorname{rank} T = m.$$

This implies that $R(\tilde{T}) \cap R(T)^{\perp} = \{0\}$. ■

In the following, we will consider the estimation of $\|\tilde{T}^{+}\|$ when $\tilde{T}$ satisfies (12) or (13). First, we give an estimate of $\delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T})$.

THEOREM 3.1. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ with $\|T^{+}\|\,\|\delta T\| < 1$. Assume that $\tilde{T}$ is a type I perturbation of T or* $\dim \operatorname{Ker} \tilde{T} = \dim \operatorname{Ker} T < \infty$. *Then*

$$\delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T}) \leqslant \frac{\|T^{+}\|\,\|\delta T\|}{1 - \|T^{+}\|\,\|\delta T\|}. \tag{14}$$

Proof. Notice that from the conditions of the theorem, $I + T^{+}\,\delta T$ is invertible. Set $S = I - (I + T^{+}\,\delta T)^{-1}T^{+}\tilde{T}$. Since $T^{+}\tilde{T} = T^{+}T + T^{+}\,\delta T$, it follows that $S = (I + T^{+}\,\delta T)^{-1}(I - T^{+}T)$.

Since $(I - T^{+}T)(I + T^{+}\,\delta T) = I - T^{+}T$, also $(I - T^{+}T)(I + T^{+}\delta T)^{-1} = I - T^{+}T$, we have $S^2 = S$. From the definition of S, $\operatorname{Ker} \tilde{T} \subseteq R(S)$.

If $\tilde{T}$ is a type I perturbation of T, then $\forall x \in R(S)$, there is a $y \in H_1$ such that $(I + T^{+}\,\delta T)^{-1}(I - T^{+}T)y = x$, that is, $(I - T^{+}T)y = (I + T^{+}\,\delta T)x$. Hence, $0 = T(I - T^{+}T)y = T(I + T^{+}\,\delta T)x$, which implies that $TT^{+}(T + \delta T)x = 0$, i.e., $TT^{+}\tilde{T}x = 0$. Thus, $\tilde{T}x \in R(\tilde{T}) \cap R(T)^{\perp} = \{0\}$, so $x \in \operatorname{Ker} \tilde{T}$ and $R(S) \subseteq \operatorname{Ker} \tilde{T}$.

On the other hand, if $\dim \operatorname{Ker} \tilde{T} = \dim \operatorname{Ker} T < \infty$, then from the definition of S, we get that

$$\dim R(S) = \dim\left[R\left((I + T^{+}\,\delta T)^{-1}(I - T^{+}T)\right)\right] = \dim[R(I - T^{+}T)]$$

$$= \dim \operatorname{Ker} T = \dim \operatorname{Ker} \tilde{T} < \infty.$$

Since $\operatorname{Ker} \tilde{T} \subseteq R(S)$, it follows that $\operatorname{Ker} \tilde{T} = R(S)$.

Finally, applying Lemma 2.1 to Ker T and Ker $\tilde{T}$, we get that

$$\begin{aligned}\delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T}) \\ &\leqslant \|(I - T^+T) - S\| = \left\|(I - T^+T) - (I + T^+ \, \delta T)^{-1}(I - T^+T)\right\| \\ &\leqslant \left\| I - (I + T^+ \, \delta T)^{-1} \right\| \leqslant \frac{\|T^+\| \, \|\delta T\|}{1 - \|T^+\| \, \|\delta T\|}. \qquad \blacksquare\end{aligned}$$

COROLLARY 3.2. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ with $\|T^+\| \, \|\delta T\| < \frac{1}{2}$. Suppose that $\tilde{T}$ is a type I perturbation of T or*

$$\dim \operatorname{Ker} \tilde{T} = \dim \operatorname{Ker} T < \infty.$$

Then $\tilde{T}$ is a type II perturbation of T.

Moreover, $\tilde{T}$ is a type I perturbation of T if and only if $\tilde{T}$ is a type II perturbation of T.

Proof. By Theorem 3.1, $\|T^+\| \, \|\delta T\| < \frac{1}{2}$ implies that

$$\delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T}) \leqslant \frac{\|T^+\| \, \|\delta T\|}{1 - \|T^+\| \, \|\delta T\|} < 1.$$

From [3, p. 201, Theorem 2.9],

$$\delta\left(\overline{R(\tilde{T}^*)}, R(T^*)\right) = \delta\left((\operatorname{Ker} \tilde{T})^{\perp}, (\operatorname{Ker} T)^{\perp}\right) = \delta(\operatorname{Ker} T, \operatorname{Ker} \tilde{T}) < 1,$$

it follows from (ii), (iii) of Lemma 3.1 that $\tilde{T}$ is a type II perturbation of T.

Replacing T by T^* and $\tilde{T}$ by $\tilde{T}^*$, we obtain that $\tilde{T}$ is a type I perturbation of T if and only if $\tilde{T}$ is a type II perturbation of T. ■

The following theorem is the main result of this section.

THEOREM 3.2. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed, and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ with $\|T^+\| \, \|\delta T\| < \frac{1}{2}(3 - \sqrt{5})$ $(< \frac{1}{2})$. Assume that $\tilde{T}$ is a type I perturbation of T. Then $\tilde{T}$ has generalized inverse $\tilde{T}^+$ with*

$$\|\tilde{T}^+\| \leqslant \frac{\|T^+\|}{1 - \frac{1}{2}(3 + \sqrt{5})\|T^+\| \, \|\delta T\|}. \tag{15}$$

Proof. By applying Lemma 2.3(ii) and Theorem 3.1, we get that

$$r(\tilde{T}) \geqslant \frac{1}{\|T^+\|}\left[1 - \frac{\|T^+\|\,\|\delta T\|}{1 - \|T^+\|\,\|\delta T\|}\right] - \|\delta T\|$$

$$= \frac{1 - 3\|T^+\|\,\|\delta T\| + \|T^+\|^2\|\delta T\|^2}{\|T^+\|(1 - \|T^+\|\,\|\delta T\|)}. \tag{16}$$

Thus, if $\|T^+\|\,\|\delta T\| < \frac{1}{2}(3 - \sqrt{5})$, then $r(\tilde{T}) > 0$, so $\tilde{T}$ has the generalized inverse $\tilde{T}^+$ with

$$\|\tilde{T}^+\| = \frac{1}{r(\tilde{T})} \leqslant \frac{\|T^+\|(1 - \|T^+\|\,\|\delta T\|)}{1 - 3\|T^+\|\,\|\delta T\| + \|T^+\|^2\|\delta T\|^2}. \tag{17}$$

Clearly, if $\|T^+\|\,\|\delta T\| < \frac{1}{2}(3 - \sqrt{5}) < \frac{1}{2}$, then

$$\frac{1 - \|T^+\|\,\|\delta T\|}{1 - 3\|T^+\|\,\|\delta T\| + \|T^+\|^2\|\delta T\|^2} \leqslant \frac{1}{1 - \frac{1}{2}(3 + \sqrt{5})\|T^+\|\,\|\delta T\|}. \tag{18}$$

Thus from (17) and (18),

$$\|\tilde{T}^+\| \leqslant \frac{\|T^+\|}{1 - \frac{1}{2}(3 + \sqrt{5})\|T^+\|\,\|\delta T\|}. \qquad \blacksquare$$

4. PERTURBATION ANALYSIS

According to [1], for $T \in L(H_1, H_2)$ with $R(T)$ closed, $x = T^+y$ is the unique solution of the problem (1). Now, let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ be a type I perturbation of T with $\|T^+\|\,\|\delta T\| < \frac{1}{2}(3 - \sqrt{5})$. Then by Theorem 3.1, $\tilde{T}$ has generalized inverse $\tilde{T}^+$. Let $y \in H_2$, and let $\tilde{y} = y + \delta y \in H_2$ be the perturbation of y. Consider the least squares problem

$$\min \|x\| \qquad \text{subject to} \quad \|\tilde{y} - \tilde{T}x\| = \min_{z \in H_1} \|\tilde{y} - \tilde{T}z\|. \tag{19}$$

Then (19) has a unique solution $\tilde{x} = \tilde{T}^+\tilde{y}$. We now estimate $\|\tilde{T}^+ - T^+\|/\|T^+\|$ and $\|\tilde{x} - x\|/\|x\|$. The condition number of T is defined by $\kappa = \|T\|\,\|T^+\|$.

THEOREM 4.1. *Let $T \in L(H_1, H_2)$ with $R(T)$ closed. Let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ be a type I perturbation of T with $\|T^+\| \, \|\delta T\| < \frac{1}{2}(3 - \sqrt{5})$, and set $\varepsilon_T = \|\delta T\| / \|T\|$. Then*

$$\|\tilde{T}^+ - T^+\| \leqslant \sqrt{3} \|\delta T\| \max\{\|T\|^2, \|\tilde{T}\|^2\},$$

$$\frac{\|\tilde{T}^+ - T^+\|}{\|T^+\|} \leqslant \varepsilon_T \kappa \left(1 + \frac{1}{\left[1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa\right]^2} \right). \tag{20}$$

Proof. According to Theorem 3.2, $\tilde{T}^+$ exists. From the identity (cf. [4, p. 345, Theorem 3.10])

$$\tilde{T}^+ - T^+ = -\tilde{T}^+ \delta T \, T^+ + \tilde{T}^+ (\tilde{T}^+)^* (\delta T)^* (I - TT^+) + (I - \tilde{T}^+ \tilde{T})(\delta T)^* (T^+)^* T^+, \tag{21}$$

we then get, by applying the orthogonality of the operators on the right side of the above equality,

$$\|\tilde{T}^+ - T^+\|^2 \leqslant \left(\|\tilde{T}^+\| \, \|T^+\| \, \|\delta T\|\right)^2 + \left(\|\tilde{T}^+\|^2 \|\delta T\|\right)^2 + \left(\|T^+\|^2 \|\delta T\|\right)^2.$$

Therefore it follows from Theorem 3.1 that

$$\frac{\|\tilde{T}^+ - T^+\|}{\|T^+\|} \leqslant \varepsilon_T \kappa \left(1 + \frac{1}{\left[1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa\right]^2} \right). \qquad \blacksquare$$

COROLLARY 4.1 (The continuity of T^+ in Hilbert spaces). *Let $T \in L(H_1, H_2)$ with generalized inverse T^+, and let $\{T_n\}$ be a sequence of operators in $L(H_1, H_2)$. Let T_n^+ be the generalized inverse of T_n $\forall n$. Suppose that $T_n \xrightarrow{\|\cdot\|} T$ (with respect to the norm $\|\cdot\|$ on $L(H_1, H_2)$). Then $T_n^+ \xrightarrow{\|\cdot\|} T^+$ if and only if $R(T_n) \cap R(T)^\perp = \{0\}$ for n large enough.*

Proof. "$\Rightarrow$" part: By Lemma 2.2, $\delta(R(T_n), R(T)) \leqslant \|T_n^+\| \, \|T_n - T\|$. Thus we have $\delta(R(T_n), R(T)) < 1$ for n large enough. Then by Lemma 3.1, $R(T_n) \cap R(T)^\perp = \{0\}$ for n large enough.

"$\Leftarrow$" part: For n large enough, we have

$$R(T_n) \cap R(T)^{\perp} = 0 \quad \text{and} \quad \|T^+\| \, \|T_n - T\| < \tfrac{1}{2}(3 - \sqrt{5}).$$

Then by applying Theorem 4.1 we obtain that $T_n^+ \overset{\|\cdot\|}{\to} T^+$. ∎

Combining Corollary 4.1 and Corollary 3.1, we deduce that in the finite dimensional case, $T_n^+ \overset{\|\cdot\|}{\to} T^+$ if and only if $\operatorname{rank} T_n = \operatorname{rank} T$ for n large enough.

THEOREM 4.2. *Suppose that* $T, \tilde{T}$ *satisfy the conditions in Theorem* 4.1, *and* y, $\tilde{y} = y + \delta y \in H_2$. *Set* $\varepsilon_T = \|\delta T\|/\|T\|$ *and* $\varepsilon_y = \|\delta y\|/\|y\|$. *Then*

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leqslant \frac{\kappa}{1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa} \times \left(\varepsilon_T + \varepsilon_y \frac{\|y\|}{\|T\| \, \|x\|} + \frac{\varepsilon_T \kappa}{1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa} \frac{\|y - Tx\|}{\|T\| \, \|x\|} \right) + \varepsilon_T \kappa. \tag{22}$$

Proof. From (21) we obtain that

$$\begin{aligned} \|\tilde{x} - x\| = \|\tilde{T}^+ \tilde{y} - T^+ y\| &= \left\| \tilde{T}^+ \, \delta y + (\tilde{T}^+ - T^+) y \right\| \\ &\leqslant \|\tilde{T}^+\| \, \|\delta y\| + \|\tilde{T}^+\| \, \|\delta T\| \, \|x\| \\ &\quad + \|\tilde{T}^+\|^2 \|\delta T\| \, \|y - Tx\| + \|\delta T\| \, \|T^+\| \, \|x\|. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\|\tilde{x} - x\|}{\|x\|} &\leqslant \|\tilde{T}^+\| \left(\varepsilon_y \frac{\|y\|}{\|x\|} + \|\delta T\| + \|\tilde{T}^+\| \frac{\|y - Tx\|}{\|T\| \, \|x\|} \right) + \|\delta T\| \, \|T^+\| \\ &\leqslant \frac{\kappa}{1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa} \\ &\quad \times \left(\varepsilon_T + \varepsilon_y \frac{\|y\|}{\|T\| \, \|x\|} + \frac{\varepsilon_T \kappa}{1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa} \frac{\|y - Tx\|}{\|T\| \, \|x\|} \right) \\ &\quad + \varepsilon_T \kappa. \end{aligned}$$

∎

COROLLARY 4.2. *If in addition $y \in R(T)$ in Theorem 4.2, then*

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leqslant \frac{\kappa}{1 - \frac{1}{2}(3 + \sqrt{5})\varepsilon_T \kappa}(\varepsilon_y + \varepsilon_T) + \varepsilon_T \kappa.$$

Proof. Since $y \in R(T)$, we have $y = Tx$ and $\|y\| \leqslant \|T\| \, \|x\|$. ■

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