Gaussian Asymptotic Properties of the Sum-of-Digits Function

Jean Marie Dumont* and Alain Thomas

Institut de Mathématiques de Luminy, Unité propre de recherche 9016, case 930, 163, avenue de Luminy, F-13288 Marseille Cédex 9, France

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We will show that the sum-of-digits function has an asymptotic Gaussian behaviour, and we deduce some new summational formulae. More precisely we consider numeration systems relative to substitutive bases, whose general case contains among others the case of Parry linear recurrent bases (as the Fibonacci sequence for instance). © 1997 Academic Press

1. INTRODUCTION

Let \((u_n)_{n \geq 0}\) be an increasing sequence of integers with \(u_0 = 1\), and for an integer \(N\), let \(d_i(N)\) be its digits on the “base” \(u\) obtained by the greedy algorithm. We assume that \(u\) is given by a linear recurrence relation satisfying the “Parry’s conditions” (see [GT91] or [DT93]). This is the case for instance for the Fibonacci sequence. Let \(\theta\) be the largest root of the polynomial associated to the linear recurrence. Then the digits belong to the finite set \(D = \mathbb{N} \cap [0, \theta]\) and for any application \(f : D \to \mathbb{N}\) we can define

\[
S_{u,f}(N) = \sum_{i \geq 0} f(d_i(N))
\]

as the “sum-of-digits function,” denoted by \(S_f(N)\).

It is known that \(\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} s(n)\) exists and its value is explicitly determined [GT91, DT93]. Moreover, we did compute in 1993 the number \(\beta\) such that for any integer \(k\)

\[
N^{-1} \sum_{n \leq N} (s_f(n) - \beta \log_\theta n)^k
= (\beta \log_\theta N)^{k/2} (k-1)(k-3) \cdots 1 + O(\log N)^{k/2-1}
\]

if \(k\) is even

\[
= O((\log N)^{(k-1)/2})
\]

if \(k\) is odd. (1)

* E-mail: dumont@lumimath.univ-mrs.fr.
Actually, we did find a very precise form of the error term, involving 1-periodic functions of \( \log_\alpha N \) and generalizing previous formulas by Delange and by Coquet.

Here we study the asymptotic value of

\[
S_\phi(N) = N^{-1} \sum_{n < N} \phi \left( \frac{x'(n) - x \log_\alpha n}{\sqrt{\beta \log_\alpha n}} \right)
\]

for a class of differentiable real functions \( \phi \). Denoting \( \mathcal{F}_\varepsilon = \{ \phi \in C^1(\mathbb{R}) \; | \; \exists K \text{ with } \forall x \in \mathbb{R}, |\phi'(x)| \leq K e^{-\varepsilon x} \} \), we find that

\[
\forall \varepsilon' > 0 \; \exists \varepsilon > 0 \quad \phi \in \mathcal{F}_\varepsilon \Rightarrow S_\phi(N) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(t) e^{-t^2/2} dt + O(\log N)^{\varepsilon - 1/2}.
\]

(2)

For instance, with \( \phi(x) = x^s \), we can recover (1) using (2) (but with a greater bound for the error term). The main tool used is the results by Statulevičius [S73] concerning the central limit theorem and the large deviations theorem for Markov chains, not necessarily homogeneous.

Actually, we state (2) in the more general framework of numeration systems associated with a primitive substitution on a finite alphabet. There is a large field of number theoretical functions that can be represented by some “sum-of-digits function” associated to a substitution. For instance, if \( \omega = (1 + \sqrt{3})/2 \) and

\[
s(n) = \# \{ k \leq n; \{ k\omega \} \in [0, \frac{1}{2}[ \} - \# \{ k \leq n; \{ k\omega \} \in [\frac{1}{2}, 1[ \}
\]

where \( \{ x \} \) denotes the fractional part of \( x \), the result (2) is true with \( \theta = 2 + \sqrt{3}, \alpha = \frac{1}{2}, \beta = \sqrt{3}/6 \). Another example is

\[
s(n) = \# \{ k \leq n; k \text{ is the sum of three squares} \} - \frac{\omega}{2} n
\]

with \( \theta = 4, \alpha = \frac{1}{2} \) [OS89, DT91] and \( \beta = \frac{25}{172} \) (applying [D90]). In this case Section 5 applies.

Recently other examples appear (see [Dr] and [D]).

2. SUM-OF-DIGITS FUNCTION ASSOCIATED WITH A SUBSTITUTION

Let \( \mathcal{A} = \{ 1, \ldots, d \} \) be a finite alphabet and \( \sigma \) a substitution on \( \mathcal{A} \), i.e., an application from \( \mathcal{A} \) to \( \mathcal{A}^+ = \bigcup_{i=1}^\infty \mathcal{A}^i \). We note also by \( \sigma \) the extension of \( \sigma \) on the set of words \( \mathcal{A}^* = \mathcal{A}^+ \cup \{ \epsilon \} \), \( \epsilon \) the empty word, and for any \( n \geq 0 \), \( \sigma^n \) is the \( n \)-times iteration of \( \sigma \). We assume the existence of an integer \( k \) such that, for any letter \( a \), all the letters occur in \( \sigma^k a \). We denote by \( |m| \) the length of the word \( m \), and \( m' < m \) means that \( m' \) is a strict prefix of \( m \). We suppose \( 1 < \sigma(1) \).
2.1 A sequence of words $m_1m_2 \cdots m_n$ is said to be $a$-admissible if there exist some letters $a_0 = a, a_1, \ldots, a_n$, such that $m_1a_i \in \sigma(a_{i-1})$ for $i = 1, 2, \ldots, n$. Clearly, the $a_i$ are unique.

2.2 [DT89]. The admissible representation of an integer $v \geq 1$ is the unique 1-admissible sequence $m_{n(v)}(v)m_{n(v)-1}(v)\cdots m_1(v)$, with $m_{n(v)}(v)$ not empty, such that

$$v = |\sigma^{n(v)-1}(m_{n(v)}(v))| + \cdots + |\sigma^0(m_1(v))|.$$

2.3 The sum-of-digits function relative to a map $f : \mathbb{N} \to \mathbb{R}$ is defined for any integer $v \geq 1$ by

$$s^f(v) = s_{n(v)}(v) = \sum_{i=1}^{n(v)} f(m_i(v)).$$

As a special case of this numeration system, we have the usual representation of integers in an integer base $g$ ($d = 1$ and $\sigma : 1 \to 1 \cdots 1$). We have also the representation of integers associated with a linear recurrence relation (Parry), defined in [B89] and [GT91], one example being the "multinacci representation of order $d$" with

$$\sigma : i \to l(i+1) \quad \text{for} \quad i = 1, \ldots, d-1$$
$$d \to 1$$

(see [DT93, Section 8]); here the 1-admissible sequences are the sequences of 1 and $A$, with no $d$ consecutive 1. The usual sum-of-digits function, in these cases, is equal to the sum-of-digits function relative to the map $f : m \to |m|$.

In the general case, the constants $\alpha$ and $\beta$ defined in the introduction are computed in [D90] in this way: Let $M(x)$ be the matrix defined by

$$(M(x))_{a,b} = \sum_{m \in \sigma b} x^{f(m)}.$$

$M(x)$ tends, when $x \to 1$, to the matrix of the substitution $M$ ($M_{a,b}$ being the number of occurrences of $a$ in $\sigma b$). $M$ has a dominating eigenvalue $\theta$, hence $M(x)$ has, for $x$ sufficiently close to 1, a dominating eigenvalue $\theta(x)$. The values of $\alpha$ and $\beta$ are

$$\alpha = \theta(1) \theta$$
$$\beta = \alpha - \alpha^2 + \theta^2(1) \theta.$$
See also [DT93, Sect. 8] for the explicit values of $\alpha$ and $\beta$ with respect to a linear Parry recurrence relation.

3. ASYMPTOTIC EQUIVALENT OF $S_\beta(n, a)$.

**Notation.** For $n \geq 1$, $a \in \mathcal{A}$, $f : \mathcal{A}^* \to \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}$, let

$$S_\beta(n, a) = S_\alpha, f, \phi (n, a) = \frac{1}{|\sigma^\gamma a|} \sum_{u \in \mathcal{U}} \phi \left( \frac{f(m_1) + \cdots + f(m_n) - \beta n}{\sqrt{\beta n}} \right)$$

where $\mathcal{U} = \mathcal{U}_{a, n}$ is the set of $a$-admissible sequences of length $n$, $u = m_1 \cdots m_n$.

We suppose that $\beta$ is not zero. This is the case, for instance, with the usual representation in an integer base $g$: $S_\alpha, f, \phi (n, a)$, with appropriate substitution $\sigma$ and map $\phi$ (see Section 2), equal to

$$g^{-n} \sum_{v=1}^{g^n} \phi \left( \frac{s_g(v) - (g - 1) n/2}{\sqrt{(g^2 - 1) n/12}} \right)$$

where $s_g$ is the ordinary sum-of-digits function in base $g$.

3.1. The Markov Chain of the Digits

We will use the results of Statulevičius about the central limit theorem and large deviations theorem [S73]; it applies to a Markov chain $\xi_t$ where the time $t$ belongs to a finite set $\{1, \ldots, n\}$ and the transition probabilities may depend on the time.

We fix an integer $n \in \mathbb{N}$ and a letter $a \in \mathcal{A}$; the Markov chain we shall define depends on $n$ and $a$.

We define the random variables $\xi_k : \mathcal{U} = \mathcal{U}_{a, n} \to \Omega = \mathcal{A}^\infty \times \mathcal{A}$ by

$$\xi_0(u) = (A, a)$$

$$\xi_k(u) = (m_k, a_k)$$

for any $u = m_1 \cdots m_n \in \mathcal{U}$ and $1 \leq k \leq n$, $A$ being the empty word. Let the transition probabilities, from the step $k - 1$ to the step $k$, be

$$p_{k-1,k}((m, b), (m', b')) = \begin{cases} \frac{|\sigma^{n-k}b'|}{|\sigma^{n-k}b|} & \text{if } m'b' \leq \sigma b \\ 0 & \text{otherwise} \end{cases}$$
and the initial distribution

\[ P_{0}(A, a) = 1 \]
\[ P_{0}(m, b) = 0 \quad \text{if} \quad (m, b) \neq (A, a). \]

Let \( P \) be the probability on \( \mathcal{A}_{n,n} \), relative to these transition and initial probabilities.

**Remarks.** The sum of \( p_{a}((m, b), (m', b')) \) for \( m'b' \leq ab \) is obviously equal to one. For each \( a \)-admissible sequence \( m_1 \cdots m_n \) we have

\[ P((A, a), (m_1, a_1), \ldots, (m_n, a_n)) = \frac{1}{|\sigma'^a|} \]

and thence there are \( |\sigma'^a| \) \( a \)-admissible sequences, as has been shown directly in [DT93].

We set also

\[ X_k(u) = f(m_k) \]
\[ S_n = X_1 + \cdots + X_n \]
\[ Z_n = S_n - ES_n. \]

### 3.2. The Central Limit Theorem of Statulevičius

The first theorem of [S73] states

\[ \sup_{x \in \mathbb{R}} |P(Z_n < x) - \Phi(x)| \leq C_1 \frac{C^{(n)}A_n}{\sqrt{\text{var} S_n}} \tag{1} \]

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \), \( C_1 \) is an absolute constant, and

\[ C^{(n)} = \max_{1 \leq k \leq n, u \in \mathcal{U}} |X_k(u)|, \]
\[ A_n = \max_{1 \leq k \leq n} \sum_{l = k}^{n} \sup_{\omega \in \mathcal{D}, A \in \Omega} |p_{k,l}(\omega, A) - p_{k,l}(\tilde{\omega}, A)|, \]

and \( p_{k,l}(\omega, A) \) is the transition probability from the state \( \omega \) at the time \( k \) to the set of states \( A \) at the time \( l \). (For \( l = k \), this transition probability is equal to 1 if \( \omega \in A \), and 0 otherwise.)

A sufficient condition for the sequence \( n \to A_n \) to be bounded is that

\[ \sup_{\omega \in \mathcal{D}, A \in \Omega} |p_{k,l}(\omega, A) - p_{k,l}(\tilde{\omega}, A)| \]

tends exponentially to 0 when \( l - k \to \infty \). The meaning is that the random variables \( \zeta_k \) and \( \zeta_l \) become independent when \( l - k \to \infty \).
We will check this condition, first in the special case $A = \{\cdot, b\}$, i.e. $b$ is a given letter and $A$ is the set of the states $(m', b)$ for any $m' \in \mathcal{A}^\ast$.

\[
p_{k,i}(m, c), (\cdot, b)) = \sum p_{k,k+1}(m, c), (m_{k+1}, a_{k+1})) \cdots p_{1,1}(m_1, a_1), (m, b))
\]

where the sum runs over the $c$-admissible sequences $(m_{k+1}, a_{k+1}) \cdots (m, b)$ of length $l - k$ such that $a_i = b$. We prove easily by recurrence on $h = l - k$ that the number of such sequences is equal to $L_\lambda(\sigma^{l-k}c)$, i.e., to the number of occurrences of $b$ in $\sigma^{l-k}c$. Thence

\[
p_{k,i}(m, c), (\cdot, b)) = \frac{|\sigma^{l-k}b|}{|\sigma^{l-k}c|} L_\lambda(\sigma^{l-k}c).
\]

Now we have the following estimates when $h \to \infty$, proved in [DT89, Lemma 6],

\[
L_\lambda(\sigma^h c) = \varepsilon(c)\lambda_0 \theta^h(1 + o(\rho^h))
\]

\[
|\sigma^h c| = \varepsilon(c)\theta^h(1 + o(\rho^h)),
\]

where $\theta$ is the dominating eigenvalue of the matrix $M$ of the substitution, the vectors $\varepsilon(1) \cdots \varepsilon(d)$ and \[
\begin{pmatrix}
\hat{\lambda}(1) \\
\vdots \\
\hat{\lambda}(d)
\end{pmatrix}
\]

are respectively left and right eigenvectors for $M$, and $\rho < 1$ is a constant such that all the eigenvalues except $\theta$ have modulus less than $\rho \theta$. We deduce

\[
p_{k,i}(m, c), (\cdot, b)) = |\sigma^{l-k}b|\lambda_0 \theta^{l-k}(1 + o(\rho^{l-k}))
\]

\[
= |\sigma^{l-k}b|\lambda_0 \theta^{l-k} + o(\rho^{l-k})
\]

\[
p_{k,i}(m, c), (\cdot, b)) - p_{k,i}(m, c), (\cdot, b)) = o(\rho^{l-k}).
\]

The same estimate holds when we replace $(\cdot, b)$ with a set of states $A$, because we have

\[
p_{k,i}(m, c), A) = \sum p_{k,i-1}(m, c), (\cdot, b)) p_{i-1,1}(\cdot, b), (m', b')),
\]

the sum over the finite set of $(b, (m', b')) \in \mathcal{A} \times A$ such that $m'b' \preceq ab$. Thence we deduce that $\sup_{n \in \mathbb{N}} A_n$ is finite.

The constant $C^{\alpha}$ of (1) being also bounded by $\max_{k \in \mathcal{A}, m \in A} |f(m)|$, we deduce from (1) the following lemma.
**Lemma 3.2.1.** There exists a constant $K = K_{n,f}$ depending only on the substitution $\sigma$ and the map $f$, such that

$$\sup_{x \in \mathbb{R}} |P(Z_n < x) - \Phi(x)| \leq \frac{K}{\sqrt{\text{var } S_n}}.$$

### 3.3. The Theorem of Large Deviations

Suppose $E X_k = 0$ for $1 \leq k \leq n$ and suppose that there exist some positive constants $C, d_1, \ldots, d_n$ such that

$$E(e^{hX_k}) \leq e^{d_k h^2} \quad (2)$$

for $0 \leq h \leq C$ and $1 \leq k \leq n$. Theorem 6 of [S73] states the existence of absolute positive constants $\alpha'$ and $\beta'$ such that

$$P(S_n > x) \leq \exp \left( -\frac{\beta' x^2}{\alpha'} \right)$$

for $0 \leq x \leq \alpha'(\mathcal{D}_n/A_n)$, with $\mathcal{D}_n = \sum_{k=1}^n d_k$.

In our case $E S_k$ is not zero, thence we use $X'_k = X_k - E X_k$. The $|X'_k|$ are bounded by the constant

$$C' = 2 \max_{h \in \mathcal{D}, m < nb} |f(m)|.$$

Using the inequality, for any $x \leq \log 2$,

$$e^x \leq 1 + x + x^2,$

we obtain for $h \leq (\log 2)/C'$,

$$e^{hX_k} \leq 1 + hX'_k + h^2 C'^2$$

$$E(e^{hX_k}) \leq 1 + h^2 C'^2$$

$$\leq e^{dh^2},$$

thence (2) is satisfied with $d_k = C'^2$. Now we can reformulate the result:

**Lemma 3.3.1.** There exist some positive constants $\alpha''$ and $\beta''$, depending only on $\sigma$ and the map $f$, such that

$$P(S_n - E S_n > x) \leq \exp \left( -\frac{\beta'' x^2}{n} \right)$$

for $0 \leq x \leq \alpha'' n$. ($\alpha''$ is equal to $\alpha'C^2/(\sup_{n \in \mathcal{N}} A_n)$.)
3.4. Relation between $S_\epsilon(n, a)$ and $E(\epsilon(Z_n))$

We denote by $\mathcal{F}_\epsilon$, $\epsilon > 0$, the set of functions $\phi \in C^1(\mathbb{R})$ for which there exists a constant $\gamma$ such that

$$|\phi'(x)| \leq \gamma e^{\epsilon x^2}.$$  

(More generally we can assume that $\phi$ is continuous on $\mathbb{R}$, but continuously differentiable apart from a finite set of points, and $|\phi'(x)| \leq \gamma e^{\epsilon x^2}$ when it is defined.)

**Lemma 3.4.1.** If $\phi \in \mathcal{F}_\epsilon$, we have for any $x, y \in \mathbb{R}$

$$|\phi(y) - \phi(x)| \leq \gamma |y - x| e^{\epsilon x^2}$$

with $z = \max(|x|, |y|)$.

**Proof.** We use $\phi(y) - \phi(x) = \int_{x}^{z} \phi'(t) \, dt$.

**Lemma 3.4.2.** We have $Z_n = (S_n - ES_n)/\sqrt{\text{var } S_n} = (S_n - \alpha n)/\sqrt{\beta n} + O(1/\sqrt{n})$ and $Z_n = O(1/\sqrt{n})$.

**Proof.** Using some estimates proved in [DT93, Lemmas 3, 6, and 7], we have

$$ES_n = \frac{1}{|a|^d} \sum_{m} (f(m_1) + \cdots + f(m_d))$$

$$= \frac{1}{|a|^d} (\alpha x(a) n + O(1)) 0^d$$

$$= \alpha n + O(1)$$

$$\text{var } S_n = E((S_n - ES_n)^2)$$

$$= E(S_n - \alpha n)^2 - (ES_n - \alpha n)^2$$

$$= \frac{1}{|a|^d} (\beta x(a) n + O(1)) 0^d - (ES_n - \alpha n)^2$$

$$= \beta n + O(1)$$

$$(\text{var } S_n)^{-1/2} = \left(\beta n \left(1 + O\left(\frac{1}{n}\right)\right)\right)^{-1/2}$$

$$= (\beta n)^{-1/2} \left(1 + O\left(\frac{1}{n}\right)\right)$$
\[
Z_n = \frac{S_n - ES_n}{\sqrt{\text{var } S_n}} = (S_n - x_n + O(1))(\beta n)^{-1/2} \left(1 + O\left(\frac{1}{n}\right)\right)
= (S_n - x_n)(\beta n)^{-1/2} + O(n^{-1/2})
\]
(using \(S_n = O(n)\)). We deduce \(Z_n = O(\sqrt{n})\).

**Lemma 3.4.3.** For \(0 < \varepsilon < 1\) and \(\phi \in \mathcal{F}_\varepsilon\), we have
\[S_\phi(n, a) = E(\phi(Z_n)) + O(n^{-1/2} E(e^{c\varepsilon^2})).\]

**Proof.** We have
\[S_\phi(n, a) = E\left(\phi\left(\frac{S_n - x_n}{\sqrt{\beta n}}\right)\right)\]
and, using Lemmas 3.4.1 and 3.4.2,
\[\phi\left(\frac{S_n - x_n}{\sqrt{\beta n}}\right) - \phi(Z_n) = O(n^{-1/2} e^{c\varepsilon^2}).\]

**3.5. Estimate of \(S_\phi(n, a)\)**

**Proposition 3.5.1.** Given \(c' > 0\), there exists \(c > 0\) such that for any \(\phi \in \mathcal{F}_\varepsilon\),
\[S_\phi(n, a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(t) e^{-ct^2} dt + O(\varepsilon^{-1/2 + c'}).\]

**Proof.** By Lemma 3.4.3, it is sufficient to estimate \(E(\phi(Z_n))\) and to bound \(E(e^{c\varepsilon^2})\).

We decompose the function \(\phi\)
\[\phi(x) = 1_{\mathcal{M}(x)}(x) \phi(x) + 1_M(x) \phi(x) + 1_{\mathcal{M}^+}(x) \phi(x),\]
where \(1_M, 1_{\mathcal{M}},\) and \(1_{\mathcal{M}^+}\) are respectively, the characteristic functions of \([-\infty, M[\), \([M, +\infty[\), and \([0, +\infty[\). We shall choose later the value of \(M = M(n) \geq 1\).

An integration by parts gives
\[E(1_{\mathcal{M}}(Z_n) \phi(Z_n)) = \int_{Z_n(u) < M} \phi(Z_n(u)) dP(u) = \phi(M) P(Z_n < M)\]
\[- \int_{-\infty}^{M} \phi'(x) P(Z_n < x) dx.\]
By another integration by parts,

\[ E(1_{\mathbb{I}_{Z_n}}(Z_n)) \phi(Z_n)) = \int_{Z_n > M} \phi(Z_n(u)) \, dP(u) = \phi(M) \, P(Z_n > M) \]

\[ + \int_M^{+\infty} \phi'(x) \, P(Z_n > x) \, dx. \]

In fact, we do not integrate on \([M, +\infty[\) but on an interval \([M, M']\), as

\[ P(Z_n > x) \]

is zero for sufficiently large \(x\).

As for \(E(1_{\mathbb{I}_{Z_n}}(Z_n))\), it is equal to \(\phi(M) \, P(Z_n = M)\). We obtain, by
adding these three expressions,

\[ E(\phi(Z_n)) = \phi(M) - \int_{-\infty}^{M} \phi'(x) \, P(Z_n < x) \, dx + \int_{-\infty}^{+\infty} \phi'(x) \, P(Z_n > x) \, dx \]

\[ = \phi(M) - \int_{-\infty}^{-M} \phi'(x) \, P(Z_n < x) \, dx - \int_{M}^{+\infty} \phi'(x) \, P(Z_n < x) \, dx \]

\[ + \int_{M}^{+\infty} \phi'(x) \, P(Z_n > x) \, dx. \quad (1) \]

So we will use the central limit theorem on the interval \([-M, M]\), and the

theorem of large deviations outside of this interval.

(a) Majoration of \(I_1 = \int_{-\infty}^{M} \phi'(x) \, P(Z_n > x) \, dx\). As \(\text{var} \, S_n\) is equivalent to \(\beta n\) (see the proof of Lemma 3.4.2), it is greater than \(\beta n/2\) for

sufficiently large \(n\) and we have

\[ P(Z_n > x) \leq P(S_n - ES_n > x\sqrt{\beta n/2}). \]

Thence, by Lemma 3.3.1, there exist some positive constants \(\lambda\) and \(\delta\) such that

\[ P(Z_n > x) \leq e^{-2\delta x^2} \]

for \(0 \leq x \leq \lambda \sqrt{n}\).

Let \(\varepsilon \leq \delta\) and \(\phi \in \mathcal{F}_\varepsilon\); by definition there exists a constant \(\gamma = \gamma_{\varepsilon}\) such that

\[ ||\phi'(x)|| \leq \gamma e^{\varepsilon x^2} \]

and thence

\[ \int_{-M}^{+\infty} \phi'(x) \, P(Z_n > x) \, dx \leq \int_{-M}^{+\infty} \gamma e^{-\delta x^2} \, dx. \]

Using the classical majoration ([F68, Vol. 1, page 175])

\[ \int_{-\infty}^{+\infty} e^{-t^2/2} \, dt < 1 \]

\[ x \quad e^{-t^2/2} \]
and the change of variable $t = u \sqrt{2\lambda}$, we deduce
\[
\left| \int_{-M}^{e^{2/\lambda}} \phi'(x) P(Z_n > x) \, dx \right| \leq C e^{-\lambda M^2}
\]
with $C$ constant.

It remains to integrate in the interval $[\lambda \sqrt{n}, +\infty[$, actually, using Lemma 3.4.2, in the interval $[\lambda \sqrt{n}, L \sqrt{n}]$ with $L$ constant:
\[
\left| \int_{\lambda \sqrt{n}}^{e^{2/\lambda}} \phi'(x) P(Z_n > x) \, dx \right| \leq P(Z_n > \lambda \sqrt{n}) \left| \phi'(x) \right| \, dx 
\leq e^{-2\lambda \delta^2 L \sqrt{n} - \lambda \sqrt{n}} \gamma e^{L^2 \lambda n}.
\]
This is in $O(e^{-\delta^2 n})$ with $\delta'$ chosen between 0 and $2\lambda \delta^2 - \lambda L^2$, and $\varepsilon < 2\lambda \lambda^2 L^{-2}$.

(b) Majoration of $I_2 = \left| \int_{-M}^{\lambda \sqrt{n}} \phi'(x) P(Z_n < x) \, dx \right|$. The computation is the same, using Lemma 3.3.1, with $-S_n = -X_1 - \cdots - X_n$ in place of $S_n$, and $-x$ in place of $x$.

(c) Majoration of $I_3 = \left| \int_{-M}^{\lambda \sqrt{n}} \phi'(x) P(Z_n < x) \, dx - \int_{-M}^{\lambda \sqrt{n}} \phi'(x) \Phi(x) \, dx \right|$. By Lemma 3.2.1,
\[
I_3 \leq \sup_{x \in \mathbb{R}} \left| P(Z_n < x) - \Phi(x) \right| \int_{-M}^{\lambda \sqrt{n}} \left| \phi'(x) \right| \, dx 
\leq \frac{K}{\sqrt{\text{var} S_n}} \int_{-M}^{\lambda \sqrt{n}} \gamma e^{\lambda^2} \, dx 
\leq \frac{K'}{\sqrt{n}} 2Me^{2M^2}
\]
with $K'$ constant.

(d) Majoration of $I = \left| E(\phi(Z_n)) - E_\phi(\phi) \right|$, $E_\phi(\phi)$ means $(1/\sqrt{2\pi}) \int_{-\infty}^{+\infty} \phi(x) e^{-x^2/2} \, dx$. Setting $I_4 = (1/\sqrt{2\pi}) \int_{|x| > M} \phi(x) e^{-x^2/2} \, dx$ we have, using (1) and an integration by parts for $E_\phi(\phi)$,
\[
I \leq I_1 + I_2 + I_3 + I_4 + |\phi(M) - \phi(M) \Phi(M) + \phi(-M) \Phi(-M)| 
\leq I_1 + I_2 + I_3 + I_4 + |\phi(M) + \phi(-M)| \Phi(-M).
\]
If \( \phi \in \mathcal{F} \) with \( \varepsilon \leq \frac{1}{2} \) we have, using Lemma 3.4.1, \( |\phi(x) - \phi(0)| \leq \varepsilon |x| e^{\varepsilon^2/4} \), thence for \( |x| \geq 1 \),

\[
|\phi(x)| \leq C|x|e^{\varepsilon^2/4}
\]

with \( C \) constant, and

\[
I_4 \leq \frac{1}{\sqrt{2\pi}} \int_{|x| \geq M} C|x|e^{-x^2/4} \, dx \\
\leq 2Ce^{-M^2/4}.
\]

We deduce from (a), (b), and (c) that

\[
I = O\left( e^{-\delta M^2} + e^{-\delta^2 n} + \frac{1}{\sqrt{n}} Me^{\varepsilon M^2} + Me^{-M^2/4} \right).
\]

We can choose \( M \) such that \( e^{-\delta M^2} + Me^{-M^2/4} = 1/\sqrt{n} \); for instance, \( M = (\lambda \log n)^{1/2} \) with \( \lambda = \max\{1/2\delta, 4\} \). Thence \((1/\sqrt{n}) Me^{\varepsilon M^2} = O(n^{-1/2 + 2\delta})\), and \( I = O(n^{-1/2 + \varepsilon}) \) if \( \varepsilon \leq \delta^2/2\lambda \).

(c) \textit{Majoration of } \mathcal{E}(e^{\varepsilon M^2}). \text{ The function } \phi_\varepsilon(x) = e^{\varepsilon^2 x^2} \text{ belongs to } \mathcal{F}_{\varepsilon,}\text{ so we can apply (d) and associate to any } \varepsilon > 0 \text{ an } \alpha > 0 \text{ such that}

\[
|\mathcal{E}(\phi_\varepsilon(Z_n)) - \mathcal{E}(\phi_\varepsilon)| = O(n^{-1/2 + \varepsilon}).
\]

Thence if \( \varepsilon \leq \frac{1}{2} \) the sequence \( n \to \mathcal{E}(\phi_\varepsilon(Z_n)) \) is bounded. This proves Proposition 3.5.1, using Lemma 3.4.3 and (d).

4. ASYMPTOTIC EQUIVALENT OF \( S_\phi(N) \)

\textbf{Notation.} For any integer \( N \geq 1 \) and \( f: \mathcal{D}^* \to \mathbb{R} \), \( \phi: \mathbb{R} \to \mathbb{R} \), let

\[
S_\phi(N) = S_{\alpha, f, \phi}(N) = \sum_{1 < v < N} \phi(z_v)
\]

with \( z_v = (s^\varepsilon(v) - \ln \log \nu \sqrt{\beta \log \nu} \),

4.1. \textit{Partition of } \( \{1, 2, \ldots, N-1\} \).

Let the representation of \( N \) be

\[
N = |\sigma^{-1}(m_1)| + \cdots + |\sigma^{-1}(m_l)|
\]

and let \( v \in \{1, 2, \ldots, N-1\} \).
If \( v \) has a representation of length \( \mathcal{N} \), let \( k \) be the greatest integer such that \( m_{k+1}(v) \neq m_{k+1} \). We have
\[
  v = |\sigma^{-1}(m_1)| + \cdots + |\sigma^{-1}(m_{k+2})| + |\sigma'(m)| + |\sigma^{k-1}(m_k(v))| + \cdots,
\]
where \( (k, m) \) belongs to the set
\[
  \mathcal{E} = \{(k, m) \in \mathbb{N} \times \mathcal{A}^* ; k \leq \mathcal{N} - 1, m < m_{k+1}, (k, m) \neq (\mathcal{N} - 1, A) \}.
\]

Let \( I(k, m) \) be the set of integers \( v \) having such a representation.

If \( v \) has a representation of length less than \( \mathcal{N} \), this representation is
\[
  v = |\sigma^{k}(m)| + |\sigma^{k-1}(m_k(v))| + \cdots,
\]
where \( (k, m) \) belongs to the set
\[
  \mathcal{E}' = \{(k, m) \in \mathbb{N} \times \mathcal{A}^* ; k \leq \mathcal{N} - 2, A < m < \sigma(1) \}.
\]

Let \( I'(k, m) \) be the set of integers \( v \) having such a representation.

**Lemma 4.1.1.** \( I(k, m) \) and \( I'(k, m) \) have \( |\sigma^{k}b| \) elements, where \( b \) is the letter such that \( mb \leq m_{k+1} \) (in the first case) or \( mb \leq \sigma(1) \) (in the second). The families of sets \( I(k, m) \) for \( (k, m) \in \mathcal{E} \) and \( I'(k, m) \) for \( (k, m) \in \mathcal{E}' \) both form a partition of \( \{1, 2, \ldots, \mathcal{N} - 1\} \).

**Proof.** Let \( (k, m) \in \mathcal{E}' \). For any \( b \)-admissible sequence \( m'_1 \cdots m'_r \),
\[
  |\sigma^{k}(m)| + |\sigma^{k-1}(m'_1)| + \cdots + |\sigma^{r}(m'_r)|
\]
is the representation of an integer \( v \in I'(k, m) \). Thence the cardinality of \( I'(k, m) \) is equal to the number of \( b \)-admissible sequences of length \( k \) and, using the remark of Section 3.1, it is \( |\sigma^{k}b| \).

The proof is the same for \( I(k, m) \).

**Lemma 4.1.2.** The length \( n(v) \) of the representation of any integer \( v \geq 2 \) satisfies
\[
  1 \leq n(v) \leq \log_{\lambda'} v + \log_{\lambda''} v \leq \log_{\lambda''} v + \log_{\lambda''} n(v)
\]
where the positive constants \( \lambda, \lambda', \) and \( \lambda'' \) depend only on the substitution.
Proof. As $|\sigma^m(v) - 1(1)| \leq v < |\sigma^m(v)|$ and $|\sigma^m(v)| \sim \epsilon(1) \theta^n$ when $n \to +\infty$, we have the first inequality and the second with $\lambda' = \log_\theta \lambda$.

Thence $\inf_{v \geq 2}(\log_\theta v/n(v))$ is positive and $\sup_{v \geq 2}(\log_\theta v/n(v))$ is finite.

4.2. Sum of $\phi(z_v)$ on One Set of the Partition

Lemma 4.2.1. For any $\varepsilon' > 0$ there exists $\varepsilon > 0$ such that, for $\phi \in \mathcal{F}_\varepsilon$ and $(k, m) \in E$,

$$\sum_{v \in \mathcal{R}(k, m)} \phi(z_v) = |\sigma^b| E_\phi(\phi) + O\left(\frac{1}{\varepsilon' + \theta^{-v}}\frac{\theta^{-v}}{\theta^{1/3} n(v)^{1/3}}\right).$$

We have the same estimate for $\sum_{v \in \mathcal{R}(k, m)} \phi(z_v)$.

Proof. Let $z_v = (s^f(v) - x \log_\theta v)/\sqrt{\theta \log_\theta v}$. Let $s^f(v, k) = f(m_1(v)) + \cdots + f(m_1(v))$ and $z_{v, k} = (s^f(v, k) - xk)/\sqrt{\theta \log_\theta v}$ (with $z_{v, k} = 0$ if $k = 0$). We have

$$z_v - z_{v, k} = \frac{s^f(v) - s^f(v, k) - xk + \log_\theta v}{\sqrt{\theta \log_\theta v}} - \frac{k - \log_\theta v}{\sqrt{\theta \log_\theta v}}.$$

If $v \in \mathcal{R}(k, m) \cup I'(k, m)$, then $n(v)$ is equal to $\lambda' + k + 1$. We deduce from the second inequality of Lemma 4.1.2 that

$$-\lambda' \leq \log_\theta v - k \leq \lambda' + n(v) - k.$$

Thence we have $|\log_\theta v - k| = O(n(v) - k)$ and, from (1) and Lemma 4.1.2,

$$z_v - z_{v, k} = O\left(\frac{k(n(v) - k)}{\sqrt{n(v)}} + \frac{k}{\sqrt{k}} \frac{n(v) - k}{\sqrt{k \log n(v)}}\right) = O\left(\frac{k(n(v) - k)}{\sqrt{n(v)}}\right).$$

As the function $x \to (x - k)/\sqrt{x}$ is non-decreasing, we deduce

$$z_v - z_{v, k} = O\left(\frac{\lambda' - k}{\sqrt{\lambda'}}\right).$$
Using Lemma 3.4.1 and the function $\psi(x) = e^{e^x}$, there exists a constant $C$ such that

$$
|\phi(z) - \phi(z_{n,k})| \leq C \frac{V - k}{\sqrt{V}} \psi \left( |z_{n,k}| + C \frac{V - k}{\sqrt{V}} \right).
$$

As $z_{n,k} = O(\sqrt{k})$ there exists a constant $D$ such that

$$
\left( |z_{n,k}| + C \frac{V - k}{\sqrt{V}} \right)^2 \leq (z_{n,k})^2 + D(V - k)
$$

thence

$$
|\phi(z) - \phi(z_{n,k})| \leq C \frac{V - k}{\sqrt{V}} \psi(z_{n,k}) e^{e^{Dk}} e^{-k}
$$

and, summing for $v \in I(k, m)$ or $v \in I'(k, m)$,

$$
\left| \sum \phi(z) - \sum \phi(z_{n,k}) \right| \leq \sum |\phi(z) - \phi(z_{n,k})| \\
\leq C \frac{V - k}{\sqrt{V}} e^{e^{Dk}} e^{-k} \sum \psi(z_{n,k}). \tag{2}
$$

$\sum \phi(z_{n,k})$ is equal to $S_\phi(k, b)$, and by Proposition 3.5.1 we can choose $\varepsilon$ such that for $\phi \in \mathcal{F}_\varepsilon$,

$$
\sum \phi(z_{n,k}) = |\sigma^b| E_{\phi}(\phi) + O(\theta^k(k + 1)^{-1/2 + \varepsilon}).
$$

The same estimate holds for the function $\psi$, which belongs to $\mathcal{F}_{2\varepsilon}$, and we deduce

$$
\sum \psi(z_{n,k}) = O(\theta^k).
$$

With (2) we obtain

$$
\sum \phi(z) - |\sigma^b| E_{\phi}(\phi) = O \left( \theta^k(k + 1)^{-1/2 + \varepsilon} + \frac{V - k}{\sqrt{V}} e^{e^{Dk}} e^{-k} \theta^k \right). \tag{3}
$$
As $(k + 1)(N - k) \geq \lambda^+$ for $k = 0, 1, \ldots, N - 1$, we majorize $(k + 1)^{-1/2 + \epsilon}$ by $(N - k/N)^{1/2 - \epsilon}$. As $(N - k/N)^{1/2 - \epsilon}$ and $(N - k)/\sqrt{N}$ are in $O(1/N^{1/2 - \epsilon})$, the estimate (3) implies

$$\sum \phi(z_{n,k}) - |\sigma^b|E_d(\phi) = O\left(\frac{1}{\sqrt{1 + \theta}^{1 + \epsilon}} e^{(1 + \theta)(N - k)} \frac{\theta^{\epsilon}}{\theta^{1/2}}\right).$$

This proves Lemma 4.2.1, choosing $\epsilon$ sufficiently small.

### 4.3. Asymptotic Estimate of $S_d(N)$

**Proposition 4.3.1.** For any $\epsilon' > 0$, there exists $\epsilon > 0$ such that for $\phi \in \mathcal{F}_s$,

$$S_d(N) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x) e^{-x^2/2} dx + O((\log N)^{\epsilon' - 1/2}).$$

**Proof.** Using the Lemmas 4.1.1 and 4.2.1,

$$NS_d(N) = \sum \sum_{x \in \{k, m\}} \phi(z_n) + \sum_{x \in \mathcal{E}} \sum_{x' \in \mathcal{E}} \phi(z_{n,k})$$

$$= \left(\sum_{x \in \mathcal{E}} |\sigma^b| \right) E_d(\phi) + O\left(\frac{\theta^{\epsilon'}}{\sqrt{1 + \theta}^{1/2 - \epsilon}} \sum_{x \in \mathcal{E}} \frac{1}{\theta^{1/2 + \epsilon - k}}\right).$$

$\sum_{x \in \mathcal{E}} |\sigma^b|$ is equal to $N - 1$ by Lemma 4.1.1, and $\sum_{x \in \mathcal{E}} (1/\theta^{1/2 + \epsilon - k})$ is bounded because, for each integer $k$, there is a bounded number of words $m$ such that $(k, m) \in \mathcal{E} \cup \mathcal{E}'$. We obtain

$$NS_d(N) = (N - 1) E_d(\phi) + O\left(\frac{\theta^{\epsilon'}}{\sqrt{1 + \theta}^{1/2 - \epsilon}}\right)$$

and, using Lemma 4.1.2 with $\nu = N$ and $n(\nu) = \lambda^+$, we obtain Proposition 4.3.1.

**Corollary 4.3.2.** We have the same estimate for $N^{-1} \sum_{1 < v < N} \phi((s^v(v) - \lambda \log v)/\sqrt{\beta \log v} N)$.

**Proof.** We modify the beginning of the proof of Lemma 4.2.1, replacing $z_{n,k}$ by $(s^v(v) - \lambda \log v)/\sqrt{\beta \log v} N - z_{n,k}$. It has the same expression as in (1), replacing $\sqrt{\beta \log v} N$ by $\sqrt{\beta \log v} N$ in the denominator; thence it is in $O((n(\nu) - k)/\sqrt{\mathcal{M}})$ and in $O((\lambda^+ - k)/\sqrt{\mathcal{N}})$. 

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Remark. Let \( \phi \) be defined by
\[
\phi(x) = 1 \quad \text{if} \quad x < x_0
\]
\[
= 0 \quad \text{otherwise}.
\]
Proposition 4.3.1 does not apply because \( \phi \) is not continuous; but given \( x > 0 \), we can easily define two continuously differentiable functions \( \phi_1 \) and \( \phi_2 \) such that
\[
0 \leq \phi_1(x) \leq \phi(x) \leq \phi_2(x) \leq 1 \quad \text{for any} \quad x \in \mathbb{R}
\]
\[
\phi_1(x) = \phi(x) = \phi_2(x) \quad \text{for} \quad x \notin [x_0, x_0 + \varepsilon].
\]
Thence we have \( S_\phi(N) \leq S_\phi(N) \leq S_\phi(N) \) and \( E_\phi(\phi_2) - E_\phi(\phi_1) < \varepsilon \).

Applying Proposition 4.3.1 to the functions \( \phi_1 \) and \( \phi_2 \) we deduce
\[
N^{-1} \int_{n < N; \frac{x^T(v) - \alpha \log_\beta \nu}{\sqrt{\beta \log_\beta \nu} < x_0}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x^2/2} dx + o(1).
\]

5. SUMS ASSOCIATED WITH THE FIXED POINT OF A SUBSTITUTION.

Let \( u_1, u_2, \ldots \) be the fixed point of the substitution \( \sigma \), with \( u_1 = 1 \), and \( M \) the matrix of \( \sigma \) (\( M_{i,j} \) is the number of occurrences of the letter \( i \) in the word \( \sigma(j) \)). We suppose that its eigenvalues satisfy
\[
\theta > \theta_2 = 1 > |\theta_3| \geq \cdots \geq |\theta_d|.
\]
We suppose also that \( f \cdot \xi = 0 \), where \( f = (f(1), \ldots, f(d)) \) is a row-vector and \( \xi \) is a column-eigenvector for \( \theta \) (i.e., \( M \xi = \theta \xi \)). Now we consider \( \alpha \) and \( \beta \) being as in [D90, Section 3], and
\[
S_\phi(N) = N^{-1} \sum_{1 < r < N} \phi(z'_r)
\]
\[
z'_r = \frac{f(u_1) + \cdots + f(u_d) - \alpha \log_\beta \nu}{\sqrt{\beta \log_\beta \nu}}.
\]

PROPOSITION 5.1. We have the same estimate for \( S_\phi(N) \), as for \( S_\phi(N) \) in Proposition 4.3.1.

Proof. Let \( E \) be the set of row-vectors \( X = (x_1, \ldots, x_d), x_i \in \mathbb{R} \). We have \( E = E_1 \oplus E_2 \oplus E_3 \), where \( E_1 \) and \( E_2 \) are the sets of row-eigenvectors associated respectively with the eigenvalues \( \theta \) and \( \theta_2 \), and where \( E_3 \) is a subspace of \( E \) stable by the map \( X \to XM \).
As $E_2 \oplus E_3$ is equal to \{ $X, X \cdot \lambda = 0$ \}, the row-vector $f$ has a decomposition

$$f = f_2 + f_3$$

with $f_2 \in E_2$ and $[ f_3 ] \in E_3$. We have also

$$\| f_3 M^n \| = O(r^n)$$

where $\| \cdot \|$ is the norm-sup and $r$ is a real number between $|\theta_3|$ and 1.

Extending the map $f$ to a morphism $f: A^* \rightarrow R$ we have

$$f(u_1) + \cdots + f(u_n) = f(u_1 \cdots u_n)$$

$$= f(\sigma^{m_1 - 1}(m_{m_1}(v))) \cdots \sigma^{0}(m_1(v)))$$

$$= \sum_{i=1}^{m_1} f_2(\sigma^{i-1}(m_1(v))) + \sum_{i=1}^{m(v)} f_3(\sigma^{i-1}(m_i(v))).$$

We remark that the row-eigenvector

$$(f_2(\sigma^{i-1}(1)), \ldots, f_2(\sigma^{i-1}(d)))$$

is equal to $f_2 M^{i-1}$, thence to $f_2$. The row-eigenvector

$$(f_3(\sigma^{i-1}(1)), \ldots, f_3(\sigma^{i-1}(d)))$$

being equal to $f_3 M^{i-1}$, its norm is in $O(r^i)$, thence

$$f(u_1) + \cdots + f(u_n) = \sum_{i=1}^{m_1} f_2(m_1(v)) + O \left( \sum_{i=1}^{m(v)} r^i \right)$$

$$= s'(v) + O(1).$$

Thence the proof of Lemma 4.2.1 and Proposition 4.3.1 applies, replacing $s'(v)$ by $f(u_1) + \cdots + f(u_n)$ and $s'(v, k)$ by $s'^{(v, k)}$.

### 6. CASE OF THE SUM-OF-DIGITS IN INTEGER BASE G.

We may precisely set the set of functions $\phi$ for which the estimates of Propositions 3.5.1, 4.3.1 and 5.1 occur.

Let $\mathcal{U}_n = \{ 0, 1, \ldots, g - 1 \}^n$. The Markov chain is defined in this case by

$$X_k(u) = \varepsilon_k$$

for any $u = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{U}_n$. The $X_k$ are independent and uniformly distributed.
We do not need to use Lemmas 3.4.1, 3.4.2, and 3.4.3; indeed, $Z_n$ is exactly equal to $(S_n - mn)/\sqrt{bn}$, and the majoration of $|\phi(x) - \phi(y)|$ in Lemma 3.4.1 is not necessary. Thence we may modify the definition of $\mathcal{F}_\varepsilon$, and let $\mathcal{F}'_\varepsilon$ be the set of continuous functions satisfying

$$|\phi(x)| \leq \gamma e^{x^2}$$

with $\gamma$ constant.

In the proof of Proposition 3.5.1, we have to majorize the integral

$$J = \left| \int_{Z_n(u) > M} \phi(Z_n(u)) \, dP(u) \right|.$$ 

Summing by parts,

$$J \leq \int_{Z_n(u) > M} \gamma e^{e^{u^2}} \, dP(u)$$

$$\leq \gamma e^{e^{M^2}} P(Z_n > M) + \int_{M}^{+\infty} 2\gamma xe^{x^2} P(Z_n > x) \, dx.$$ 

For any $x > 0$, $P(Z_n > x)$ is in $O(e^{-1/2 + x^2})$ when $x \to +\infty$ and $x = o(1/\sqrt{n})$ (see for instance \cite{F68}, Vol. 2, page 553). Thence by a method similar to that in Proposition 3.5.1, we obtain

$$J = O(e^{-1/2 + x^2}M^2).$$

The estimates of Propositions 3.5.1, 4.3.1, and 5.1 are valid for the functions $\phi$ which belong to $\mathcal{F}'_\varepsilon$ for some $\varepsilon < 1/2$, the error terms are in $O(n^{-1/2 + x^2})$ and thus in $O((\log N)^{-1/2 + x^2})$ for any $x > 0$.

**Examples in base $g = 2$.**

(a) For $\phi_1(x) = e^{1/4}x^2$,

$$S_{g_1}(N) = \sqrt{2} + O\left(\frac{1}{(\log N)^{1/2 - x}}\right)$$

for all $x > 0$. The numerical computation for $N = 2000$ gives

$$S_{g_1}(2000) = 1, 37 \pm 0, 01.$$

(b) For $\phi_2(x) = e^x$, we have by Proposition 4.3.1

$$S_{g_2}(N) = \sqrt{e} + O\left(\frac{1}{(\log N)^{1/2 - x}}\right)$$

for all $x > 0$. The numerical computation for $N = 2000$ gives

$$S_{g_2}(2000) = 1, 37 \pm 0, 01.$$
for all \( \alpha > 0 \); but for \( N = 2000 \),

\[
S_{\beta^2}(2000) = 2, 28 \pm 0, 01.
\]

We will obtain a better error term for \( S_{\beta^2}(N) \) by modifying the definition of \( z_* \). In the case \( N = 2^r \), we have

\[
S_{\beta^2}(N) = \frac{1}{2^r} \sum_{\nu < 2^r} e^{\nu r}
\]

with \( z_* = (x_1(v) - \alpha \log_2 v)/\sqrt{\beta \log_2 v} \). Here we have \( \lambda = \frac{1}{2} \) and \( \beta = \frac{1}{2} \). If we set \( z_* = (x_1(v) - \alpha n)/\sqrt{\beta n} \) or \( z_* = (x_2(v) - \alpha \lfloor \log_2 v \rfloor)/\sqrt{\beta n} \), we can easily compute this sum and obtain

\[
\sqrt{e} \left( 1 - \frac{1}{12n} \right) + o \left( \frac{1}{n} \right)
\]

in the first case, and

\[
\sqrt{e} \left( 1 + \frac{2}{\sqrt{n}} + \frac{11}{12n} \right) + o \left( \frac{1}{n} \right)
\]

in the second. It shows that the estimate is better in the first case, and the error term \( O(1/(\log N)^{1/2}) \) cannot be improved in the second.

REFERENCES


