On Sample Path Properties of Semistable Processes

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This paper contains three main results: In the first result a correspondence principle between semistable measures on $L_p$, $1 \leq p < \infty$, and Banach space valued semistable processes is established. In the second result it is shown that the paths of a Banach space valued semistable process belong to $L_p$ with probability zero or one, and necessary and sufficient conditions for the two alternatives to hold are given. In the third result necessary and sufficient conditions are given for almost sure path absolute continuity for certain Banach space valued semistable processes.

1. INTRODUCTION AND PRELIMINARIES

In this paper three main results are presented: First, we show that, for a fixed $0 < r < 1$ and $0 < \alpha < 2$, the set of $r$-semistable index $\alpha$ ($r$-SS($\alpha$)) probability (p.) measures on a separable Banach space $B$ is closed under weak limits. Second, we show that the now well-known correspondence principle between Gaussian (Vakhania [18], Rajput [13], Byczkowski [3]) and stable index $\alpha(S(\alpha))$ (Weron [19] and Louie [10]) processes and measures on $L_p$ extends to Banach valued $r$-SS($\alpha$) processes and measures on $L_p$, $1 \leq p < \infty$. Third, we prove that the paths of a $B$-valued $r$-SS($\alpha$) process belong to $L_p$ with probability zero or one, and give necessary and sufficient conditions, in terms of the integrability of the
moments of the process, for the two alternative to hold; these last two results when specialized to $S(x)$ processes yield and in fact, in some cases, improve the corresponding results for $S(x)$ processes obtained by Weron [19], Cambanis and Miller [6] and Louie [10], as long as $p \geq 1$. Finally, we give several necessary and sufficient conditions for almost sure path absolute continuity for $r$-SS($a$) $B$-valued separable processes, under certain conditions on $B$. This result is motivated from and is an extension of the work of Cambanis and Miller [6] and Louie [10], who obtained similar conditions for real (symmetric) $S(x)$ and $B$-valued $S(x)$ processes, respectively.

In the rest of this section, we record certain notation, definitions, and conventions. Throughout the paper $r$ and $a$ will denote, respectively, real numbers satisfying $0 < r < 1$ and $0 < a < 2$. All measures on a topological space $S$ are assumed to be defined on $B(S)$, the $\sigma$-algebra of its Borel sets. If $B$ is a Banach space, then $B^*$ and $\langle \cdot , \cdot \rangle$ will denote, respectively, its topological dual and the natural duality between $B$ and $B^*$. Throughout, $B$ will denote a real separable Banach space. If $\mu$ is a p. measure on $B$, then $\bar{\mu}$ will denote its characteristic (ch.) function; and the symbol $\rightarrow_w$ will denote the weak convergence of measures. Finally, if $X$ is a random (r.) object taking values in $B$, then its law on $B$ will be denoted by $\mathcal{L}_X$. Now we give the definition of an $r$-SS($a$) process. Let $X = \{X_t : t \in \mathbb{A} \}$ be a $B$-valued process, then we say $X$ is an $r$-SS($a$) (resp. a centered $r$-SS($a$)) process if for every $n$, and $\lambda_1,...,\lambda_n$ in $\mathbb{A}$, $\mathcal{L}_{(X_{\lambda_1},...,X_{\lambda_n})}$ is an $r$-SS($a$) (resp. a centered $r$-SS($a$)) p. measure on $B^n$ (for definition and properties of $r$-SS($a$) p. measures on Banach spaces, we refer the reader to [7, 14, 15]). As noted in [14], if $a \neq 1$, then every $r$-SS($a$) process $X$ can be written as $X_t = Y_t + \theta(\lambda)$, where $Y = \{Y_t : \lambda \in \mathbb{A} \}$ is a centered $r$-SS($a$) process and $\theta$ is the centering function; a similar phenomenon holds for $r$-SS($a$) p. measures on separable Banach spaces (see, again, [7, 14, 15]). Whenever convenient we shall write $X(\lambda)$ for $X_t$.

2. CORRESPONDENCE PRINCIPLE AND A ZERO–ONE LAW FOR $r$-SS($a$) PROCESSES

Before we can discuss our results of this section, we shall need a few more notation and conventions: If $(S, \mathscr{F}, G)$ is a $\sigma$-finite measure space and $F$ a real or complex Banach space with norm $\| \cdot \|$, then, throughout, $L_p(S,F)$ will denote the usual Banach space of $1 \leq p < \infty$, and the usual metric topological vector space if $0 < p < 1$. The norm of an element $f$ in $L_p(S,F)$, $1 \leq p < \infty$, will be denoted by $\|f\|_p = (\int \|f\|^p dG)^{1/p}$. Throughout, unless stated otherwise, $(A, \mathscr{A})$ will denote a measurable
space Borel isomorphic to \([0, 1], \mathcal{B}[0, 1]\) (e.g., an uncountable Borel subset of a complete separable metric space endowed with its Borel \(\sigma\)-algebra); further, throughout this section, it will be assumed that a finite measure \(\nu\) is given on \((A, \mathcal{A})\). Let \(\phi: [0, \infty) \rightarrow \mathbb{R}\), the reals, be a continuous nondecreasing subadditive function such that \(\phi(t) = 0\) if and only if \(t = 0\), then the vector space of equivalence classes of measurable functions \(f: S \rightarrow F\) satisfies

\[
\|f\|_\phi = \int_S \phi(\|f\|) \, dG < \infty
\]

is a complete metric topological space under the metric \(d(f, g) = \|f - g\|_\phi\). We shall denote this space by \(L_\nu(S, F)\).

In the statement of Theorem 2.5, we require that \(L_p^*(A, B) = L_q(A, B^*)\), where \(1 \leq p < \infty\) and \(q\) is conjugate of \(p\); there are a variety of sufficient conditions for this to hold, e.g., \(B^*\) is separable or even \(B^*\) has the Radon-Nikodym property [8, p. 79]. Theorem 3.1 is proved assuming that the measure \(\nu\) on \((A, \mathcal{A})\) is finite; this assumption on \(\nu\) is not necessary and \(\nu\) can be replaced by any \(\sigma\)-finite measure. Indeed, if \(\gamma\) is a \(\sigma\)-finite measure on \((A, \mathcal{A})\), then one defines

\[
\nu(A) = \sum_j (2^{j/2} \gamma(A_j))^{-1} \gamma(A \cap A_j),
\]

where \(A_j\)'s are disjoint, \(\bigcup_j A_j = A\) and \(0 < \gamma(A_j) < \infty\); and one proves the theorem first for the finite measure \(\nu\), and then, using the isomorphism:

\[
\psi \in L_p(A, \mathcal{A}, \nu; B) \rightarrow \psi^{1/p} \in L_p(A, \mathcal{A}, \gamma; B),
\]

where \(\psi = d\nu/d\gamma\), one obtains the theorem for the \(\sigma\)-finite measure \(\gamma\). We finish this paragraph by recording one more notation: Let \(\{X_\lambda: \lambda \in \Lambda\}\) be a measurable \(B\)-valued process with almost all paths in \(L_p(A, B)\); then \(L_p(A, B)\) is separable (as \(B\) is) and the map \(\omega \rightarrow X(\cdot, \omega)\), if \(X(\cdot, \omega) \in L_p(A, B)\), \(\omega \rightarrow 0\) otherwise, is Borel measurable; this map will be denoted by \(\hat{X}\). We shall also use similar notation for the map \(\omega \rightarrow X(\cdot, \omega)\) from \(\Omega \rightarrow B^A\), the space of all \(B\)-valued functions on \(A\).

Now we are ready to state and prove our first result of this section.

**Proposition 2.1.** Let \(\mu, \mu_n, n = 1, 2, \ldots, \) be \(p\) measures on \(B\) with \(\mu_n\)'s i.d. If \(\mu_n \rightarrow \omega_\mu\), then \(\mu\) is i.d. and \(\mu_n' \rightarrow \omega_\mu',\) for all \(t > 0\), where \(\mu_n'\) and \(\mu'\) denote, respectively, the \(t\)th roots of \(\mu_n\) and \(\mu\) (see [7] for the definition and properties of roots of \(\sigma\)-finite measures). Further, if \(\mu_n\)'s are \(r\)-SS(\(\alpha\)), then \(\mu\) is \(r\)-SS(\(\alpha\)); moreover, if \(\alpha \neq 1\) and \(\mu_n = \gamma_n \ast \delta_0\) and \(\mu = \gamma \ast \delta_0\), where \(\theta_n\) and \(\theta\) are, respectively, the centering elements of \(\mu_n\) and \(\mu\), then \(\theta_n \rightarrow \theta\) and, hence, also \(\gamma_n \rightarrow \gamma\).

**Proof.** The fact that \(\mu\) is i.d. follows easily from a result of Tortrat [17, p. 320]: indeed, according to this result, to show that \(\mu\) is i.d., it is sufficient to prove that \(\mu \circ (y_1, \ldots, y_k)^{-1}\) is i.d. on \(R^k\), for any choice of \(k\) and \(y_1, \ldots, y_k\) in \(B^*\). But this follows, since \(\mu_n \circ (y_1, \ldots, y_k)^{-1}\) are i.d. on \(R^k\) and \(\mu_n \circ (y_1, \ldots, y_k)^{-1} \rightarrow \omega_\mu \circ (y_1, \ldots, y_k)^{-1}\).
To prove that $\mu_n \to^* \mu^t$, $t > 0$, it is sufficient to show this only for $t \in (0, 1)$. Fix such a $t \in (0, 1)$; since $\mu_n = \mu_n^t * \mu_n^{1-t}$, $\{\mu_n^t\}$ is shift tight; thus, to show that $\mu_n^t \to^* \mu^t$, it is sufficient to prove that $\mu_n^t(y) \to \mu^t(y)$ uniformly on the unit ball $A$ of $B^*$ (see [1]). If this were not true, then, for some $\varepsilon > 0$, there would exist a sequence $\{y_n\}$ in $A$ and a $y \in A$ such that $y_n \to y$ in the weak topology of $B$ and

$$|\hat{\mu}_n^t(y) - \hat{\mu}^t(y)| \geq \varepsilon. \tag{1}$$

But $\mu_n \to^* \mu$ and hence $\hat{\mu}_n(y) \to \hat{\mu}(y)$ uniformly on $A$ [1]; thus we also have

$$|\hat{\mu}_n(y_n) - \hat{\mu}(y)| \to 0. \tag{2}$$

Taking a continuous version of the logarithm in a neighborhood of $\hat{\mu}(y)$ and using (2), we have $t \log \hat{\mu}_n(y_n) \to t \log \hat{\mu}(y)$, which, upon taking exponents, contradicts (1). To see that $\mu$ is $r$-SS$(\alpha)$, one uses the above and the fact (see [7]),

$$\mu_n^t = r^{1/\alpha} \cdot \mu_n * \delta_x,$$

to conclude

$$\mu^t = r^{1/\alpha} \cdot \mu * \delta_x,$$

where $x$ is the limit of $\{x_n\}$.

To prove the last part, one notes that if $\alpha \neq 1$, then $\theta_n = x_n(r - r^{1/\alpha})^{-1}$ (see [7]); hence, as $\{x_n\}$ converges, so does $\{\theta_n\}$, say, to $\theta$. Then, as $\mu_n \to^* \mu$, we have $\gamma_n \to^* \gamma$ and $\mu = \gamma * \delta_\theta$.

The following three results will be needed for the proof of the main result of this section, namely Theorem 2.5.

**Proposition 2.2.** Let $p \geq 1$ and let $L_p(A, B^*) = L_q(A, B^*)$. Let $Y \equiv \{Y_\lambda: \lambda \in A\}$ be a centered $r$-SS$(\alpha)$ $B$-valued measurable process with paths in $L_p(A, B)$ a.s. Let $f \in L_q(A, B^*)$, and $L(Y_f)$ be the closure in probability of the linear span of $\{\langle Y(\lambda), f(\lambda) \rangle: \lambda \in A\}$. Then $\int \langle U(\lambda), f(\lambda) \rangle \, dv$ belongs to $L(Y_f)$.

**Proof.** Let $v = v_0 + v_1$, where $v_0$ and $v_1$ are, respectively, the continuous and discrete parts of $v$. Let $v_1$ be concentrated on $A_1 = \{\lambda_j\}$ with $v_1(\lambda_j) > 0$ and let $A_0 = A \setminus A_1$. Let $\tau$ be Borel isomorphism: $A_0 \setminus S_0$ onto $[0, 1] \setminus A_0$ such that $v_0 \circ \tau^{-1} = \text{Leb}$, where $S_0$ and $A_0$ are measurable sets with $v_0(S_0) = 0 = \text{Leb}(A_0)$ (see Royden [16, p. 270]). Then
\[ \int \langle Y(\lambda), f(\lambda) \rangle \, d\nu = \int_{A_0} \langle Y(\lambda), f(\lambda) \rangle \, d\nu_0 + \int_{A_1} \langle Y(\lambda), f(\lambda) \rangle \, d\nu_1 \]

\[ = \int_{[0,1]} \langle Y(\tau^{-1}(\lambda)), f(\tau^{-1}(\lambda)) \rangle \, d\text{Leb} + \sum_j \langle Y(\lambda_j), f(\lambda_j) \rangle \, \nu_1(\lambda_j), \]

where \( Y(\tau^{-1}(\lambda)) \) is defined to be 0 for \( \lambda \in A_0 \).

Therefore, according to Theorem 2.8 of [9], the first integral on the right side of (3) belongs to \( L(Y) \). Therefore, as the second term on the right of (3) is clearly in \( L(Y) \), the proof of the proposition is complete.

**Corollary 2.3.** Under the hypotheses of Proposition 2.2, \( \tilde{Y} \) is Borel measurable and the induced p. measure on \( L_\rho(A, B) \) is r-SS(\( \sigma \)).

**Proof.** First, we show that \( \mu \equiv \mathscr{L}_\tilde{Y} \) is i.d. Using Tortrat [17], it is enough to show that \( \tilde{\mu} = (S, ..., f_k)^{-1} \) is i.d. on \( R_k \) for any \( k \) and \( f_1, ..., f_k \) in \( \mathcal{B}(A, B) \). But this follows from Proposition 2.1 and from the proof of Proposition 2.2. That \( \mu \) is r-SS(\( \sigma \)) also follows using the same two propositions.

**Lemma 2.4.** Let \( X = \{X_\lambda: \lambda \in \Lambda \} \) be a B-valued r-SS(\( \sigma \)) process, where \( \Lambda \) is any index set. Let \( \mathcal{F} = \mathscr{B}(B^\Lambda) \) denote the \( \sigma \)-algebra generated by cylinder sets of \( B^\Lambda \). Let \( \tilde{X} \) be the natural map: \( \Omega \to B^\Lambda \) (see paragraph prior to Proposition 2.1), then the induced p. measure \( \mu \equiv \mathscr{L}_\tilde{X} \) on \( (B^\Lambda, \mathcal{F}) \) is r-SS(\( \sigma \)) in the sense of [11], i.e., \( \mu \) is i.d. and there exists a semigroup \( \{\mu^t: t > 0\} \) of p. measures on \( \mathcal{F} \) satisfying:

\[ \mu^t = \mu \quad \text{and} \quad \mu^t = r^{1/\lambda} \cdot \mu \ast \delta_x, \quad (4) \]

for some \( x \in B^\Lambda \).

**Proof.** Fix \( t > 0 \) and define, for every \( n \) and \( \lambda_1, ..., \lambda_n \in \Lambda \), a finite-dimensional p. distribution on \( B^\nu \) by

\[ F_{\lambda_1, ..., \lambda_n}^\nu(D) = (\mu \circ \pi_{\lambda_1, ..., \lambda_n}^{-1})^\nu(D), \]

for every Borel set \( D \) of \( B^\nu \), where \( \pi_{\lambda_1, ..., \lambda_n} \) is the natural projection. One easily verifies that \( \{F_{\lambda_1, ..., \lambda_n}\} \) is a consistent family of finite dimensional p. distributions, and hence, via Kolmogorov’s existence theorem, there exists a unique p. measure \( \mu' \) on \( \mathcal{F} \) having \( F_{\lambda_1, ..., \lambda_n} \) as its finite-dimensional projection measures. That \( \{\mu^t: t > 0\} \) is a semigroup and satisfies (4) follow now easily by recalling that the set of measures \( \{((\mu \circ \pi_{\lambda_1, ..., \lambda_n})^t: t > 0\} \) have
similar properties, for all $n$ and $\lambda_1, \ldots, \lambda_n$. This completes the proof of the lemma.

In the statement of Theorem 2.5 both the $r$-SS($\alpha$) process $X$ and the measure $\mu$ are assumed (except in the zero–one law part) to be centered when $\alpha = 1$.

**Theorem 2.5.** (a) (Correspondence principle). Let $p > 1$ and $B$ be such that $L_p^r(\Lambda, B) = L_q(\Lambda, B^*)$. Let $X = \{X_\lambda : \lambda \in \Lambda\}$ be a $B$-valued measurable $r$-SS($\alpha$) process. If $X(\cdot, \omega) \in L_p(\Lambda, B)$ a.s., then the map $\overline{X}$ is Borel measurable and the induced p. measure $\mu_{\overline{X}} = \mathcal{L}_{\overline{X}}$ is r-SS($\alpha$). Conversely, if $\mu$ is an r-SS($\alpha$) p. measure on $L_p(\Lambda, B)$ (here the hypothesis $L_p^r(\Lambda, B) = L_q(\Lambda, B^*)$ is not assumed), then there exists a B-valued measurable $r$-SS($\alpha$) process inducing the measure $\mu$ on $L_p(\Lambda, B)$.

(b) (zero–one law). Let $X = \{X_\lambda : \lambda \in \Lambda\}$ be a $B$-valued measurable $r$-SS($\alpha$) process. Then $X(\cdot, \omega) \in L_p(\Lambda, B)$ with probability zero or one, and if $L_p^r(\Lambda, B^*) = L_q(\Lambda, B^*)$, $p \geq 1$, then the second alternative occurs if

$$E \left( \int \|X(\lambda)\|^p \, dv \right)^{\frac{1}{p}} < \infty,$$

for some (equivalently all) $0 < \delta < \alpha$, where $\theta$ is the centering function of $X$ and $\{Y_\lambda : \lambda \in \Lambda\}$ is the corresponding centered process of $X$.

**Proof of (a).** Measurability of $\overline{X}$ follows from Fubini’s theorem and the separability of $L_p(\Lambda, B)$.

Now we write $X_\lambda = Y_\lambda + \theta(\lambda)$, where $\theta$ is the centering function and $Y = \{Y_\lambda : \lambda \in \Lambda\}$ is the corresponding centered $r$-SS($\alpha$) process. We will now show that $\theta$ is Borel measurable: $\Lambda \to B$. Let $y \in B^*$, then $\langle X_\lambda, y \rangle = \langle Y_\lambda, y \rangle + \langle \theta(\lambda), y \rangle$ and $\theta(\lambda) \equiv \langle \theta(\lambda), y \rangle$ is the centering function of the real r-SS($\alpha$) measurable process $\langle X_\lambda, y \rangle$. Then it follows from Remark 3.2(ii) of [14] and Fubini’s theorem, that $\theta(\lambda)$ is real Borel measurable on $\Lambda$, which shows $\theta$ is Borel measurable. Now we shall show that $\mu_{\overline{X}}$ is r-SS($\alpha$), that $\theta \in L_p(\Lambda, B)$ and that it is the centering element of $\mu_{\overline{X}}$.

For each $n = 1, 2, \ldots$, let $X^{(n)}(\lambda) = Y_\lambda I_{\{0(\lambda) \leq n\}} + \theta(\lambda) I_{\{0(\lambda) \leq n\}}$, then each $X^{(n)}$ is an $r$-SS($\alpha$) process with centering function $\theta^{(n)}(\lambda) \equiv \theta(\lambda) I_{\{0(\lambda) \leq n\}}$ and the corresponding centered measurable $r$-SS($\alpha$) process $Y^{(n)}_\lambda = Y_\lambda I_{\{0(\lambda) \leq n\}}$. Since $v$ is finite, $\theta^{(n)} \in L_p(\Lambda, B)$, for each $n$; hence
\[ Y^{(n)}(\cdot, \omega) \in L_p(A, B) \text{ a.s.} \]

Therefore, according to Corollary 2.3, \( Y_n \), the p. measure induced by \( Y^{(n)} \), is a centered \( r - \mathrm{SS}(\chi) \) p. measure on \( L_p(A, B) \). Let \( \mu_n \) be the p. measure induced by \( X^{(n)} \); then, clearly,

\[ \mu_n = \gamma_n \ast \delta_{\theta_n}. \]  

Equation (7) implies that \( \mu_n \) is \( r - \mathrm{SS}(\chi) \) with centering element \( \theta_n \). Now as \( X^{(n)}(\cdot, \omega) \rightarrow X(\cdot, \omega) \), for all \((\lambda, \omega)\); it follows, from the dominated convergence theorem, that \( X^{(n)}(\cdot, \omega) \rightarrow X(\cdot, \omega) \) in \( L_p(A, B) \) for almost all \( \omega \). Hence \( \mu_n \rightarrow \omega \mathcal{L}_X \); therefore, by Proposition 2.1, \( \mathcal{L}_X \) is \( r - \mathrm{SS}(\chi) \); further, as \( \theta_n(\lambda) \rightarrow \theta(\lambda) \) pointwise and \( \{ \theta_n \} \) converges in \( L_p(A, B) \), \( \theta_n \rightarrow \theta \) in \( L_p(A, B) \), and \( \theta \) is the centering element of \( \mathcal{L}_X \).

To prove the converse, we define the measure \( \nu_0 \) and \( \nu_1 \), the map \( \tau \), and the sets \( A_0, A_1 \), and \( S_0, A_0 \) as in the proof of Proposition 2.2.

Define the process \( \{ X_{\lambda} : \lambda \in A \} \) on the probability space \( (L_p(A, B), \mu) \) as follows:

\[
X(\lambda, f) = \begin{cases} 
  f(\lambda) & \text{if } \lambda \in A_1, \\
  \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{\tau^{-1}(\tau(\lambda) - \varepsilon, \tau(\lambda) + \varepsilon)} f(u) \nu_0(du), & \text{if } \lambda \in A_0 \setminus S_0 \text{ and the limit exists,} \\
  0 & \text{elsewhere.}
\end{cases}
\]

Clearly \( X \) is a measurable process and since \( X(\cdot, f) = f \) a.e. [\nu], \( X \) induces the measure \( \mu \) on \( L_p(A, B) \). To see that \( X \) is an \( r - \mathrm{SS}(\chi) \) process one uses the facts that, for a fixed \( y \in B^* \), \( g_y(u) = I_{\tau^{-1}(\tau(\lambda) - \varepsilon, \tau(\lambda) + \varepsilon)}(u) \cdot y \) is an element of \( L_p^*(A, B) \) and that

\[
\left\langle \int_{\tau^{-1}(\tau(\lambda) - \varepsilon, \tau(\lambda) + \varepsilon)} f(u) \nu_0(du), y \right\rangle = \int \langle f(u), g_y(u) \rangle \nu_0(du),
\]

and applies Proposition 2.2.

**Proof of (b).** Let the measures \( \nu_0, \nu_1 \), the map \( \tau \) and the sets \( S_0, A_0, A_1 \), and \( A_0 \) be as in the proof of Proposition 2.2; and let \( X^{(0)} \) and \( X^{(1)} \) be, respectively, the restrictions of \( X \) to \( A_0 \) and \( A_1 \). Then to prove the zero–one law part of the theorem, it is sufficient to prove

\[
\int \|X^{(j)}(\lambda)\|^p \, dv_j < \infty,
\]

with probability zero or one for \( j = 0, 1 \). First, we prove this for \( j = 0 \). Since for \( j = 0 \),

\[
\int_0^1 \|X^{(0)}(\tau^{-1}(\lambda))\|^p \, d\text{Leb} = \int_{A_0} \|X^{(0)}(\lambda)\|^p \, dv_0,
\]

(9)
showing that (8) holds with probability zero or one is equivalent to showing that the left side of (9) is finite with probability zero or one (one can define $X^0_t$ and $X^{(0)}_{t, (\omega)}$ to be zero on $S_0$ and $A_0$, respectively). Let $\xi = \{ \xi_t = X^{(0)}_{t, (\omega)} : t \in [0, 1] \}$ then $\xi$ is a measurable r-SS($\alpha$) process and $\{ n_{\omega} = \| \xi_t \| : t \in [0, 1] \}$ is a real measurable process. Let $\xi$ be the natural map: $\Omega \rightarrow B^{(0, 1)}$, then, according to Lemma 3.4, $\mu_\xi \equiv \mathcal{L}_\xi$ is an r-SS($\alpha$) p. measure on $(B^{(0, 1)}, \mathcal{F}(B^{(0, 1)}))$.

For each $k = 1, 2, \ldots$, let $q_k(t) = t I_{1, \omega \in A_k, t \in [0, \infty)}$, and $\eta^{(k)}_t = (g_k(\| \xi_t \|))_t$, then $\eta^{(k)}_t(\lambda, \omega) \uparrow \eta(\lambda, \omega)$ for all $(\lambda, \omega)$; and, clearly, $\eta^{(k)}$'s are all real bounded measurable processes on $[0, 1]$. Then, using Theorem 2.8 of [9], for positive integers $n$ and $k$, we can choose finitely many points $\{ \lambda(n, k) \}$ in $[0, 1]$ and real constants $\{ c(n, k) \}$ (independent of $\omega$) such that

$$\sum_j^n \eta^{(k)}(\lambda_j) c_j \rightarrow \int_0^1 \eta^{(k)}(\lambda) d \text{Leb}$$

a.s. as $n \rightarrow \infty$. Hence, for almost all $\omega$,

$$\lim_{k} \lim_{n} \sum_j^n \eta^{(k)}(\lambda_j) c_j = \int_0^1 \eta(\lambda) d \text{Leb}. \quad (10)$$

Now let, for every $x \in B^{(0, 1)}$,

$$\psi(x) = \lim_{k} \lim_{n} \sum_j^n (g_k(\| x(\lambda_j) \|))_t$$

then $\psi$ is an extended Borel measurable map on $(B^{(0, 1)}, \mathcal{F}(B^{(0, 1)}))$, and

$$G = \{ x : \psi(x) < \infty \}$$

is an $\mathcal{F}$-measurable linear manifold in $B^{(0, 1)}$. Hence, as we have shown above, $\mu_\xi$ is r-SS($\alpha$); according to Theorem 3.1 of [13], $\mu_\xi(G) = 0$ or 1. But, from (10), $G = \{ \omega : \int_0^1 \eta(\lambda, \omega) d \text{Leb} < \infty \}$ modulo a $P$-null set; hence the left side of (9) is finite with probability zero or one. The fact that \( \int \| X^{(1)}(\lambda) \|_p d \nu_j \leq \infty \) holds with probability zero or one can be proved by noting that the law of the r. object $X^{(1)}(\omega) \rightarrow \{ X_{t, (\omega)}^{(1)} \}$ is an r-SS($\alpha$) p. measure on $(B^\infty, \mathcal{F}(B^\infty))$ and using Theorem 3.1 of [12].

Now let $X(\cdot, \omega) \in L_p(\Lambda, B)$ a.s., then by part (a), $\mu_\xi$ is an r-SS($\alpha$) p. measure on $L_p(\Lambda, B)$. Hence according to Theorem 3.1 of [1], (5) holds for all $\delta$ with $0 < \delta < \alpha$; the equivalence of (5) and (6) follows from the facts that $X(\cdot, \omega) \in L_p(\Lambda, B) \Leftrightarrow Y(\cdot, \omega) \in L_p(\Lambda, B)$ a.s. and $\theta \in L_p(\Lambda, B)$ (see proof of part (a) and Theorem 3.1 of [11]). Finally, if (5) holds for some $0 < \delta < \alpha$, then trivially $X(\cdot, \omega) \in L_p(\Lambda, B)$ a.s. That (6) implies $X(\cdot, \omega) \in L_p(\Lambda, B)$ is also trivial.
Remark 2.6. As our proof shows, the first half of part (b) of Theorem 2.5 also holds for any $L_q(A, B)$ space; in particular, it holds for $L_p(A, B)$ $0 < p < 1$.

Remark 2.7. The correspondence principle for real centered $S(\alpha)$ processes is recently established by Weron [19] which holds for all $0 < p < \infty$ and $0 < \alpha < 2$. Earlier, Louie [10] had proved this principle for $B$-valued $S(\alpha)$ processes; his proof, though yielded the principle for the noncentered case also for all $p > 0$: he had to assume that $p < \alpha$ and $\alpha \neq 1$. Since every $S(\alpha)$ p. measure is an $r$-SS($\alpha$) p. measure, our result yields the correspondence principle for $B$-valued $S(\alpha)$ processes without the unnatural and restrictive hypothesis $p < \alpha$, as long as $1 \leq p$ and $\alpha \neq 1$. The reason our methods are not applicable to investigate a similar principle for $r$-SS($\alpha$) processes when $0 < p < 1$ is that, unlike in locally convex (l.c.) spaces [7], at present no suitable characterization of $r$-SS($\alpha$) p. measures on nonlocally convex topological vector spaces is available. Also, an investigation of this principle for noncentered $r$-SS($\alpha$) processes when $\alpha = 1$ seems to be beyond the methods used here; and, it appears, that different methods need to be used to tackle this case. It may be pointed out here that the methods of proof used in [10, 19] are similar to those developed by Byczkowski [3] who proved the correspondence principle for real Gaussian processes with paths in $L_p$ for all $0 < p < \infty$; his methods of proof do not apply in the present semistable case.

Remark 2.8. The zero–one law and the criterions for the two alternatives similar to (5) for $B$-valued $S(\alpha)$ processes, under the condition $1 < p < \alpha$, are proved by Louie [10]. The criterions for the two alternatives in the real symmetric $S(\alpha)$ case under the condition $1 < p < \alpha$ was also proved by Cambanis and Miller [6]. The proofs in both of these papers follow a truncation argument and a method first used in [13]. The zero–one law proved in Theorem 2.5 yields the zero–one law for $B$-valued $S(\alpha)$ processes mentioned above for any $p \geq 1$ and $0 < \alpha < 2$; thus extending the above known zero–one laws in the stable case. Similar remark applies for the criterion for the two alternatives in the zero–one law.

3. Absolute Continuity of Paths of $r$-SS($\alpha$) Processes

In this section, we obtain several necessary and sufficient conditions for almost sure absolute continuity of paths of $B$-valued $r$-SS($\alpha$) processes. The proof of the equivalence of the first two conditions of the theorem follow from the methods of proof of the equivalence of the corresponding conditions for $B$-valued $S(\alpha)$ processes given by Louie [10] who, in turn, used modifications of techniques of proof of Cambanis developed in [4, 5]...
where equivalence of corresponding statements for second order processes are proved. We begin with a few conventions: If \( X_\lambda = \xi_\lambda + i\eta_\lambda, \lambda \in \Lambda \), where \( \{\xi_\lambda\} \cup \{\eta_\lambda\} \) is a \( B \)-valued process, then we say \( X \) is a complex \( B \)-valued process; further, if the imaginary component is zero, then, to distinguish such a process from that defined above, we call \( X \) (in this section) to be a real \( B \)-valued process. Before we state our theorem, we may note that the definitions of absolute continuity and differentiability used below are analogous to the corresponding definitions for real valued functions.

**Theorem 3.1.** Assume \( B \) is (separable) and reflexive and \( X = \{X_\lambda : \lambda \in \Lambda = [0, 1]\} \) be a separable \( r\)-SS(\( \alpha \)) complex \( B \)-valued process and \( 1 < p < \alpha \), then the following two statements are equivalent:

(i) sample paths of \( X \) are absolutely continuous a.s. \( [P] \)

(ii) the map \( \dot{X} : \Lambda \to L_p(\Omega, B) \) defined by \( \dot{X}(t) = X_t \) is absolutely continuous.

Further, if one (and hence both) of the above statements holds then

\[
\frac{dX}{d\lambda} \in L_p(\Lambda, B) \quad \text{a.s.} \quad [P] \Leftrightarrow \frac{d\dot{X}}{d\lambda} \in L_1(\Lambda, L_p(\Lambda, B))
\]

(recall, from Brezis [2], that under (i) (resp. (ii)) \( dX/d\lambda \) (resp. \( d\dot{X}/d\lambda \)) exists a.e. \( [v] \)).

If the process takes values in \( C \), the complex field, and has the representation \( X_\lambda = \int_\Gamma f_\lambda dM \), where \( M \) is a complex \( r\)-SS(\( \alpha \)) random measure with the associated marginal control measure \( \gamma \) on Borel subsets of an interval \( T \) of \( R \) and \( f_\lambda \in L_s(T, C) \) (see [14, 15] for details on these representations), then each of the above is equivalent to:

(iii) the map \( \lambda \to f_\lambda \) from \( \Lambda \) into \( L_s(\Lambda, C) \) is absolutely continuous.

Finally, if \( f_\lambda \) is of the special form \( f_\lambda(t) = g(t) \int_0^t \theta(t, s) ds \), where \( \theta \) is a complex function such that \( \theta(t, \cdot) \) is continuous for every \( t \in T \) and \( |\theta(t, s)| = 1 \), then each of the above is equivalent to:

(iv) \( \int_T |g(t)|^p dt < \infty \).

**Proof.** As noted above, the proof of (1) and that of the equivalence of (i) and (ii) for \( B \)-valued \( S(\alpha) \) processes is given by Louie [10] using modifications of techniques of Cambanis [4] (see also [5]) who proved these results for second order processes. Using Lemma 2.4 and Theorem 3.1 of [11], precisely the same methods as in [10] can be used to prove (1) and the equivalence of (i) and (ii) for the complex \( B \)-valued \( r\)-SS(\( \alpha \)) processes. In fact, these same proofs yield these results for any \( p \)th order.
complex $B$-valued process of which the canonical induced measure of every measurable subspace of $B^4$ is zero or one.

We now prove the equivalence (ii) and (iii). We recall from [15] that if $\zeta = \int_T f dM$, $f \in L_\infty(T, C)$, then the following inequality holds:

$$C_0 \left( \int_T |f|^2 \, d\gamma \right)^{1/\alpha} \leq \|\xi\|_p \leq C_1 \left( \int_T |f|^2 \, d\gamma \right)^{1/\alpha}, \quad (3)$$

where $C_0$ and $C_1$ are universal positive constants. Using (3), for any finite collection $\{(\lambda_j, \lambda'_j)\}$ of disjoint intervals of $A$, we have

$$C_0 \sum_j \left( \int |f_{\lambda_j} - f_{\lambda'_j}|^2 \, d\nu \right)^{1/\alpha} \leq \sum_j \|X_{\lambda_j} - X_{\lambda'_j}\|_p \leq C_1 \sum_j \left( \int |f_{\lambda_j} - f_{\lambda'_j}|^2 \, d\nu \right)^{1/\alpha};$$

which proves the equivalence of (ii) and (iii).

Now let $f_j(t) = g(t) \int_0^t \theta(t, s) \, ds$ and let (ii) holds; then $d\hat{X}/d\lambda$ exists a.e. [v]. Let $\lambda_0$ be such that $\hat{X}$ is differentiable at $\lambda_0$, then we have

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{-1} \|X_\lambda - X_{\lambda_0}\|_p < \infty. \quad (4)$$

Now, using (3), we have

$$C_0 |\lambda - \lambda_0|^{-1} \left( \int |g(t)|^2 \left| \int_0^\lambda \theta(t, s) \, ds \right|^\alpha \gamma(dt) \right)^{1/\alpha}$$

$$= C_0 (\lambda - \lambda_0)^{-1} \left( \int |g(t)|^2 \left| \int_0^\lambda \theta(t, s) \, ds - \int_0^{\lambda_0} \theta(t, s) \, ds \right|^\alpha \gamma(dt) \right)^{1/\alpha}$$

$$\leq |\lambda - \lambda_0|^{-1} \|X_\lambda - X_{\lambda_0}\|_p;$$

hence, using (4), Fatou's lemma and the fact $\lim_{\lambda \to \lambda_0} |\lambda - \lambda_0|^{-1} \left| \int_0^\lambda \theta(t, s) \, ds \right| = 1$, we have

$$C_0 \left( \int |g(t)|^x \, d\gamma \right)^{1/\alpha}$$

$$= C_0 \left( \int |g(t)|^x \liminf_{\lambda \to \lambda_0} |\lambda - \lambda_0|^{-1} \left| \int_0^\lambda \theta(t, s) \, ds \right|^\alpha \gamma(dt) \right)^{1/\alpha}$$

$$\leq \liminf_{\lambda \to \lambda_0} |\lambda - \lambda_0|^{-1} \|X_\lambda - X_{\lambda_0}\|_p < \infty.$$

Finally, assume (2) holds, then using the right-side inequality of (3), we have, for any finite collection $\{(\lambda_j, \lambda'_j)\}$ of disjoint intervals of $A$,
this shows that the map $\tilde{X}$ is absolutely continuous. This completes the proof.

Remark 3.1. If $\theta(t, s) = e^{it\lambda}$ and $g(t) = t$ in the above, then $f_j(t) = e^{it\lambda}$, and one obtains that the $r$-SS($\alpha$) process $X_t = \int e^{it\lambda} dM$ is absolutely continuous a.s. $\int |t|^\alpha \, dy < \infty$. This result for real symmetric stable processes was proved by Cambanis and Miller [6] using properties of covariance function; the present proof appears to be simpler.

Remark 3.2. In addition to the hypotheses stipulated in Theorem 3.1 on $B$ and $X$, assume that $B$ is of stable type $p > 1$ and $X$ is a (real) symmetric $B$-valued $S(\alpha)$ process with $p < \alpha$. Then, using the stochastic integrals $\int f \, dM$, where $f \in L_p(T, B)$ and $M$ is a real symmetric $S(\alpha)$ random measure (see Remark 4.3 of [15]), and using the analog of inequality (3), the same proof as above yields the equivalence (i) to (iv) for symmetric real $B$-valued $S(\alpha)$ processes. Of course, here it is tacitly assumed that $f_j \in L_p(T, B)$, $g$ is real, and $\theta$ is $B$-valued.

Note added in proof. We direct the attention of the reader to a recent paper “On stochastic integral representation of stable processes with sample paths in Banach spaces” by J. Rosinski to appear in this journal. Several results of this paper are related to some of the results presented in our paper.

REFERENCES


