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Procedia Computer Science 74 (2015) 118 – 123

Procedia Computer Science

International Conference on Graph Theory and Information Security

On Total Irregularity Strength of Double-Star and Related Graphs

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Abstract

Let $G = (V, E)$ be a simple and undirected graph with a vertex set V and an edge set E. A totally irregular total k-labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ is a labeling of vertices and edges of G in such a way that for any two different vertices x and x', their weights $wt_f(x) = f(x) + \sum_{xy \in E} f(xy)$ and $wt_f(x') = f(x') + \sum_{x'y' \in E} f(x'y')$ are distinct, and for any two different edges xy and $x'y'$ their weights $f(x) + f(xy) + f(y)$ and $f(x') + f(x'y') + f(y')$ are also distinct. A total irregularity strength of graph G, denoted by ts(G), is defined as the minimum k for which G has a totally irregular total k-labeling. In this paper, we determine the exact value of the total irregularity strength for double-star $S_{n,m}, n, m \geq 3$ and graph related to it, that is a caterpillar $S_{n,2,n}, n \geq 3$. The results are $ts(S_{n,m}) = \left\lceil \frac{n+m-1}{2} \right\rceil$ and $ts(S_{n,2,n}) = n$.

Peer-review under responsibility of the Organizing Committee of ICGTIS 2015

Keywords: Totally irregular total k-labeling, total irregularity strength, weight, double-star, caterpillar.
2010 MSC: 05C78

1. Introduction

Let us consider a simple, connected and undirected graph G with a vertex set ($V(G)$ and an edge set $E(G)$). A labeling of a graph G is a mapping that carries a set of graph elements into a set of integers, called labels (see Wallis[14]). If the domain of mapping is a vertex set, or an edge set, or a union of vertex and edge sets, then the labeling is called *vertex labeling*, *edge labeling*, or *total labeling*, respectively. In his survey, Gallian[5] shows that there are various kinds of labelings on graphs, and one of them is an irregular total labeling.

For a graph G, Bača *et al.*[4] defined a labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be a *vertex irregular total k-labeling* if for every two different vertices x and y the *vertex-weights* $wt_f(x) \neq wt_f(y)$, where the vertex-weight $wt_f(x) = f(x) + \sum_{xz \in E} f(xz)$. A minimum k for which G has a vertex irregular total k-labeling is defined as the *total vertex irregularity strength* of G and denoted by $tvs(G)$. They obtained the exact values of the total vertex irregularity strength for cycles, stars, complete graphs and prisms. Moreover, Nurdin *et al.*[11] proved the exact value of the total

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1877-0509
Peer-review under responsibility of the Organizing Committee of ICGTIS 2015
doi:10.1016/j.procs.2015.12.086

vertex irregularity strength for any tree T with n pendant vertices and no vertex of degree two, that is

$$tvs(T) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{1}$$

For a graph G, Bača *et al.*[4] also define a labeling $g : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G the *edge-weights* $wt_g(xy) = g(x) + g(xy) + g(y)$ and $wt_g(x'y') = g(x') + g(x'y') + g(y')$ are different. The *total edge irregularity strength* denoted by $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. They also obtained the exact values of the *tes* for paths, cycles, stars, wheels and friendship graphs. The *tes* of generalized helm and generalized web graphs have been determined by Indriati *et al.*[6] ,[7]. Moreover, Ivančo and Jendrol[9] proved that for any tree T, satisfy

$$tes(T) = \max\left\{\lceil(|E(T)| + 2)/3\rceil, \lceil(\Delta(T) + 1)/2\rceil\right\}. \tag{2}$$

Combining the ideas of vertex irregular total k-labeling and edge irregular total k-labeling, Marzuki *et al.*[10] introduced another irregular total k-labeling called the totally irregular total k-labeling.
A labeling $h : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be a *totally irregular total k-labeling* of the graph G if for every two different vertices x and y the *vertex-weights* $wt_h(x) \neq wt_h(y)$, where the vertex-weight $wt_h(x) = h(x) + \sum_{xz\in E} h(xz)$ and also for every two different edges xy and $x'y'$ of G the *edge-weights* $wt_h(xy) = h(x) + h(xy) + h(y)$ and $wt_h(x'y') = h(x') + h(x'y') + h(y')$ are different. The *total irregularity strength*, $ts(G)$, is defined as the minimum k for which G has a totally irregular total k-labeling. For the total irregularity strength of a graph G, they observed that

$$ts(G) \geq \max\{tes(G), tvs(G)\}. \tag{3}$$

They determined the total irregularity strength of cycles and paths. Ramdani and Salman[12] obtained the total irregularity strength of some cartesian product graphs, namely $K_{1,n}\Box P_2$, $P_n\Box P_2$, $(P_n+P_1)\Box P_2$, and $C_n\Box P_2$. In[13], Ramdani *et al.* determined the total irregularity strength of Gear graphs $G_n, n \geq 3$, fungus graphs $F_{g_n}, n \geq 3$ and disjoint union of stars $mS_n, n, m \geq 2$.
In order to find the total irregularity strength for tree in general, we have started the investigation for star $K_{1,n}$ and double star $S_{n,n}$ (see[8]). In this paper, we continue to investigate the total irregularity strength of double-star $S_{n,m}$ for $n, m \geq 3$ and related graphs, a caterpillar $S_{n,2,n}$ for $n \geq 3$.

2. Double-Star Graphs

The next theorem presents an exact value of the total irregularity strength of double stars, $S_{n,m}$, that is a tree obtained by connecting the centers of two disjoint stars $K_{1,n}$ and $K_{1,m}$ for $n, m \geq 3$. One of the pendant vertex of $K_{1,n}$ and $K_{1,m}$ is embedded to the center of $K_{1,m}$ and $K_{1,n}$ respectively.

Theorem 1. *Let $S_{n,m}$, $n, m \geq 3$, be a double star graphs. Then*

$$ts(S_{n,m}) = \left\lceil \frac{n+m-1}{2} \right\rceil.$$

Proof. First, assume that $n \leq m$. Therefore, if $m < n$, then we could change m by n, for example $S_{5,3}$ could be written as $S_{3,5}$. The maximal degree of this graph is $\Delta = m$. A double-star $S_{n,m}$ is a tree T with $n + m - 1$ edges, $n + m - 2$ pendant vertices with no vertex of degree two and no isolated vertex. According to (1), we know that

$$tvs(S_{n,m}) = \left\lceil \frac{n+m-1}{2} \right\rceil, \tag{4}$$

and from (2) we obtain

$$\text{tes}(S_{n,m}) = \max\left\{\left\lceil \frac{m+1}{2} \right\rceil, \left\lceil \frac{n+m+1}{3} \right\rceil\right\}$$

There are two possibilities value of tes($S_{n,m}$).

1. Suppose that tes($S_{n,m}$) $= \left\lceil \frac{m+1}{2} \right\rceil$. According to (4) , $n + m - 1 = m + 1 + n - 2 > m + 1$, for $n \geq 3$. Therefore, $\max\{tes(S_{n,m}), tvs(S_{n,m})\} = tvs(S_{n,m})$.
2. Suppose that tes($S_{n,m}$) $= \left\lceil \frac{n+m+1}{3} \right\rceil$. By (4) it was resulted that $\left\lceil \frac{m+n-1}{2} \right\rceil > \left\lceil \frac{m+n+1}{3} \right\rceil =$ tes($S_{n,m}$).
Then, $\max\{tes(S_{n,m}), tvs(S_{n,m})\} = tvs(S_{n,m})$.
For the all possibilities, according to (3), we conclude that $ts(S_{n,m}) \geq \left\lceil \frac{n+m-1}{2} \right\rceil$.
In order to obtain the total irregularity strength of $S_{n,m}$, we prove the upper bound of this parameter as follows.

Assume the set of vertices of $S_{n,m}$ is $V(S_{n,m}) = \{v_i^1 : 1 \leq i \leq n-1\} \cup \{v_i^2 : 1 \leq i \leq m-1\} \cup \{v^j : 1 \leq j \leq 2\}$ and the set of edges is $E(S_{n,m}) = \{v^1 v_i^1 : 1 \leq i \leq n-1\} \cup \{v^2 v_i^2 : 1 \leq i \leq m-1\} \cup \{v^1 v^2\}$.

Case 1: For both n and m are even or odd.
Let $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. Define the total labeling g by the following way.

$$
\begin{aligned}
g(v_i^j) &= \begin{cases} i, & \text{for} 1 \leq i \leq n-1,\ j = 1, \\ \min\{n-1+i, k\}, & \text{for} 1 \leq i \leq m-1,\ j = 2, \end{cases} \\
g(v^j) &= k,\ \text{for } j = 1, 2, \\
g(v^1 v^2) &= k, \\
g(v^1 v_i^1) &= 1,\ \text{for } 1 \leq i \leq n-1, \\
g(v^2 v_i^2) &= \begin{cases} 1, & \text{for } 1 \leq i \leq \left\lceil \frac{|n-m|+1}{2} \right\rceil, \\ i - \left\lceil \frac{|n-m|+1}{2} \right\rceil + 1, & \text{for} \left\lceil \frac{|n-m|+1}{2} \right\rceil + 1 \leq i \leq m-1. \end{cases}
\end{aligned}
$$

It can be seen that under the total labeling g, all vertex and edge labels are at most $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. Therefore, g is a total k-labeling. Then, the vertex and edge-weights are as follows:

$$
\begin{aligned}
wt_g(v_i^j) &= \begin{cases} i + 1, & \text{for } 1 \leq i \leq n-1,\ j = 1, \\ n + i, & \text{for } 1 \leq i \leq m-1,\ j = 2. \end{cases} \\
wt_g(v^j) &= \begin{cases} 2k + n - 1, & \text{for } j = 1, \\ 2k + m - 1 + \frac{\left(m-1-\left\lceil \frac{|n-m|+1}{2} \right\rceil\right)\left(m-\left\lceil \frac{|n-m|+1}{2} \right\rceil\right)}{2}, & \text{for } j = 2. \end{cases} \\
wt_g(v^j v_i^j) &= \begin{cases} k + i + 1, & \text{for } 1 \leq i \leq n-1,\ j = 1, \\ k + n + i, & \text{for } 1 \leq i \leq m-1,\ j = 2. \end{cases} \\
wt_g(v^1 v^2) &= 3k.
\end{aligned}
$$

Since the vertex-weights of v_i^j form a set of consecutive integers from 2 up to $n+m-1$ and the weights of v^j, $j = 1, 2$ are distinct and greater than $n + m - 1$, then it indicates that the weight of each pair of vertices is distinct. The weight of pendant edges also construct a set of consecutive integers from $k + 2$ up to $k + n + m - 1$ and the weight of $v^1 v^2$ is $3k$ which is greater than the weight of all pendant edges. Therefore, it conclude that g is a totally irregular total k-labeling with $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. Then, $ts(S_{n,m}) = k = \left\lceil \frac{n+m-1}{2} \right\rceil$.

Case 2: For n and m have a different parity.
Suppose $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. Define a total labeling g as follows.

$$g(v_i^1) = \begin{cases} 1, & \text{for } 1 \le i \le k, \\ i-k+1, & \text{for } k+1 \le i \le m-1. \end{cases}$$
$$g(v_i^2) = k+i+1-n, \text{ for } 1 \le i \le n-1.$$
$$g(v^j) = \begin{cases} k-1, & \text{for } j=1, \\ k, & \text{for } j=2. \end{cases}$$
$$g(v^1v^2) = k-n+2.$$
$$g(v^1v_i^1) = \begin{cases} i, & \text{for } 1 \le i \le k, \\ k, & \text{for } k+1 \le i \le m-1. \end{cases}$$
$$g(v^2v_i^2) = k, \text{ for } 1 \le i \le n-1.$$

Under the labeling g, it can be seen that the greatest label of edges and vertices is $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. It means that g is an irregular total k-labeling with $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. The weight of vertices and edges are as follows.

$$\begin{aligned} wt_g(v^1) &= \sum_{i=1}^{k} i + (m-1-k)k + 2k + 1 - n \\ &= \frac{k(k+1)}{2} + (m+1-k)k + 1 - n. \end{aligned}$$
$$wt_g(v^2) = (n+1)k - n + 2.$$
$$wt_g(v_i^j) = \begin{cases} i+1, & \text{for } 1 \le i \le m-1,\ j=1, \\ 2k+i-n+1, & \text{for } 1 \le i \le n-1,\ j=2. \end{cases}$$
$$wt_g(v^jv_i^j) = \begin{cases} k+i, & \text{for } 1 \le i \le m-1,\ j=1, \\ 3k+i-n+1, & \text{for } 1 \le i \le n-1,\ j=2. \end{cases}$$
$$wt_g(v^1v^2) = 3k-n+1.$$

The weight of pendant vertices v_i^j form a set of consecutive integers from 2 up to $2k$ and the weight of v^j, $j = 1, 2$ are distinct and greater than $2k$. Therefore, the weight of every pair of vertices are distinct. The edge-weights also form a set of consecutive integers from $k+1$ up to $3k$, which is indicated that the weight of every pair of edges is distinct. Therefore, we conclude that g is a totally irregular total k-labeling with $k = \left\lceil \frac{n+m-1}{2} \right\rceil$. Then, $ts(S_{n,m}) = k = \left\lceil \frac{n+m-1}{2} \right\rceil$. □

3. Caterpillar $S_{n,2,n}$

A caterpillar $S_{n,2,n}$ is a class of graph constructed from the double-star $S_{n,n}$ by inserting one vertex on the bridge connecting of the two centers of two stars. Therefore, this caterpillar contains three stars with the center of the two end-stars have degree n and the center of star in the middle has degree two. This graph is a tree with $2n+1$ vertices, $2n$ edges and $2n-2$ pendant vertices. Maximal degree of the graph is $\Delta = n$.

According to (3), the lower bound of its total irregularity strength is the maximum value between its total edge irregularity strength and its total vertex irregularity strength. The total edge irregularity strength of graph $S_{n,2,n}$ can be found by (2), that is

$$tes(S_{n,2,n}) = \max\left\{\left\lceil \frac{\Delta+1}{2} \right\rceil, \left\lceil \frac{|E|+2}{3} \right\rceil\right\} = \max\left\{\left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{2n+2}{3} \right\rceil\right\} = \left\lceil \frac{2n+2}{3} \right\rceil. \quad (5)$$

The next theorem gives the total vertex irregularity strength of $S_{n,2,n}$.

Theorem 2. *Let consider a caterpillar $S_{n,2,n}$, $n \geq 3$. Its total vertex irregularity strength is*

$$tvs(S_{n,2,n}) = n.$$

Proof. $S_{n,2,n}$ is a tree with $2n+1$ vertices, $2n$ edges and $2n-2$ pendant vertices. Because of one vertex as a center of the middle star has degree two, then (1) can not be used for determining the total vertex irregularity strength of the graph. $S_{n,2,n}$ has $2n-2$ pendant vertices, one vertex of degree two and two vertices of degree $n \geq 3$. In order to obtain as small as possible label, start the labeling from the vertex with smallest degree (in this situation, pendant vertices have a smallest degree). After that, we continue the labeling for vertices with greater degree than before, and so on until all vertices are labeled. The smallest vertex-weight is two, then with a consecutive weight, the smallest weight of $2n-2$ pendant vertices is not smaller than $2n-2+1 = 2n-1$ which is a sum of two labels, namely the label of pendant vertex and the label of edge incident to this vertex. Then, the greatest label of pendant vertices is not smaller than $\left\lceil \frac{2n-1}{2} \right\rceil$. There is a vertex of degree two, therefore the greatest vertex-label is not smaller than $\left\lceil \frac{2n}{3} \right\rceil$. For two vertices of degree n, the greatest label is not smaller than $\left\lceil \frac{2n+2}{n+1} \right\rceil$. Therefore, the greatest label of all vertices is not smaller than

$$\max\left\{ \left\lceil \frac{2n-1}{2} \right\rceil, \left\lceil \frac{2n}{3} \right\rceil, \left\lceil \frac{2n+2}{n+1} \right\rceil \right\} = \left\lceil \frac{2n-1}{2} \right\rceil = n.$$

Let the vertex set of this graph be $V(S_{n,2,n}) = \{v_i^j : 1 \leq i \leq n-1, j = 1,3\} \cup \{v^j : j = 1,2,3\}$ and the edge set be $E(S_{n,2,n}) = \{v^j v_i^j : 1 \leq i \leq n-1, j = 1,3\} \cup \{v^j v^{j+1} : j = 1,2\}$. To determine the exact value of tvs, define the vertex irregular total k-labeling h as follows.

$$h(v_i^j) = \begin{cases} 1, & \text{for } 1 \leq i \leq n-1,\ j = 1, \\ i, & \text{for } 1 \leq i \leq n-1,\ j = 3. \end{cases}$$

$$h(v^j) = \begin{cases} n-1, & \text{for } j = 1, \\ n, & \text{for } j = 2,3. \end{cases}$$

$$h(v^j v_i^j) = \begin{cases} i, & \text{for } 1 \leq i \leq n-1,\ j = 1, \\ n, & \text{for } 1 \leq i \leq n-1,\ j = 3. \end{cases}$$

$$h(v^j v^{j+1}) = \begin{cases} 1, & \text{for } j = 1, \\ n, & \text{for } j = 2. \end{cases}$$

Under the total labeling h it is shown that the greatest label for all vertices is n. It means that h is a total k-labeling with $k = n$. The weight of vertices are as follows.

$$wt(v_i^j) = \begin{cases} i+1, & \text{for } 1 \leq i \leq n-1,\ j = 1, \\ n+i, & \text{for } 1 \leq i \leq n-1,\ j = 3. \end{cases}$$

$$wt(v^1) = \begin{cases} \frac{n+1}{2} n, & \text{for } n \geq 3 \text{ and odd}, \\ \frac{n}{2} n + \frac{n}{2}, & \text{for } n \geq 4 \text{ and even}. \end{cases}$$

$$wt(v^j) = \begin{cases} 2n+1, & \text{for } n \geq 3,\ j = 2, \\ n(n+1), & \text{for } n \geq 3,\ j = 3. \end{cases}$$

The weight of vertices v_i^j for j=1 and 3 form a consecutive integers from 2 up to n and from $n+1$ up to $2n-1$, respectively. The weight of v^2 is $2n+1$ which is smaller than the weight of v^1, and the weight of v^1 is a half of the weight of v^3. Then, it indicates that the weight of every pair of vertices are distinct. Therefore, we conclude that h is a vertex irregular total k-labeling. Then, $tvs(S_{n,2,n}) = k = n$. □

The next theorem proved the total irregularity strength of graph $S_{n,2,n}$ as follows.

Theorem 3. *Let $S_{n,2,n}$, $n \geq 3$ be a caterpillar graph. Then,*

$$ts(S_{n,2,n}) = n.$$

Proof. As in Theorem 2, $S_{n,2,n}$ is a tree having $2n+1$ vertices, $2n$ edges and $2n-2$ pendant vertices. Theorem 2 proves that $\text{tvs}(S_{n,2,n}) = n$. Total edge irregularity strength of this graph is in (5). According to (3), the lower bound of total irregularity strength is

$$ts(S_{n,2,n}) \geq \max\{tes(S_{n,2,n}), tvs(S_{n,2,n})\} = \max\left\{\left\lceil \frac{2n+2}{3} \right\rceil,\ n\right\} = n.$$

To prove the exact value of ts, we show the existence of totally irregular total k-labeling with $k = n$. The similar definition of vertex and edge set of $S_{n,2,n}$ as presented in Theorem 2. In fact, the vertex irregular total k-labeling which is obtained in Theorem 2 satisfies the condition of totally irregular total k-labeling. The weight of the edges are as follows.

$$wt(v^j v_i^j) = \begin{cases} n+i, & \text{for } 1 \leq i \leq n-1,\ j = 1, \\ 2n+i, & \text{for } 1 \leq i \leq n-1,\ j = 3. \end{cases}$$

$$wt(v^j v^{j+1}) = \begin{cases} 2n, & \text{for } n \geq 3,\ j = 1, \\ 3n, & \text{for } n \geq 3,\ j = 2. \end{cases}$$

It can be seen that the weight of $v^1 v_i^1$ and $v^3 v_i^3$ form a consecutive integers from $n+1$ up to $2n-1$ and from $2n+1$ up to $3n-1$, respectively. The weight of $v^1 v^2$ is $2n$ and the weight of $v^2 v^3$ is $3n$. Therefore, the weight of each pair of edges are distinct. It means that h is also totally irregular total k-labeling with the total irregularity strength, $ts(S_{n,2,n}) = k = n$. □

Furthermore, we conclude this paper with the following conjecture for the direction of further research which is still in progress.

Conjecture: The total irregularity strength of caterpillar $S_{n,2,m}$ for $n, m \geq 3$ is $\left\lceil \frac{n+m-1}{2} \right\rceil$.

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